Announcements

- Homework 8 due this Friday 10/26/2007

Approximations with Differentials

Example: approximate $\frac{1}{3.98}$ using differentials.

Let $f(x) = \frac{1}{x}$, so $f'(x) = -x^{-2}$.

Recall

$$f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)dx$$

Let $x_0 = 4$ and $\Delta x = dx = -0.02$

$$\begin{aligned} f(3.98) &\approx f(4) + f'(4)(-0.02) \\ &= \frac{1}{4} + \frac{-1}{16}(-0.02) \\ &= 0.25 + 0.00125 \\ &= 0.25125 \end{aligned}$$

Compare to calculator value:

$$f(3.98) \approx 0.2512562814$$

Error Propagation

Example: you measure the side of a cube as $10cm$ long; therefore conclude that the volume of the cube is $1000cm^3$.

If the original measurement is accurate to within 2%, approximately how accurate is the calculation of the volume?

$$\Delta V = V(10 + \Delta x) - V(10) \approx V'(10)(\pm 0.2)$$

If x is the measured value and $x + \Delta x$ represents the exact value, then Δx is the **error in measurement**. The difference between $f(x + \Delta x)$ and $f(x)$ is called the **propagated error** at x:

$$\Delta f = f(x + \Delta x) - f(x)$$

The **relative error** is $\frac{\Delta f}{f} \approx \frac{df}{f}$.

The **percentage error** is $100(\frac{\Delta f}{f})\%$.

Newton-Raphson for Root Approximation

The Newton-Raphson method uses tangent lines to estimate roots (zeros) of equations.

Theorem: Newton-Raphson Method: To approximate $f(x) = 0$, start with a preliminary estimate (or guess) x_0, and generate a sequence of estimates $x_1, x_2, x_3, \ldots$ using the formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0$$

Either this sequence of approximations will approach a limit that is a root of the equation, or the sequence does not have a limit.

Newton-Raphson procedure

To find a solution of $f(x) = 0$ using N-R:

1. Choose a number $\epsilon > 0$ for the allowable tolerance for the estimated solution.
2. Compute $f'(x)$ and choose and initial estimate x_0 (with $f'(x_0) \neq 0$) "close" to a solution.
3. Compute a new approximation using
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0$$
4. Repeat step 3 until $|x_{n+1} - x_n| < \epsilon$.

Extreme Values

Let $f(x)$ be a function on an interval I containing number c.

- $f(c)$ is an **absolute maximum** of f on I if $f(c) \geq f(x)$ for all x in I.
- $f(c)$ is an **absolute minimum** of f on I if $f(c) \leq f(x)$ for all x in I.
- The absolute maximum and minimum are called **extreme values** or **absolute extrema** of f on I.
- *Note that a function does not necessarily have a maximum or a minimum on a given interval.*

Extreme Value Theorem: A function f has both an absolute maximum and an absolute minimum on any closed, bounded interval $[a, b]$ where it is continuous.

Note: if f is discontinuous or the interval is not both closed and bounded, we cannot conclude that f has an absolute max and min.

Relative Max and Min

- A function f has a **relative maximum** at a point c if $f(c) \geq f(x)$ for all x in an open interval containing c.
- A function f has a **relative minimum** at a point d if $f(d) \leq f(x)$ for all x in an open interval containing d.
- The relative maxima and minima are called **relative extrema** of f on I.

- Suppose f is defined at c and either $f'(c) = 0$ or $f'(c)$ does not exist. Then c is called a **critical number** of f, and the point $P(c, f(c))$ on the graph is called a **critical point**.
- **Critical number theorem:** If a continuous function f has a relative extremum at c, then c must be a critical number of f. (I.e., either the derivative is 0 or it does not exist at c.)

Procedure for finding absolute extrema

To find the absolute extrema of a continuous function f on $[a, b]$:

1. Compute $f'(x)$ and find all critical numbers of f on $[a, b]$.
2. Evaluate f at the endpoints a and b and at each critical number c.
3. Compare the values from (2).
   Largest is the absolute maximum of f on $[a, b]$.
   Smallest is the absolute minimum of f on $[a, b]$.