## Announcements

- Homework 10 & 11 due today.
  - Solutions will be posted this afternoon.
- **Exam 3: Monday 11/19/2007**.
  - Strauss sections 3.8, 4.1 through 4.6, 5.1.
- Discussion sections canceled Tuesday 11/20/2007.

## Area under a function

Suppose $f$ is continuous and $f(x) \geq 0$ throughout the interval $[a, b]$.

We can approximate the area under the curve by summing rectangles.

E.g.,

$$A \approx f(a + \Delta x)\Delta x + f(a + 2\Delta x)\Delta x + \cdots + f(a + n\Delta x)\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.

## Area as the limit of a sum

Suppose $f$ is continuous and $f(x) \geq 0$ throughout the interval $[a, b]$. Then the **area** of the region under the curve $y = f(x)$ over the interval $[a, b]$ is

$$A = \lim_{\Delta x \to 0} [f(a + \Delta x) + f(a + 2\Delta x) + \cdots + f(a + n\Delta x)]\Delta x,$$

where $\Delta x = \frac{b-a}{n}$.

## Summation notation

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

Also called **sigma notation**.

We can rewrite the area formula in summation notation:
Let

$$\begin{aligned} S_n &= [f(a+\Delta x) + f(a+2\Delta x) + \cdots + f(a+n\Delta x)]\Delta x \\ &= \sum_{k=1}^{n} f(a+k\Delta x)\Delta x \end{aligned}$$

where $\Delta x = \frac{b-a}{n}$.

Then,

$$\begin{aligned} A &= \lim_{n\to+\infty} S_n \\ &= \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(a+k\Delta x)\Delta x \end{aligned}$$

## Basic rules for sums

| | |
|---|---|
| Constant term rule: | $\sum_{k=1}^{n} c = c + c + \cdots + c = nc$ |
| Sum rule: | $\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$ |
| Scalar mult. rule: | $\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$ |
| Linearity rule: | $\sum_{k=1}^{n}(ca_k + db_k) = c\sum_{k=1}^{n} a_k + d\sum_{k=1}^{n} b_k$ |
| Subtotal rule: | If $1 < m < n$, then |
| | $\sum_{k=1}^{n} a_k = \sum_{k=1}^{m} a_k + \sum_{k=m+1}^{n} a_k$, |
| Dominance rule: | If $a_k \leq b_k$ for $k = 1, 2, \ldots, n$, then |
| | $\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k$ |

## Summation formulas

$$\sum_{k=1}^{n} 1 = n$$

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

## Riemann Sums

Suppose a bounded function $f$ is given along with a closed interval $[a, b]$ on which $f$ is defined.

1. Partition the interval $[a, b]$ into $n$ subintervals by choosing points $[x_0, x_1, \ldots, x_n]$ arranged so that

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Call this partition $P$.
For $k = 1, 2, 3, \ldots, n$, the $k^{th}$ subinterval width is $\Delta x_k = x_k - x_{k-1}$. The largest of these widths is called the **norm** of the partition P and is denoted $\|P\|$.

$$\|P\| = \max_{k=1,2,\ldots,n} \{\Delta x_k\}$$

2. Choose a number $x_k^*$ arbitrarily from each subinterval $[x_{k-1}, x_k]$. This number is called the $k^{th}$ **subinterval representative** of the partition $P$.

3. Form the sum

$$\begin{aligned} R_n &= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n \\ &= \sum_{k+1}^{n} f(x_k^*)\Delta x_k \end{aligned}$$

This is the **Riemann sum** associated with $f$, the given partition $P$ and the chosen subinterval representatives $x_1^*, x_2^*, \ldots, x_n^*$.

Note that the Riemann sum does not require that the function be nonnegative.

## Definite Integral

If $f$ is defined on the closed interval $[a, b]$, we say that $f$ is **integrable on $[a, b]$** if

$$I = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

exists.

This limit is called the **definite integral** of $f$ from $a$ to $b$. The definite integral is denoted by

$$I = \int_a^b f(x) dx$$

Definite integral at a point:

$$\int_a^a f(x)dx = 0$$

Interchanging the limits of a definite integral:

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

**Theorem:** If $f$ is continuous on an interval $[a, b]$, then $f$ is integrable on $[a, b]$.

**Area as an integral:** Suppose $f$ is continuous and $f(x) \geq 0$ on the closed interval $[a, b]$. Then the area under the curve $y = f(x)$ on $[a, b]$ is given by the definite integral of $f$ on $[a, b]$. I.e.,

$$\text{AREA} = \int_a^b f(x)dx$$