Announcements

- **Homework 2 due this Friday in class.**
  - Please write your Section number on your homework.
  - Staple pages together.
- Two volunteer note takers needed.

Recall:

- A **function** f is a rule that assigns to each element x of a set X a unique element y of a set Y.

$$f(x) = y$$

- The element y is called the **image** of x under f and is denoted $f(x)$. The set X is called the **domain** of f, and the set of all images of elements of X is called the **range** of the function.
- If the range of f consists of all of Y, then f is said to map X **onto** Y. If each element in the range is the image of one and only one element in the domain, then f is said to be a **one-to-one** function.
- A function f is **bounded** on $[a, b]$ if there exists a number B such that $|f(x)| \leq B$ for all x in $[a, b]$.
- y is called the **dependent variable** and x is called the **independent variable**.

Domain Convention

- Domain of f is assumed to be the set of real numbers for which the function is defined (unless otherwise specified).
- Examples: find the domain:
  $f(x) = 2x - 1$
  $g(x) = 2x - 1, \quad x \neq -3$
  $h(x) = \frac{(2x-1)(x+3)}{x+3}$
  $F(x) = \sqrt{x+2}$
  $G(x) = \frac{4}{5-\cos x}$

Function equality

- Two functions f and g are **equal** if and only if
  1. f and g have the same domain
  2. $f(x) = g(x)$ for all x in the domain
- Example:
  $f(x) = 2x - 1$
  $g(x) = 2x - 1, \quad x \neq -3$
  $h(x) = \frac{(2x-1)(x+3)}{x+3}$

Function composition

The **composite function** $f \circ g$ is defined by

$$(f \circ g)(x) = f[g(x)]$$

for each x in the domain of g for which $g(x)$ is in the domain of f.

- Example: $f(x) = 3x + 5$, $g(x) = \sqrt{x}$

  $(f \circ g)(x) = f[g(x)] = f(\sqrt{x}) = 3\sqrt{x} + 5$

  $(g \circ f)(x) = g[f(x)] = g(3x + 5) = \sqrt{3x + 5}$

We can also express a given function as a composition of two functions.

- Example:
  $f(x) = (x^2 + 5x + 1)^5$
  $u(x) = x^2 + 5x + 1, \qquad g(u) = u^5$

Example application

Suppose a study showed that when the population is p hundred thousand people, the average daily level of carbon monoxide in the air is given by

$$L(p) = 0.70\sqrt{p^2 + 3}$$

parts per million (ppm).

A second study showed that t years from now, the population will be

$$p(t) = 1 + 0.02t^3$$

hundred thousand people.

What will the level of air pollution be in t years?

Solution:

$$\begin{aligned} (L \circ p)(t) &= L[p(t)] \\ &= L(1 + 0.02t^3) \\ &= 0.70\sqrt{(1 + 0.02t^3)^2 + 3} \end{aligned}$$

- The **graph** of a function consists of points whose coordinates (x, y) satisfy $y = f(x)$, for all x in the domain of f.
- Vertical line test for function
- **Intercepts** of a graph: **y-intercept** is the point $(0, b)$ where $b = f(0)$. An **x-intercept** is a point $(a, 0)$ where $f(a) = 0$.
- Note that there can be at most one y-intercept. There may be more than one x-intercept.
- To find x-intercepts, solve $f(x) = 0$ for x.
- To find y-intercept, calculate $f(0)$ (if 0 is in the domain).

Symmetry

- **symmetric with respect to the y-axis:** $f(-x) = f(x)$ (**even function**)
  Examples: $f(x) = \cos x$, $f(x) = x^2$
- **symmetric with respect to the origin:** $f(-x) = -f(x)$ (**odd function**)
  Examples: $f(x) = \sin x$, $f(x) = x^3$
- Some functions are even, some are odd, some are neither.
- See sample graphs in Table 1.3.

Classification of Functions

Algebraic functions:

- **Polynomial:**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

- a_n is the **leading coefficient**
- a_0 is the **constant term**
- If $a_n \neq 0$, n is the **degree** of the polynomial.
  E.g., a constant function is zero degree,
  a linear function is first degree,
  a quadratic function is second degree,
  etc.

- A **rational function** is the quotient of two polynomial functions, $p(x)$ and $q(x)$,

$$f(x) = \frac{p(x)}{q(x)}, \qquad q(x) \neq 0$$

- **Power** functions: $f(x) = x^r$ (r nonzero, real)
  - integer powers: $f(x) = x^n = n \cdot n \cdot \cdots \cdot n$, (pos. integer n)
  - reciprocal powers: $f(x) = x^{-n} = \frac{1}{x^n}$
  - roots: $f(x) = x^{\frac{m}{n}} = \sqrt[n]{x^m}$

Transcendental functions:

- **Trigonometic** functions: $f(x) = \cos x$
- **Exponential** functions: $f(x) = b^x$
- **Logarithmic** functions: $f(x) = \log_b x$

Section 1.4: inverse functions

- $f(x_0) = y_0$
- If f has an **inverse** f^{-1}, it is the function that reverses the effect of f:

$$f^{-1}(y_0) = x_0$$

- If the inverse of a function is itself a function, we have the following definition:
  Let f be a function with domain D and range R. Then the function f^{-1} with domain R and range D is the **inverse** of f if

$$f^{-1}[f(x)] = x \text{ for all } x \text{ in } D$$

and

$$f[f^{-1}(y)] = y \text{ for all } y \text{ in } R$$

- If f has an inverse, the inverse is unique.

- Not every function has an inverse. (E.g., $f(x) = x^2$.)
- A function f will have an inverse f^{-1} on the interval I when there is exactly one number in the domain associated with each number in the range.
- I.e., a function has an inverse only if it is **one-to-one**.
- "Horizontal line test"

- A function is **strictly increasing** on an interval I if its graph is always rising on I
  $x_1 > x_2$ implies $f(x_1) > f(x_2)$
- A function is **strictly decreasing** on an interval I if its graph always falls on I
  $x_1 > x_2$ implies $f(x_1) < f(x_2)$
- A function is **strictly monotonic** on an interval I if it is either strictly increasing or strictly decreasing throughout that interval.

Theorem: A strictly monotonic function has an inverse.

Lef f be a a function that is strictly monotonic on I. Then f^{-1} exists and is strictly monotonic on I (strictly increasing if f is strictly increasing and strictly decreasing if f is strictly decreasing).

Graphing inverse functions

- If (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}.
- You can graph f^{-1} by reflecting the graph of f about the line $x = y$

Inverse Trig Functions

- The trigonometric functions are not one-to-one, so they do not in general have inverses.
- But we can restrict them to intervals on which they are one-to-one and do have inverses.
- E.g., $\sin(x)$ is strictly increasing on $[-\frac{\pi}{2}, \frac{\pi}{2}]$. If we restrict $\sin(x)$ to that interval, it does have an inverse, $\sin^{-1}$.
  $y = \sin^{-1}(x)$ if and only if $x = \sin(y)$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- Similar process for the other trigonometric functions. See Table 1.4 in Strauss.

Inversion Formulas

- $\sin(\sin^{-1} x) = x$ for $-1 \leq x \leq 1$
- $\sin^{-1}(\sin y) = y$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
- $\tan(\tan^{-1} x) = x$ for all x
- $\tan^{-1}(\tan y) = y$ for $-\frac{\pi}{2} < y < \frac{\pi}{2}$