Announcements

- Homework 6 due next Monday, 10/8/2007

Rates of Change

$$\text{Average rate of change } = \frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\text{Instantaneous rate of change } = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Suppose $f(x)$ is differentiable at $x = x_0$. Then the **instantaneous rate of change** of $y = f(x)$ with respect to x at x_0 is

$$f(x_0) = \left.\frac{dy}{dx}\right|_{x=x_0}$$

The **relative rate of change** of $y = f(x)$ at $x = x_0$ is given by the ratio

$$\frac{\text{Instantaneous rate of change}}{\text{size of the quantity}} = \frac{f'(x_0)}{f(x_0)}$$

Mathematical modeling

Developing a mathematical framework based on certain assumptions in order to study a real world problem (usually one too complex to be expressed in a single simple formula).

- Real world problem
- Use abstraction to develop a mathematical model
- Use model to derive results
- Make predictions
- Compare with real world and interpret results
- Improve model
- Repeat

Example: identifying models

x	**y**
-4	2.00
-3	1.28
-2	0.72
-1	0.32
0	0.08
1	0.00
2	0.08
3	0.32
4	0.72

x	**y**
0.0	0.0
1.0	1.7
1.5	2.0
2.5	1.2
3.5	-0.7
4.5	-1.9
5.5	-1.4
6.0	-0.6
6.5	0.4

Example: rectilinear motion (modeling in physics)

An object that moves along a straight line with **position** $s(t)$ has **velocity** $v(t) = \frac{ds}{dt}$ and **acceleration** $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ when these derivatives exist.

The **speed** of the object is $|v(t)|$.

Note that speed indicates how fast an object is moving. Velocity indicates speed and direction (relative to a coordinate system).

Examples:

- Position, velocity, and acceleration of a moving object whose position is given by:

$$s(t) = 3t^3 - 40.5t^2 162t$$

- Falling body:

$$h(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$$

The Chain Rule

Example: level of pollution L is a function of the population P, which in turn is a function of time t.

What is the rate of change of pollution L with respect to time t?

$$\begin{bmatrix}\text{Rate of change of L}\\ \text{with respect to t}\end{bmatrix} = \begin{bmatrix}\text{Rate of change of L}\\ \text{with respect to P}\end{bmatrix}\begin{bmatrix}\text{Rate of change of P}\\ \text{with respect to t}\end{bmatrix}$$

$$\frac{dL}{dt} = \frac{dL}{dP}\frac{dP}{dt}$$

Theorem: Chain Rule

If $y = f(x)$ is a differentiable function of u, and u is a differentiable function of x, then $y = f(u(x))$ is a differentiable function of x and its derivative is given by

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Theorem: Alternative form of Chain Rule

If u is differentiable at x and f is differentiable at $u(x)$, then the composite function $f \circ u$ is differentiable at x and

$$\frac{d}{dx} f[u(x)] = \frac{d}{du} f(u) \frac{du}{dx}$$

Extended derivative formulas

If u is a differentiable function of x, then

Extended Power Rule

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

Extended Trig Rules

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx} \qquad \frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}\cot u = -\csc^2 \frac{du}{dx}$$

$$\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx} \qquad \frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}$$

Extended Exponential and Logarithmic Rules

$$\frac{d}{dx}e^u = e^u\frac{du}{dx} \qquad \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$$