Announcements

- Tuesday 10/30 is the last day to drop
- Homework 9 due this Friday 11/2/2007

Monotone function theorem: Let f be differentiable on the open interval (a, b).

- If $f'(x) > 0$ on (a, b), then f is strictly increasing on (a, b).
- If $f'(x) < 0$ on (a, b), then f is strictly decreasing on (a, b).

1st Derivative Test for Relative Extrema

1. Find all critical numbers of a continuous function f.
   ($f'(c) = 0$ or $f'(c)$ does not exist)
2. Classify each critical point as follows:
   - $(c, f(c))$ is a **relative maximum** if
     $f'(x) > 0$ for all x in an open interval (a, c) to the left of c,
     and
     $f'(x) < 0$ for all x in an open interval (c, b) to the right of c
   - $(c, f(c))$ is a **relative minimum** if
     $f'(x) < 0$ for all x in a open interval (a, c) to the left of c,
     and
     $f'(x) > 0$ for all x in an open interval (c, b) to the right of c
   - $(c, f(c))$ is **not an extremum** if the derivative $f'(x)$ has the same sign for all x in the open intervals (a, c) and (c, b) on each side of c.

- If the graph of a function f lies above all of its tangents on an interval I, then it is said to be **concave up** on I.
- If the graph of a function f lies below all of its tangents on an interval I, then it is said to be **concave down** on I.

- The graph of a function f is **concave up** on any open interval I where $f''(x) > 0$, and **concave down** where $f''(x) < 0$.
- A point $P(c, f(c))$ on a curve is called an **inflection point** if the graph is concave up on one side of P and concave down on the other side.

2nd Derivative Test for Relative Extrema

Let f be a function such that $f'(c) = 0$ and the second derivative exists on an open interval containing c.

- If $f''(c) > 0$, there is a **relative minimum** at $x = c$.
  (f is concave up on an interval around c)
- If $f''(c) < 0$, there is a **relative maximum** at $x = c$.
  (f is concave down on an interval around c)
- If $f''(c) = 0$, the 2nd derivative test **fails**. (The point could be either a max, a min, or neither.)

Limits to Infinity

$$\lim_{x \to +\infty} f(x) = L$$

means that $f(x)$ approaches the number L as x increases without bound.

$$\lim_{x \to -\infty} f(x) = M$$

means that $f(x)$ approaches the number M as x decreases without bound.

Formal definition of limits to infinity

The limit statement $\lim_{x\to+\infty} f(x) = L$ means that for any number $\epsilon > 0$, there exists a number N_1 such that

$$|f(x) - L| < \epsilon \quad \text{whenever} \quad x > N_1$$

for x in the domain of f.

Similarly, $\lim_{x\to-\infty} f(x) = M$ means that for any number $\epsilon > 0$, there exists a number N_2 such that

$$|f(x) - M| < \epsilon \quad \text{whenever} \quad x < N_2$$

for x in the domain of f.

Limit Rules

If $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} g(x)$ exist, then for constants a and b:

Power rule:

$$\lim_{x\to+\infty}[f(x)]^n = [\lim_{x\to+\infty} f(x)]^n$$

Linearity rule:

$$\lim_{x\to+\infty}[af(x) + bg(x)] = a\lim_{x\to+\infty} f(x) + b\lim_{x\to+\infty} g(x)$$

Product rule:

$$\lim_{x\to+\infty}[f(x)g(x)] = [\lim_{x\to+\infty} f(x)][\lim_{x\to+\infty} g(x)]$$

Quotient rule:

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to+\infty} f(x)}{\lim_{x\to+\infty} g(x)} \quad \text{if} \quad \lim_{x\to+\infty} g(x) \neq 0$$

Similarly for $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to-\infty} g(x)$ if they exist.

Special limits

If A is a real number and r is a positive rational number, then

$$\lim_{x\to+\infty} \frac{A}{x^r} = 0$$

If r is such that x^r is defined for $x < 0$, then

$$\lim_{x\to-\infty} \frac{A}{x^r} = 0$$

Infinite Limits

$$\lim_{x\to c} f(x) = +\infty$$

means that f increases without bound as x approaches c (from either side).

$$\lim_{x\to c} g(x) = -\infty$$

means that g decreases without bound as x approaches c (from either side).

Formal definition of infinite limits

$$\lim_{x \to c} f(x) = +\infty$$

if for any number $N > 0$ (no matter how large), it is possible to find a number $\delta > 0$ such that $f(x) > N$ whenever $0 < |x - c| < \delta$.

Similarly,

$$\lim_{x \to c} g(x) = -\infty$$

if for any number $N > 0$, it is possible to find a number $\delta > 0$ such that $f(x) < -N$ whenever $0 < |x - c| < \delta$.

Note that ∞ is not a number. A limit that goes to infinity does not exist (in the sense that it does not approach a *number*). But this specifies that way in which the limit does not exist.

Asymptotes

The line $x = c$ is a **vertical asymptote** of the graph of f if either of the one-sided limits

$$\lim_{x \to c^-} f(x) \quad \text{or} \quad \lim_{x \to c^+} f(x)$$

is infinite ($+\infty$ or $-\infty$).

The line $y = L$ is a **horizontal asymptote** of the graph of f if

$$\lim_{x \to +\infty} f(x) = L \quad \text{or} \quad \lim_{x \to -\infty} f(x) = L$$

Vertical tangents and cusps

Suppose the function f is continuous at the point $P(c, f(c))$. Then the graph of f has

- a **vertical tangent** at P if $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^+} f'(x)$ are either both $+\infty$ or both $-\infty$.
- a **cusp** at P if $\lim_{x\to c^-} f'(x)$ and $\lim_{x\to c^+} f'(x)$ are both infinite with opposite signs (one $+\infty$ and the other $-\infty$).

Summary of Graphing Strategy

- Simplify the equation as much as possible
- Find 1st derivatives and critical numbers
- Determine intervals of increase and decrease
- Apply 2nd derivative test
- Determine concavity and points of inflection
- Apply 1st derivative test
- Find asymptotes, vertical tangents, cusps
- Plot points
- Sketch the curve

(See detailed summary on last page of section 4.3 of Strauss. Page 226 of 5th edition.)