

Announcements

- Homework 12 due Friday (posted).

Recall: Area under a function

Suppose f is continuous and $f(x) \geq 0$ throughout the interval $[a, b]$.

We can approximate the area under the curve by summing rectangles.

Let

$$\begin{aligned} S_n &= [f(a + \Delta x) + f(a + 2\Delta x) + \cdots + f(a + n\Delta x)]\Delta x \\ &= \sum_{k=1}^n f(a + k\Delta x)\Delta x \end{aligned}$$

where $\Delta x = \frac{b-a}{n}$.

Then,

$$\begin{aligned} A &= \lim_{n \rightarrow +\infty} S_n \\ &= \lim_{\Delta x \rightarrow 0} \sum_{k=1}^n f(a + k\Delta x)\Delta x \end{aligned}$$

Summation formulas

$$\sum_{k=1}^n 1 = n$$

$$\sum_{k=1}^n k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Riemann Sums

Suppose a bounded function f is given along with a closed interval $[a, b]$ on which f is defined.

1. Partition the interval $[a, b]$ into n subintervals by choosing points $[x_0, x_1, \dots, x_n]$ arranged so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Call this partition P .

For $k = 1, 2, 3, \dots, n$, the k^{th} subinterval width is

$\Delta x_k = x_k - x_{k-1}$. The largest of these widths is called the **norm** of the partition P and is denoted $\|P\|$.

$$\|P\| = \max_{k=1,2,\dots,n} \{\Delta x_k\}$$

2. Choose a number x_k^* arbitrarily from each subinterval $[x_{k-1}, x_k]$. This number is called the k^{th} **subinterval representative** of the partition P .

3. Form the sum

$$\begin{aligned} R_n &= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n \\ &= \sum_{k=1}^n f(x_k^*)\Delta x_k \end{aligned}$$

This is the **Riemann sum** associated with f , the given partition P and the chosen subinterval representatives

$$x_1^*, x_2^*, \dots, x_n^*.$$

Note that the Riemann sum does not require that the function be nonnegative.

Definite Integral

If f is defined on the closed interval $[a, b]$, we say that f is **integrable on $[a, b]$** if

$$I = \lim_{\|P\| \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

exists. (x_k^* = subinterval representatives.)

This limit is called the **definite integral** of f from a to b . The definite integral is denoted by

$$I = \int_a^b f(x) dx$$

Definite integral at a point:

$$\int_a^a f(x) dx = 0$$

Interchanging the limits of a definite integral:

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

Theorem: If f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$.

Area as an integral: Suppose f is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$. Then the area under the curve $y = f(x)$ on $[a, b]$ is given by the definite integral of f on $[a, b]$. I.e.,

$$\text{AREA} = \int_a^b f(x) dx$$

If f is continuous and sometimes positive and sometimes negative, then

$$\text{AREA} = \int_a^b f(x) dx = A_1 - A_2$$

where

- A_1 is the sum of all areas of regions above the x-axis and below the graph of f (i.e., where $f(x) \geq 0$) and
- A_2 is the sum of all areas of the regions below the x-axis and above the graph of f (i.e., where $f(x) \leq 0$).

Properties of the definite integral

- **Linearity:** If f and g are integrable on $[a, b]$, then so is $rf + sg$, for constants r and s .

$$\int_a^b [rf(x) + sg(x)] dx = r \int_a^b f(x) dx + s \int_a^b g(x) dx$$

- **Dominance:** If f and g are integrable on $[a, b]$ and $f(x) \leq g(x)$ throughout the interval, then

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx$$

- **Subdivision:** For any number c such that $a < c < b$,

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

Distance as an integral

The **total distance traveled** by an object with continuous velocity $v(t)$ along a straight line from time $t = a$ to $t = b$ is

$$S = \int_a^b |v(t)| dt$$