# QUANTUM MECHANICS AND NON-ABELIAN THETA FUNCTIONS FOR THE GAUGE GROUP $SU(2)$

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ABSTRACT. This paper outlines an approach to the non-abelian theta functions of the $SU(2)$-Chern-Simons theory with the methods used by A. Weil for studying classical theta functions. First we translate in knot theoretic language classical theta functions, the action of the finite Heisenberg group, and the discrete Fourier transform. Then we explain how the non-abelian counterparts of these arise in the framework of the quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface, in the guise of the non-abelian theta functions, the action of a skein algebra, and the Reshetikhin-Turaev representation of the mapping class group. We prove a Stone-von Neumann theorem on the moduli space of flat $SU(2)$-connections on the torus, and using it we deduce the existence and the formula for the Reshetikhin-Turaev representation on the torus from quantum mechanical considerations. We show how one can derive in a quantum mechanical setting the skein that allows handle slides, which is the main ingredient in the construction of quantum 3-manifold invariants.

CONTENTS



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## 1. Introduction

This paper outlines a study of the non-abelian theta functions that arise in Chern-Simons theory by adapting the method used by André Weil for studying classical theta functions [45]. We discuss the case of the gauge group $SU(2)$, which is important because it corresponds to the Witten-Reshetikhin-Turaev topological quantum field theory, and is therefore related to the Jones polynomial of knots [47], [35]. The methods can be applied to more general gauge groups, which will be done in subsequent work.

In Weil's approach, classical theta functions come with an action of the finite Heisenberg group and a projective representation of the mapping class group of the torus. By analogy, our point of view is that the theory of non-abelian theta functions consists of:

- the Hilbert space of non-abelian theta functions, namely the holomorphic sections of the Chern-Simons line bundle;
- an irreducible representation on the space of theta functions of the algebra generated by quantized Wilson lines (i.e. of the quantizations of traces of holonomies of simple closed curves);
- a projective representation of the mapping class group of the surface on the space of non-abelian theta functions.

The representation of the mapping class group intertwines the quantized Wilson lines; in this sense the two representations satisfy the exact Egorov identity.

Non-abelian theta functions, quantized Wilson lines, and the projective representations of the mapping class groups of surfaces from topological quantum field theories have each been studied in their own right; we suggest that they should be studied together. One should recall one instance when these were considered together, namely in the proof of the asympotic faithfulness of the Reshetikhin-Turaev representation of the mapping class group [11], [2].

Our prototype is the quantization of a one-dimensional particle. The paradigm is that *the quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface and the Reshetikhin-Turaev representation of the mapping class group are the analogues of the Schrödinger representation of the Heisenberg group and of the metaplectic representation.* The Schrödinger representation arises from the quantization of the position and the momentum of a one-dimensional free particle, and is a consequence of a fundamental postulate in quantum mechanics. It is a unitary irreducible representation of the Heisenberg group, and the Stone-von Neumann theorem shows that it is unique. This uniqueness implies that linear changes of coordinates (which act as outer automorphisms of the Heisenberg group) are also quantizable, and their quantization yields an infinite dimensional representation of the metaplectic group.

André Weil [45] observed that when looking at a particle with periodic position and momentum, that is when working on the torus, one obtains the classical theta functions with characteristic, the Schrödinger representation of the finite Heisenberg group, and the well known Hermite-Jacobi action of the modular group $SL(2,\mathbb{Z})$. Then it was observed that classical theta functions are holomorphic sections of a line bundle over the moduli space of flat $U(1)$-connections on the torus, and by analogy, the holomorphic sections of the similar line bundle over the moduli space of flat $G$-connections over a surface (where $G$ is a compact simple Lie group) were called non-abelian theta functions. Edward Witten [47] placed non-abelian theta functions in the context of Chern-Simons theory, related them to the Jones polynomial [18] and conformal field theory, and gave new methods for studying them. This had a tremendous impact in the guise of the Verlinde formula which computes the dimension of the space of non-abelian theta functions. We will show below how within Witten's theory one can find the non-abelian analogues of Weil's constructs. We will do this by analyzing the quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface.

Because this work joins concepts and results from the theory of theta functions, quantum mechanics, representation theory, and low dimensional topology, we made it as self-contained as possible. At the heart of the paper lies a comparision between classical theta functions and the non-abelian theta functions of the gauge group $SU(2)$.

We start by reviewing the prototype: the Schrödinger and metaplectic representations. Then, we proceed with a detailed discussion of classical theta functions with characteristic. We obtain theta functions and the action of the finite Heisenberg group from Weyl quantization. We point out two important properties of the finite Heisenberg group: the Stone-von Neumann theorem, and the fact that its group algebra is symmetric with respect to the action of the mapping class group of the torus. Each of them is responsible for the existence of the Hermite-Jacobi action by discrete Fourier transforms on the space of theta functions. These properties have non-abelian counterparts, which we reveal later in the paper.

We express the Schrödinger representation of the group algebra of the finite Heisenberg group on the space of theta functions as the left action of this algebra on a quotient of itself. As a new approach to the theory of theta functions, we rephrase theta functions, the Schrödinger representation, and the discrete Fourier transform in a topological language. We do this using skein modules - algebraic structures defined by knots and links and combinatorial relations among them, which were first introduced by Turaev [41] for cylinders over surfaces, and by Przytycki [32] for general 3-dimensional manifolds. The skein modules used here are related to the linking number.

The fact that skein modules can be used to describe classical theta functions is a corollary of Witten's Feynman integral approach to the Chern-Simons theory with gauge group $U(1)$. In fact, Witten has explained in [47] that the topological quantum field theory of any Lie group gives rise to skein relations, hence it should have a skein theoretic version. For example, the skein theoretic version for the gauge group $SU(2)$ was constructed in [6]. But we do not rely on quantum field theory to introduce our skein modules, instead we show that they arise naturally in the quantum mechanical framework. Moreover, our analysis of the discrete Fourier transform shows the existence of a skein that allows handle slides and points towards the existence of a topological quantum field theory associated to the linking number, to be constructed in [15].

We then recall the construction of the space of non-abelian theta functions as the Hilbert space of the quantization of the moduli space of flat connections on a surface, and present in detail the case of the torus, in which case the moduli space is the pillow case. The paper continues with a description of the quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface, which is rephrased into the language of the skein modules. We deduce that the quantum group quantization is defined by the left action of a skein algebra on a quotient of itself. Because we are interested in the multiplicative structure, we use the skein relations of the Reshetikhin-Turaev invariant instead of those of the Kauffman bracket, since the latter introduce sign discrepancies. We recall our previous result [16] that on the pillow case, the quantum group and Weyl quantizations coincide. Then we prove a Stone-von Neumann theorem on the pillow case.

An application of our method is to deduce the existence of the Reshetikhin-Turaev representation on the torus as a consequence of this Stone-von Neumann theorem. We show how the explicit formula for this representation can be computed from quantum mechanical considerations. As such we arrive at the element $\Omega$ by which one colors knots in order to allow Kirby moves along them. This element is the fundamental building block in the construction of quantum 3-manifold invariants (see [35], [6], [26]), and we give a natural way to derive its existence.

We conclude the paper by explaining how the Reshetikhin-Turaev representation of the mapping class group of a surface is a nonabelian analogue of the Hermite-Jacobi action given by discrete Fourier transforms. For that we point out that exactly how the mapping class group of the torus acts by automorphisms on the algebra of linear operators on classical theta functions, giving rise to the Hermite-Jacobi action, the mapping class group of a surface acts by automorphisms on the algebra of linear operators on non-abelian theta functions, and hence gives rise to the Reshetikhin-Turaev representation.

## 2. The prototype

In this section we review briefly the Schrödinger and the metaplectic representations. For a detailed discussion we suggest [29] and [31].

In the canonical formalism, a classical mechanical system is described by a symplectic manifold $(M^{2n}, \omega)$, which is the phase space of the system. The classical observables are $C^\infty$ functions on $M$. To each $f \in C^\infty(M^{2n}, \mathbb{R})$ one associates a Hamiltonian vector field $X_f$ on $M^{2n}$ by $df(\cdot) = \omega(X_f, \cdot)$. This vector field defines a Hamiltonian flow on the manifold which preserves the form $\omega$.

The symplectic form $\omega$ defines a Poisson bracket by $\{f, g\} = \omega(X_f, X_g)$. There is a special observable $H$, called the Hamiltonian (total energy) of the system. The time evolution of an observable is described by the equation

$$\frac{df}{dt} = \{f, H\}.$$

Quantization replaces the symplectic manifold by a Hilbert space, real-valued observables $f$ by self-adjoint operators $\mathrm{Op}(f)$ called quantum observables, Hamiltonian flows by one-parameter groups of unitary operators, and the Poisson bracket of $\{f, g\}$ by $\frac{\hbar}{i}[\mathrm{Op}(f), \mathrm{Op}(g)]$, where $\hbar$ is Planck's constant and $[\cdot, \cdot]$ is the commutator of operators. This is done so that Dirac's conditions hold: $\mathrm{Op}(1) = Id$ and

$$\mathrm{Op}(\{f, g\}) = \frac{i}{\hbar}[\mathrm{Op}(f), \mathrm{Op}(g)] + O(\hbar).$$

The second condition is phrased by that the quantization is performed in the direction of the Poisson bracket. The time evolution of a quantum observable is described by Schrödinger's equation

$$i\hbar \frac{d\mathrm{Op}(f)}{dt} = [\mathrm{Op}(f), \mathrm{Op}(H)].$$

2.1. **The Schrödinger representation.** A fundamental example is that of a particle in a 1-dimensional space, which we will discuss in the case where Planck's constant is set equal to 1. The phase space is $\mathbb{R}^2$, with coordinates the position $x$ and the momentum $y$, and symplectic form $\omega = dx \wedge dy$. The associated Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

The symplectic form $\omega$ induces a nondegenerate antisymmetric bilinear form on $\mathbb{R}^2$, also denoted by $\omega$, which is explicitly given by

$$\omega((x,y),(x',y')) = \begin{vmatrix} x & y \\ x' & y' \end{vmatrix} = xy' - x'y.$$

The Lie algebra of observables has a subalgebra generated by $Q(x,y) = x$, $P(x,y) = y$, and $E(x,y) = 1$, called the Heisenberg Lie algebra. Abstractly, this algebra is defined by $[Q,P] = E, [P,E] = [Q,E] = 0$.

It is one of the postulates of quantum mechanics that the quantization of the position, the momentum, and the constant functions is the representation of the Heisenberg Lie algebra on $L^2(\mathbb{R}, dx)$ defined by

$$Q \to M_x, \quad P \to \frac{1}{i}\frac{d}{dx}, \quad E \to iId.$$

Here $M_x$ denotes the operator of multiplication by the variable: $\phi(x) \to x\phi(x)$. This is the Schrödinger representation of the Heisenberg Lie algebra.

The Lie group of the Heisenberg Lie algebra is the Heisenberg group. It is defined as $\mathbb{R}^3$ with the multiplication rule

$$(x,y,t)(x',y',t') = \left(x + x', y + y', t + t' + \frac{1}{2}\omega((x,y),(x',y'))\right).$$

It is standard to denote $\exp(xQ+yP+tE) = (x,y,t)$. By exponentiating the Schrödinger representation of the Lie algebra one obtains the Schrödinger representation of the Heisenberg group:

$$\exp(Q) \to e^{2\pi i M_x}, \quad \exp(P) \to e^{-\frac{d}{dx}}, \quad \exp(E) \to e^{2\pi i Id}.$$

Specifically, for $\phi \in L^2(\mathbb{R}, dx)$,

$$\begin{aligned} \exp(y_0 P)\phi(x) &= \phi(x - y_0), \\ \exp(x_0 Q)\phi(x) &= e^{2\pi i x x_0}\phi(x), \\ \exp(tE)\phi(x) &= e^{2\pi i t}\phi(x), \end{aligned}$$

meaning that $\exp(y_0 P)$ acts as a translation, $\exp(x_0 Q)$ acts as the multiplication by a character, and $\exp(tE)$ acts as the multiplication by a scalar. On the one hand this gives the quantization rules for the evolution operators of the position and the momentum, on the other hand this is a rule for quantizing the exponential functions. Specifically, $\exp(x_0 Q + y_0 P + tE)$ is the quantization of the function $f(x,y) = \exp(2\pi i(x_0 x + y_0 y + t))$.

Extending by linearity one obtains the quantization of the group ring of the Heisenberg group. This was further generalized by Hermann Weyl, who gave a method for quantizing all functions $f \in C^\infty(\mathbb{R}^2)$ by using the Fourier transform

$$\hat{f}(\xi, \eta) = \iint f(x, y) \exp(-ix\xi - iy\eta) dx dy$$

and then defining

$$\mathrm{Op}(f) = \iint \hat{f}(\xi, \eta) \exp(\xi Q + \eta P) d\xi d\eta,$$

where for $\exp(\xi Q + \eta P)$ he used the Schrödinger representation.

**Theorem** (Stone-von Neumann) The Schrödinger representation of the Heisenberg group is the unique irreducible unitary representation of this group such that $\exp(tE)$ acts as $e^{2\pi i t} Id$ for all $t \in \mathbb{R}$.

There are two other important realizations of the irreducible representation that this theorem characterizes. One comes from the quantization of the plane in a holomorphic polarization. The Hilbert space is the Bargmann space,

$$\mathrm{Bargmann}(\mathbb{C}) = \left\{ f : \mathbb{C} \to \mathbb{C} \text{ holomorphic }, \int_{\mathbb{C}} |f(z)|^2 e^{-2\pi |\mathrm{Im}\ z|^2} dz \wedge d\bar{z} < \infty \right\},$$

where the Heisenberg group acts by

$$\begin{aligned} \exp(x_0 P) f(z) &= f(z - x_0) \\ \exp(y_0 Q) f(z) &= e^{\pi(y_0^2 - 2iy_0 z)} f(z + iy_0) \\ \exp(tE) f(z) &= e^{2\pi i t} f(z). \end{aligned}$$

For the other one has to choose a Lagrangian subspace $\mathbf{L}$ of $\mathbb{R}P + \mathbb{R}Q$ (which in this case is just a one-dimensional subspace). Then $\exp(\mathbf{L} + \mathbb{R}E)$ is a maximal abelian subgroup of the Heisenberg group. Consider the character of this subgroup defined by $\chi_{\mathbf{L}}(\exp(l + tE)) = e^{2\pi i t}, l \in \mathbf{L}$. The Hilbert space of the quantization, $\mathcal{H}(\mathbf{L})$, is defined as the space of functions $\phi(u)$ on $\mathbf{H}(\mathbb{R})$ satisfying

$$\phi(uu') = \chi_L(u')^{-1} \phi(u) \text{ for all } u' \in \exp(\mathbf{L} + \mathbb{R}E)$$

and such that $u \to |\phi(u)|$ is a square integrable function on the left equivalence classes modulo $\exp(\mathbf{L} + \mathbb{R}E)$. The representation of the Heisenberg group is given by

$$u_0 \phi(u) = \phi(u_0^{-1} u).$$

If we choose an algebraic complement $\mathbf{L}'$ of $\mathbf{L}$, meaning that we write $\mathbb{R}P + \mathbb{R}Q = \mathbf{L} + \mathbf{L}' = \mathbb{R} + \mathbb{R}$, then $\mathcal{H}(\mathbf{L})$ can be realized as $L^2(\mathbf{L}') \cong L^2(\mathbb{R})$.

Under a natural isomorphism,

$$\begin{aligned} \exp(x_0)\phi(x) &= \phi(x - x_0), \quad x, x_0 \in \mathbf{L}' \\ \exp(y_0)\phi(x) &= e^{2\pi i\omega(x,y_0)}\phi(x), \quad x \in \mathbf{L}', y_0 \in \mathbf{L} \\ \exp(tE)\phi(x) &= e^{2\pi it}\phi(x) \end{aligned}$$

where $\omega$ is the standard symplectic form on $\mathbb{R}P+\mathbb{R}Q$. For $\mathbf{L} = \mathbb{R}P$ and $\mathbf{L}' = \mathbb{R}Q$, one obtains the standard Schrödinger representation in the position representation. For $L = \mathbb{R}Q$ and $L' = \mathbb{R}P$, one obtains the Schrödinger representation in the momentum representation: $\exp(y_0P)\phi(x) = e^{-2\pi ixy_0}\phi(x)$, $\exp(x_0Q)\phi(x) = \phi(x - x_0)$, $\exp(tE)\phi(x) = e^{2\pi it}\phi(x)$.

2.2. **The metaplectic representation.** A corollary of the Stone-von Neumann theorem is that, if we compose the Schrödinger representation with an automorphism of the Heisenberg group and then quantize, we obtain a unitarily-equivalent representation. A linear symplectic transformation of $\mathbb{R}^2$ induces an outer automorphism of the Heisenberg group. Therefore, the Stone-von Neumann theorem implies that linear symplectomorphisms can be quantized, giving rise to unitary operators, despite the fact that they do not arise from hamiltonian flows. Irreducibility implies, by Schur's lemma, that these operators are unique up to a multiplication by a constant. Hence we have a projective representation $\rho$ of the linear symplectic group, which in this case is $SL(2,\mathbb{R})$, on $L^2(\mathbb{R})$. This can be made into a true representation by passing to the double cover of $SL(2,\mathbb{R})$, namely to the metaplectic group $Mp(2,\mathbb{R})$. The representation of the metaplectic group is known as the *metaplectic representation* or the Segal-Shale-Weil representation.

The fundamental symmetry that Weyl quantization has is that, if $h \in Mp(2,\mathbb{R})$, then

$$\mathrm{Op}(f \circ h^{-1}) = \rho(h)\mathrm{Op}(f)\rho(h)^{-1},$$

for every observable $f \in C^\infty(\mathbb{R}^2)$, where $\mathrm{Op}(f)$ is the operator associated to $f$ through Weyl quantization. For other quantization models this relation holds only mod $O(\hbar)$, (*Egorov's theorem*). When it is satisfied with equality, as it is in our case, it is called the *exact* Egorov identity.

An elegant way to define the metaplectic representation is to use the third version of the Schrödinger representation discussed in the previous section, which identifies the metaplectic representation as a Fourier transform (see [29]). Let $h$ be a linear symplectomorphism of the plane, then let $\mathbf{L}_1$ be a Lagrangian subspace of $\mathbb{R}P + \mathbb{R}Q$ and $\mathbf{L}_2 = h(L_1)$. Define the quantization of $h$ as $\rho(h) : \mathcal{H}(\mathbf{L}_1) \to \mathcal{H}(\mathbf{L}_2)$,

$$(\rho(h)\phi)(u) = \int_{\exp \mathbf{L}_2/\exp(\mathbf{L}_1\cap\mathbf{L}_2)} \phi(uu_2)\chi_{\mathbf{L}_2}(u_2)d\mu(u_2),$$

where $d\mu$ is the measure induced on the space of equivalence classes by the Haar measure on $\mathbf{H}(\mathbb{R})$.

To write explicit formulas for $\rho(h)$ one needs to choose the algebraic complements $\mathbf{L}_1'$ and $\mathbf{L}_2'$ of $\mathbf{L}_1$ and $\mathbf{L}_2$ and unfold the isomorphism $L^2(\mathbf{L}') \cong L^2(\mathbb{R})$. For example, for

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

if we set $\mathbf{L}_1 = \mathbb{R}P$ with variable $y$ and $L_2 = S(\mathbf{L}_1) = \mathbb{R}Q$ with variable $x$ and $\mathbf{L}_1' = \mathbf{L}_2$ and $\mathbf{L}_2' = S(\mathbf{L}_1') = \mathbf{L}_1$, then

$$\rho(S)f(x) = \int_{\mathbb{R}} f(y)e^{-2\pi ixy}dy,$$

the usual Fourier transform, which establishes the unitary equivalence between the position and the momentum representations. Similarly, for

$$T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

if we set $\mathbf{L}_1 = \mathbf{L}_2 = \mathbb{R}P =$, $\mathbf{L}_1' = \mathbb{R}Q$, and $\mathbf{L}_2' = \mathbb{R}(P+Q)$, then

$$\rho(T_a)f(x) = e^{2\pi ix^2a}f(x)$$

which is the multiplication by the exponential of a quadratic function. These are the well known formulas that define the action of the metaplectic group on $L^2(\mathbb{R}, dx)$.

For $S$, $\mathbf{L}_1 \cap \mathbf{L}_2 = \{0\}$, so the integration takes place over the entire $\exp \mathbf{L}_2$, while for $T$, $\mathbf{L}_1 \cap \mathbf{L}_2 \neq \{0\}$, so the integration takes place over a quotient of $\exp \mathbf{L}_2$; in this sense it is a partial Fourier transform.

The cocycle of the projective representation of the symplectic group is

$$c_L(h', h) = e^{-\frac{i\pi}{4}\boldsymbol{\tau}(\mathbf{L}, h(\mathbf{L}), h' \circ h(\mathbf{L}))}$$

where $\boldsymbol{\tau}$ is the Maslov index. This means that

$$\rho(h'h) = c_{\mathbf{L}}(h', h)\rho(h')\rho(h)$$

for $h, h' \in SL(2, \mathbb{R})$.

## 3. Classical theta functions

**3.1. Classical theta functions from the quantization of the torus.** For an extensive treatement of theta functions the reader can consult [30], [29], [31]. From the point of view of this paper, we are quantizing a phase space which is the 2-dimensional torus $\mathbb{T}^2$ in the complex polarization of an arbitrary complex structure. The moduli space of complex structures on the torus is the upper half-plane factored by the group $PSL(2, \mathbb{Z})$. Two complex numbers in the upper half-plane yield the same complex structure if they correspond through a Möbius transformation. To a complex number $\tau = a + bi$ with positive imaginary part one associates the complex torus obtained as the quotient of $\mathbb{C}$ by the lattice $\mathbb{Z} + \mathbb{Z}\tau$. The complex variable is $z = x + \tau y$, where $(x, y)$ are the coordinates of the plane in the basis $\{1, \tau\}$.

The torus is endowed with the symplectic form $\omega = \frac{\pi i}{b} dz \wedge d\bar{z}$, which is $2\pi$ times the generator of $H^2(\mathbb{T}^2, \mathbb{Z})$.

To obtain the Hilbert space of the quantization we apply the procedure of geometric quantization. We start by setting $\hbar = \frac{1}{N}$, with $N$ a positive integer. Here, in order to have the equality $Sp_\theta(2,\mathbb{Z}) = SL(2,\mathbb{Z})$ and to establish a relationship with Chern-Simons theory, we restrict $N$ to be an *even* integer, say $N = 2r$.

The Hilbert space consists of the holomorphic sections of a line bundle which is the tensor product of a line bundle of curvature $N\omega$ and a half-density (the need of the half-density in a general situation is explained in [39]). For the half-density we choose the trivial line bundle defined by the 1-form $dx$ in the cotangent bundle, which can be thought of as $\sqrt{dx \wedge dy}$, and ignore it.

A holomorphic line bundle with curvature $N\omega$ can be obtained by factoring the trivial line bundle on $\mathbb{R}^2$ by the cocycle $\Lambda : \mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*$,

$$\Lambda((x,y),(m,n)) = \alpha(m,n) e^{-2\pi i N(\frac{\tau}{2}n^2 + n(x+\tau y))} = \alpha(m,n) e^{-2\pi i N(\frac{\tau}{2}n^2+nz)}.$$

where $\alpha$ is a character of $\mathbb{Z}^2$ which parametrizes a flat line bundle on the torus, i.e. $\alpha(m,n) = e^{2\pi i(m\mu + n\nu)}$ for some real numbers $\mu$ and $\nu$. The holomorphic sections of this line bundle are the *classical theta functions* of characteristic $N$. They can be identified with holomorphic functions on the plane satisfying the periodicity condition

$$f(z + m + n\tau) = \alpha(m,n) e^{-2\pi i N(\tau n^2 + 2nz)} f(z).$$

To simplify the discussion, we restrict ourselves to the case $\mu = \nu = 0$, when $\alpha \equiv 1$. An orthonormal basis of the space of classical theta functions is given by the *theta series*:

$$\begin{aligned} \theta_j^\tau(z) &= \sum_{n\in\mathbb{Z}} e^{2\pi i N\left[\frac{1}{2}\left(\frac{j}{N}+n\right)^2(a+bi) + (x+ay+biy)\left(\frac{j}{N}+n\right)\right]} \\ &= \sum_{n\in\mathbb{Z}} e^{2\pi i N\left[\frac{\tau}{2}\left(\frac{j}{N}+n\right)^2 + z\left(\frac{j}{N}+n\right)\right]}, \quad j = 0,1,2,\ldots,N-1. \end{aligned} \tag{3.1}$$

It will be convenient to extend this definition to all indices $j$ by the periodicity condition $\theta_{j+N}^\tau(z) = \theta_j^\tau(z)$, namely to take indices modulo $N$.

Given a torus with a Kähler structure one doesn't get automatically the theta series. In addition one needs a pair of generators of the fundamental group (see Figure 1). The torus endowed with these two generators gives rise to an element in the Teichmüller space of the torus, that is, to a complex number $\tau$ in the upper half-plane, and then the formulae above apply. The two generators of the fundamental group of the torus can be represented by two simple closed curves intersecting at one point. These curves define what we will call a *oriented rigid structure*. This consists of two curves, the first of which is oriented and cuts the torus into an annulus, while the second keeps track of twistings.

Let us now turn to the operators. The only exponentials on the plane that are double periodic, and therefore give rise to functions on the torus,

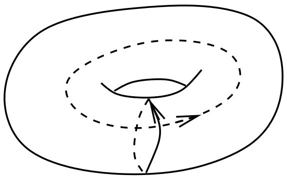

Figure 1

are of the form

$$f(x,y) = \exp 2\pi i(mx+ny), \quad m,n \in \mathbb{Z}.$$

We quantize them using Weyl quantization. In the complex polarization Weyl quantization is defined as follows (see [10]): A fundamental domain of the torus is the unit square $[0,1]\times[0,1]$ (this is done in the $(x,y)$ coordinates, in the complex plane it is actually a parallelogram). The value of a theta function is completely determined by its values on this unit square. The Hilbert space of classical theta functions can be isometrically embedded into $L^2([0,1]\times[0,1])$ with the inner product

$$\langle f,g\rangle = \left(\frac{2i}{\pi N(\tau-\bar\tau)}\right)^{1/2} \int_0^1\int_0^1 f(x,y)\overline{g(x,y)}e^{iN(\tau-\bar\tau)\pi y^2}dxdy.$$

The theta series form an orthonormal basis.

The operator associated by Weyl quantization to a function $f$ on the torus is the Toeplitz operator with symbol $e^{-\frac{\hbar\Delta_\tau}{4}}f$, where $\Delta_\tau$ is the Laplacian, which in the $(x,y)$ coordinates is given by the formula

$$\begin{aligned} \Delta_\tau &= \tfrac{1}{2\pi b}\left[(a^2+b^2)\tfrac{\partial^2}{\partial x^2} - 2a\tfrac{\partial}{\partial x\partial y} + \tfrac{\partial^2}{\partial y^2}\right] \\ &= \tfrac{i}{\pi(\tau-\bar\tau)}\left[\tau\bar\tau\tfrac{\partial^2}{\partial x^2} - (\tau+\bar\tau)\tfrac{\partial^2}{\partial x\partial y} + \tfrac{\partial^2}{\partial y^2}\right]. \end{aligned} \tag{3.2}$$

This means that the Weyl quantization of $f$ maps a classical theta function $g$ to the orthogonal projection onto the Hilbert space of classical theta functions of $(e^{-\frac{\Delta_\tau}{4N}}f)g$.

**Proposition 3.1.** *The Weyl quantization of the exponentials is given by*

$$Op\left(e^{2\pi i(px+qy)}\right)\theta_j^\tau(z) = e^{-\frac{\pi i}{N}pq - \frac{2\pi i}{N}jq}\theta_{j+p}^\tau(z).$$

*Proof.* Let us first compute

$$\begin{aligned} &(2\pi iN(\tau-\bar\tau))^{-\frac{1}{2}}\left\langle e^{2\pi i(px+qy)}\theta_j^\tau(z), \theta_k^\tau(z)\right\rangle \\ &= \int_0^1\int_0^1 e^{2\pi i(px+qy)}\sum_n e^{2\pi iN\left[\frac{\tau}{2}\left(\frac{j}{N}+n\right)^2 + (x+\tau y)\left(\frac{j}{N}+n\right)\right]} \\ &\quad\times\sum_m e^{-2\pi iN\left[\frac{\bar\tau}{2}\left(\frac{k}{N}+m\right)^2+(x+\bar\tau y)\left(\frac{k}{N}+m\right)\right]}e^{\pi iN(\tau-\bar\tau)y^2}dxdy. \end{aligned}$$

Changing the order of summation and separating the variables we obtain

$$\sum_{m,n}\int_0^1 e^{2\pi i N\left[\frac{p}{N}+\frac{j}{N}+n-\frac{k}{N}-m\right]x}dx$$
$$\times\int_0^1 e^{2\pi i N\left[\frac{\tau}{2}\left(\frac{j}{N}+n\right)^2-\frac{\bar\tau}{2}\left(\frac{k}{N}+m\right)^2+\frac{qy}{N}+\tau\left(\frac{j}{N}+n\right)y-\bar\tau\left(\frac{k}{N}+m\right)y+\frac{\tau-\bar\tau}{2}y^2\right]}dy.$$

The first integral is nonzero if and only if

$$\frac{p}{N}+\frac{j}{N}-\frac{k}{N}+n-m=0,$$

that is if and only if $p+j-k=(m-n)N$, in which case the integral is equal to 1. In fact, because of the periodicity $\theta_j^\tau=\theta_{j+N}^\tau$ for all $j$, the indices $j$ and $k$ can be taken modulo $N$. We must have $k\equiv p+j(\mathrm{mod}\ N)$, and we may set $k=p+j$, $m=n$.

Returning to the integral and switching again the order of summation, we obtain

$$\int_0^1\sum_n e^{2\pi i N\left[\frac{\tau}{2}\left(\frac{j}{N}+n\right)^2-\frac{\bar\tau}{2}\left(\frac{j+p}{N}+n\right)^2+\frac{qy}{N}+\tau\left(\frac{j}{N}+n\right)-\bar\tau\left(\frac{j+p}{N}+n\right)+\frac{\tau-\bar\tau}{2}y^2\right]}dy.$$

By expanding the exponent and then completing the square we obtain

$$e^{\pi i\left(\bar\tau-\frac{\bar\tau^2}{\tau-\bar\tau}\right)\frac{p^2}{N}}\int_0^1 e^{2\pi i qy}\times\sum_n e^{2\pi i N\left(y+\frac{j}{N}+n-\frac{\bar\tau}{\tau-\bar\tau}\frac{p}{N}\right)^2}dy.$$

Using the Poisson summation formula ($\sum_n f(n)=\sum_n\hat f(n)$) for the function $e^{-x^2}$, we transform the above sum of exponentials into

$$(2\pi i N(\tau-\bar\tau))^{-\frac{1}{2}}\sum_n e^{-\frac{\pi i n^2}{N(\tau-\bar\tau)}}e^{2\pi i\left(y+\frac{j}{N}-\frac{\bar\tau}{\tau-\bar\tau}\frac{p}{N}\right)n}. \tag{3.3}$$

The integral

$$\int_0^1 e^{2\pi i(q+n)y}dy$$

is nonzero if and only if $n=-q$, in which case it is equal to 1. So when integrating, only the term with $n=-q$ survives. Hence (3.3) equals

$$(2\pi i N(\tau-\bar\tau))^{-\frac{1}{2}}e^{-\frac{\pi i q^2}{N(\tau-\bar\tau)}-2\pi i\frac{jq}{N}-\pi i\left(\bar\tau-\frac{\bar\tau^2}{\tau-\bar\tau}\right)\frac{p^2}{N}+2\pi i\frac{\bar\tau}{\tau-\bar\tau}\frac{pq}{N}}.$$

We rewrite this as

$$(2\pi i N(\tau-\bar\tau))^{-\frac{1}{2}}e^{\frac{\pi i}{N}\left[-\tau\bar\tau p^2-(\tau+\bar\tau)pq-q^2\right]}e^{-\frac{2\pi i}{N}jq-\frac{\pi i}{N}pq}.$$

Note on the other hand that

$$\Delta_\tau e^{2\pi i(px+qy)}=-\frac{4\pi i}{\tau-\bar\tau}\left[\tau\bar\tau p^2-(\tau+\bar\tau)pq+q^2\right]e^{2\pi i(px+qy)}.$$

We conclude that the Toeplitz operator with symbol $e^{-\frac{\Delta}{4N}}e^{2\pi i(px+qy)}$ maps $\theta_j^\tau$ to $e^{-\frac{2\pi i}{N}qj-\frac{\pi i}{N}pq}\theta_j^\tau$ and the proposition is proved. □

The Weyl quantization of the exponentials gives rise to the Schrödinger representation of the Heisenberg group with integer entries $\mathbf{H}(\mathbb{Z})$ onto the space of theta functions. We define this Heisenberg group as

$$\mathbf{H}(\mathbb{Z}) = \{(p,q,k),\ p,q,k \in \mathbb{Z}\}$$

with multiplication

$$(p,q,k)(p',q',k') = (p+p', q+q', k+k'+(pq'-qp')).$$

Then the previous proposition implies that

$$(p,q,k) \mapsto \text{ the Weyl quantization of } e^{\frac{\pi i}{N}k}\exp 2\pi i(px+qy)$$

is a group morphism. This is the Schrödinger representation.

The Schrödinger representation of $\mathbf{H}(\mathbb{Z})$ is far from faithful. We want to factor it out by its kernel.

**Proposition 3.2.** *The set of elements in $\mathbf{H}(\mathbb{Z})$ that act on theta functions as identity operators is the normal subgroup consisting of the $N$th powers of elements of the form $(p,q,k)$ with $k$ even. The quotient group is isomorphic to a finite Heisenberg group.*

*Proof.* If

$$(p,q,k)\theta_j^\tau = e^{-\frac{\pi i}{N}pq - \frac{2\pi i}{N}qj + \frac{\pi i}{N}k}\theta_{j-p}^\tau$$

equals $\theta_j^\tau$ for all $j$, then $p = \ell N$ for some integer $\ell$, and

$$-\ell q + \frac{2qj}{N} + \frac{k}{N} \in 2\mathbb{Z}$$

for $j = 0, \ldots, N-1$. This easily implies that both $q$ and $k$ must be multiples of $N$, say $q = mN$ and $k = nN$. We now have that $-\ell mN + 2mj + n \in 2\mathbb{Z}$. Setting $j = \frac{N}{2}$, we conclude that $n$ is even (recall that $N$ is even), so $k$ is a multiple of $2N$. Since $(a,b,c)^N = (Na, Nb, Nc)$, any such element $(p,q,k) \in \mathbf{H}(\mathbb{Z})$ is the $N$th power of an element with the third entry even. Conversely, the computation

$$\begin{aligned}(p,q,k)^N\theta_j^\tau &= e^{2\pi ijq - 2\pi i\frac{N(N-1)}{2N}pq - N\frac{\pi ipq}{N} + N\frac{\pi ik}{N}}\theta_{j-N}^\tau \\ &= e^{2\pi ijq - \pi iNpq + \pi ik}\theta_j^\tau = e^{\pi ik}\theta_j^\tau\end{aligned}$$

shows that the $N$th powers of elements with third entry even act as the identity operator. This proves the first part of the statement.

Next, define the finite ("polarized") Heisenberg group

$$\mathbf{H}(\mathbb{Z}_N) = \{(p,q,k),\ p,q \in \mathbb{Z}_N, k \in \mathbb{Z}_{2N}\}$$

by the multiplication rule

$$(p,q,k)(p',q',k') = (p+p', q+q', k+k'+2pq').$$

It is straightforward to check that the map $F : \mathbf{H}(\mathbb{Z}) \to \mathbf{H}(\mathbb{Z}_N)$ given by

$$F((p,q,k)) = (p,q,k+pq) \tag{3.4}$$

is a group homomorphism, and because $N$ is even, its kernel is the set of elements of the form $(p, q, k)$ with $p$ and $q$ divisible by $N$ and $k$ divisible by $2N$. Hence the desired quotient group is isomorphic to a finite Heisenberg group. □

**Notation:** From now on we identify $\mathbf{H}(\mathbb{Z}_N)$ with the image of $\mathbf{H}(\mathbb{Z})$ under the Schrödinger representation (isomorphic to the quotient of $\mathbf{H}(\mathbb{Z})$ by the kernel of the Schrödinger representation, under the isomorphism induced by (3.4)), and for $p,\, q,\, k \in \mathbb{Z}$ w denote $\exp(pP + qQ + kE) = F((p, q, k))$. Note that

$$\exp(pP)\theta_j^\tau = \theta_{j+p}^\tau, \quad \exp(qQ)\theta_j^\tau = e^{-\frac{2\pi i}{N}qj}\theta_j^\tau, \quad \exp(kE)\theta_j^\tau = e^{\frac{\pi i}{N}k}\theta_j^\tau$$

for all $p, q, k, j \in \mathbb{Z}$.

**Theorem 3.3.** *(Stone-von Neumann) The Schrödinger representation of* $\mathbf{H}(\mathbb{Z}_N)$ *is the* unique *irreducible unitary representation of this group with the property that* $\exp(kE)$ *acts as* $e^{\frac{\pi i}{N}k} Id$ *for all* $k \in \mathbb{Z}$.

*Proof.* There is a standard proof based on characters, which classifies all irreducible representations of $\mathbf{H}(\mathbb{Z}_N)$, but there is a well known elementary proof which goes as follows. Let $X = \exp(P)$, $Y = \exp(Q)$, $Z = \exp(E)$. Because, $Y^N = 1$, all eigenvalues of $Y$ are $N$th roots of unity. Let $v$ be an eigenvector of $Y$ with eigenvalue $\lambda$. Then $YXv = Z^{-2}XYv = e^{-\frac{2\pi i}{N}}\lambda Xv$. So $Xv$, which is nonzero because $X$ is unitary, is also an eigenvector of $Y$ with eigenvalue $e^{-\frac{2\pi i}{N}}\lambda$. Applying $X$ repeatedly we obtain an eigenvector $v_0$ of $Y$ with eigenvalue 1. Then $v_0, Xv_0, \ldots, X^{N-1}v_0$ is a basis, and $X, Y$, and $Z$ act on it as in the Schrödinger representation. □

We must mention another important representation of the finite Heisenberg group, which by the Stone-von Neumann theorem is unitary equivalent to the one above. It arises from the quantization of the torus in a real polarization. An orthonormal basis of the Hilbert space is given by Bohr-Sommerfeld leaves of the polarization, and so the Hilbert space can be identified with $L^2(\mathbb{Z}_N)$. If the polarization is given by "$Q$ curves", the finite Heisenberg group acts by

$$\exp(pP)f(j) = f(j-p), \exp(qQ)f(j) = e^{-\frac{2\pi i}{N}qj}f(j), \exp(kE)f(j) = e^{\frac{\pi i}{N}k}f(j).$$

Thus $\exp(pP)$ acts as translation and $\exp(qQ)$ acts as a multiplication by a character of $\mathbb{Z}_N$. The characteristic functions of the singletons: $\delta_i(j) = \delta_{ij}$, $i = 0, 1, 2, \ldots, N-1$ correspond to the theta series $\theta_0^\tau, \theta_1^\tau, \ldots, \theta_{N-1}^\tau$ through the unitary isomorphisms that identifies the two representations. Note that a left shift in the index corresponds to a right shift in the variable.

*Remark* 3.4. There is a sign discrepancy between these formulas and the ones from §2.1, which shows up in the exponent of the formula for $\exp(qQ)$. This is due to an unfortunate disagreement between the standard notations in quantum mechanics and topology, in which the roles of $p$ and $q$ are exchanged. Because Chern-Simons theory has been studied extensively by topologists, we use the convention for $p$ and $q$ from topology. We point out that the above formulas describe the action of the Heisenberg group in the the momentum representation.

The Schrödinger representation of the finite Heisenberg group can be extended by linearity to a representation of the group algebra with coefficients in $\mathbb{C}$ of the finite Heisenberg group, $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$. Since the elements of $\exp(\mathbb{Z}E)$ act as multiplications by constants, this is in fact a representation of the algebra $\mathcal{A}_N$ obtained by factoring $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$ by the relations $\exp(kE) - e^{\frac{\pi i}{N}k}$ for all $k \in \mathbb{Z}$. By abuse of language, we will call this representation the Schrödinger representation as well. The Schrödinger representation of $\mathcal{A}_N$ defines the quantizations of trigonometric polynomials on the torus.

**Proposition 3.5.** *a) The algebra of Weyl quantizations of trigonometric polynomials is the algebra of all linear operators on the space of theta functions.*
*b) The Schrödinger representation of the algebra $\mathcal{A}_N$ on theta functions is faithful.*

*Proof.* We consider the operators $\mathrm{Op}\left(e^{\frac{\pi i}{N}pq}e^{2\pi i(px+qy)}\right)$, $p, q = 0, 1, \ldots, N-1$, and denote their matrices in the basis $(\theta_j^\tau)_j$ by $M_{p,q}$ respectively. The $j, k$ entry of $M_{p,q}$ is nonzero if and only if $k \equiv j+p(\mathrm{mod}\ N)$, and in that case it is equal to $e^{-\frac{2\pi i}{N}jq}$. If we fix $p$ and write the nonzero entries of $M_{p,q}$ in a vector, then the matrix of the vectors $M_{p,q}$, $q = 0, 1, \ldots, N-1$, is Vandermonde, hence $M_{p,q}$, $q = 0, 1, \ldots, N-1$, are linearly independent. Consequently the matrices $M_{p,q}$, $p, q = 0, 1, \ldots, N-1$ are linearly independent, so they form a basis of the space of $N \times N$ matrices.

On the other hand, $\exp(pP + qQ)$, $p, q = 0, 1, \ldots, N-1$, form a basis of $\mathcal{A}_N$ as a vector space. The conclusion follows. □

Because of this result, we may identify $\mathcal{A}_N$ with the algebra of Weyl quantizations of trigonometric polynomials. $\mathcal{A}_N$ can also be described in terms of the noncommutative torus. (The relevance of the noncommutative torus for Chern-Simons theory was first revealed in [12] for the gauge group $SU(2)$.) The ring of trigonometric polynomials in the noncommutative torus is $\mathbb{C}_t[U^{\pm 1}, V^{\pm 1}]$, the ring of Laurent polynomials in the variables $U$ and $V$ subject to the noncommutation relation $UV = t^2VU$. The noncommutative torus itself is a C*-algebra in which $\mathbb{C}_t[U^{\pm 1}, V^{\pm 1}]$ is dense, viewed as a deformation quantization of the algebra of smooth functions on the torus [36], [9]. The group algebra with coefficients in $\mathbb{C}$ of the Heisenberg group $\mathbf{H}(\mathbb{Z})$ is isomorphic to the ring of trigonometric polynomials in the noncommutative torus, with the isomorphism given by

$$(p, q, k) \to t^{k-pq}\, U^pV^q, \text{ for } p, q, k \in \mathbb{Z}.$$

If we set $U^N = V^N = 1$ and $t = e^{\frac{\pi i}{N}}$, we obtain the noncommutative torus at a root of unity $\widetilde{\mathbb{C}_t}[U^{\pm 1}, V^{\pm 1}]$. To summarize:

**Proposition 3.6.** *The algebra, $\mathcal{A}_N$, of the Weyl quantizations of trigonometric polynomials on the torus is isomorphic to $\widetilde{\mathbb{C}_t}[U^{\pm 1}, V^{\pm 1}]$.*

3.2. **The Schrödinger representation in an abstract setting.** We will formulate the Schrödinger representation of quantized exponentials on the space of theta functions in an abstract setting by rephrasing the standard construction based on a lattice and a Lagrangian subspace of the plane (see [31]), which is similar to the third construction in §2.1.

Given a two-dimensional torus $\mathbb{T}^2$ with a symplectic form, the analogue of a Lagrangian subspace of the plane is a simple closed curve. Such a curve generates a one-dimensional free $\mathbb{Z}$-submodule, $\mathbf{L}$, of $H_1(\mathbb{T}^2, \mathbb{Z})$. Realizing the torus as the quotient of the plane by the lattice $\mathbb{Z} + \mathbb{Z}\tau$, the curve that generates $\mathbf{L}$ can be taken to be a curve of slope $p/q$, for some coprime integers $p$ and $q$. We denote such a curve by $(p, q)$. To $\mathbf{L}$ we associate the maximal abelian subgroup of the finite Heisenberg group which is the image of the abelian subgroup of $\mathbf{H}(\mathbb{Z})$,

$$\{\, (np,\, nq,\, k) \in \mathbf{H}(\mathbb{Z}) \mid n,\ k \in \mathbb{Z}\,\}$$

under the Schrödinger representation. We denote this group by $\exp(\mathbf{L}+\mathbb{Z}E)$. Also, we define the character $\chi_{\mathbf{L}} : \exp(\mathbf{L}+\mathbb{Z}E) \to \mathbb{C}$ by $\chi_{\mathbf{L}}(\exp(npP+nqQ+kE)) = e^{\frac{\pi i}{N}k}$.

The Hilbert space of the quantization consists of the functions $\phi : \mathbf{H}(\mathbb{Z}_N) \to \mathbb{C}$ satisfying the equivariance condition

$$\phi(uu') = \chi_L(u')^{-1}\phi(u) \text{ for all } u' \in \exp(\mathbf{L} + \mathbb{Z}E).$$

The action of the finite Heisenberg group on this space is by the left action

$$u_0\phi(u) = \phi(u_0^{-1}u).$$

In order to verify that this gives rise to the same Schrödinger representation, we need to explicate an algebraic complement $\mathbf{L}'$ of $\mathbf{L}$ in $H_1(\mathbb{T}^2, \mathbb{Z})$. Then $\mathbf{L}'$ is also spanned by a simple closed curve on the torus, which intersects the generating curve of $L$ at one point.

To make specific computations, let $\mathbf{L} = \mathbb{Z}(p, q)$ and $\mathbf{L}' = \mathbb{Z}(p', q')$, with $p'q - pq' = 1$. The finite Heisenberg group is generated by

$$\Lambda := \exp(\mathbb{Z}(p, q, 0)) \quad \text{and} \quad \Lambda' := \exp(\mathbb{Z}(p', q', 0)).$$

The Hilbert space can be identified with the functions on $\Lambda'$, and the action of the Heisenberg group is given by

$$\begin{aligned}
&\exp(x_0p'P + x_0q'Q) \cdot \phi\left(\exp(xp'P + xq'Q)\right)\\
&\quad = \phi(\exp(x_0p'P + x_0q'Q)^{-1}\exp(xp'P + xq'Q))\\
&\quad = \phi(\exp((x - x_0)p'P + (x - x_0)q'Q)),
\end{aligned}$$

$$\begin{aligned}
&\exp(x_0pP + x_0qQ\cdot)\phi\,(\exp(xp'P + xq'Q)) \\
&\quad = \phi(\exp(x_0pP + x_0qQ)^{-1}\exp(xp'P + xq'Q)) \\
&= \phi(\exp(2xx_0E)\exp(xp'P + xq'Q)\exp(x_0pP + x_0qQ)^{-1}) \\
&= \phi(\exp(xp'P + xq'Q)\exp(2xx_0E)) = e^{-\frac{2\pi i}{N}xx_0}\phi(\exp(xp'P + xq'Q)),
\end{aligned}$$

$$\exp(kE)\cdot\phi\,(\exp(xp'P + xq'Q)) = e^{\frac{\pi i}{N}k}\phi(\exp(xp'P + xq'Q))$$

for all $x, x_0 \in \mathbb{Z}$. Identifying further $\Lambda'$ with $\mathbb{Z}_N$ we obtain the action

$$\begin{aligned}
&\exp(x_0p'P + x_0q'Q)\cdot\phi(x) = \phi(x - x_0) \\
&\exp(x_0pP + x_0qQ)\cdot\phi(x) = e^{-\frac{2\pi i}{N}xx_0}\phi(x) \\
&\exp(kE)\cdot\phi(x) = e^{\frac{\pi i}{N}k}\phi(x),
\end{aligned}$$

and we recognize the Schrödinger representation in the real polarization (in the momentum representation).

This representation of the finite Heisenberg group extends by linearity to a representation of the group algebra of the finite Heisenberg group, $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$. Because of finiteness, the group algebra of the finite Heisenberg group can be identified with the space of functions on the group, with multiplication given by convolution. At the same time, the space of the Schrödinger representation (in the abstract setting) is the subspace $\mathcal{H} \subset \mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$ of the functions on the finite Heisenberg group that transform according to the character $\chi$ under the right action of $G_{\mathbf{L}} := \exp(\mathbf{L} + \mathbb{Z}E)$. As is well-known, the orthogonal projector (with respect to counting measure),

$$\Pi : \mathbb{C}[\mathbf{H}(\mathbb{Z}_N)] \to \mathcal{H},$$

is given by

$$\Pi(\phi)(g) = \frac{1}{2N^2}\sum_{k\in G_L}\chi(k)\,\phi(gk),$$

and $\mathcal{H}$ can be identified with the quotient $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]/\ker\Pi$. Thus the Schrödinger representation is the action of the group algebra of the finite Heisenberg group on a quotient of itself. More precisely, we have:

**Proposition 3.7.** *Given a submodule* $\mathbf{L}$ *of* $H_1(\mathbb{T}^2, \mathbb{Z})$ *generated by a simple closed curve, there is an isomorphism that intertwines the Schrödinger representation of the finite Heisenberg group on the space of theta functions and the representation given by the left action of the finite Heisenberg group on the quotient of its group algebra,* $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$*, by the submodule that contains all elements of the form* $\chi(h)^{-1}\psi\cdot h - \psi$ *for some* $\psi \in \mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$ *and* $h \in \exp(\mathbf{L} + \mathbb{Z}E)$*. (Note: Under the identification of the group algebra with functions,* $\psi\cdot h$ *is the function* $(\psi\cdot h)(g) = \psi(gh^{-1})$*.)*

*Proof.* By the discussion preceding the proposition all we need to show is that the kernel of $\Pi$ is spanned by elements of the form above. A calculation shows that for any $\psi$ and $h$ as above $\Pi(\chi(h)^{-1}\psi\cdot h - \psi) = 0$.

For the converse, it suffices to concentrate on one conjugacy class $\mathcal{C}$ modulo $G_{\mathbf{L}}$. We denote by $\Pi_{\mathcal{C}}$ the restriction of $\Pi$ to functions defined on the cojugacy class $\mathcal{C}$. Also, to simplify the proof we eliminate the dependency on the cocycle $\chi$ by choosing $g_0 \in \mathcal{C}$, and defining, for a function $\phi : \mathcal{C} \to \mathbb{C}$, the function $\overline{\phi}(g_0 g) = \chi(g)\phi(g_0 g)$.

As such we have

$$\Pi_{\mathcal{C}}(\overline{\phi})(g_0) = \sum_{g\in\mathcal{C}} \overline{\phi}(g_0 g),$$

and we are left to show that the vector space $V$ spanned by functions of the form $\overline{\psi} \cdot h - \overline{\psi}$ coincides with the kernel of $\Pi_{\mathcal{C}}$.

Letting $\overline{\psi}$ to be a $\delta$-function, we see that $V$ contains all functions $\lambda_{g_1,g_2} : \mathcal{C} \to \mathbb{C}$,

$$\lambda_{g_1,g_2}(g) = \begin{cases} 1 & \text{if } g = g_1 \\ -1 & \text{if } g = g_2 \end{cases}$$

for any $g_1, g_2 \in \mathcal{C}$. And the kernel of $\Pi_{\mathcal{C}}$ is spanned by the functions $\lambda_{g_1,g_2}$, a fact which follows from the exactness of the Koszul sequence

$$\mathbb{C}^{|\mathcal{C}|} \oplus \mathbb{C}^{|\mathcal{C}|} \xrightarrow{\delta_2} \mathbb{C}^{|\mathcal{C}|} \xrightarrow{\delta_1} \mathbb{C} \to 0,$$

where $\delta_2(v, w) = v - w$ and $\delta_1(v)$ is the sum of the coordinates of $v$ in the standard basis. This concludes the proof. $\square$

3.3. **Classical theta functions from a topological perspective.** The considerations in the previous section imply that quantized exponentials can be modeled by curves on the torus. In the multiplication formula

$$\exp(pP + qQ)\exp(p'P + q'Q) = e^{\frac{\pi i}{N}\left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|}\exp((p + p')P + (q + q')Q),$$

the determinant in the exponent is the algebraic intersection number of such curves. Thus the finite Heisenberg group can be described in purely topological terms. Moreover, the factorization relation from Proposition 3.7 is of topological nature, it expresses the fact that equivalent curves on the torus become isotopic after filling the interior of the torus. So we can rephrase the theory of classical theta functions in topological language. We do this using skein modules.

Let $M$ be an orientable 3-dimensional manifold (which in our considerations will be either the cylinder over a torus or a solid torus). A link in $M$ is a smooth embedding of a disjoint union of finitely many circles, while a framed link is a smooth embedding of a disjoint union of finitely many annuli. We consider framed oriented links. By viewing each link component as an embedded $S^1 \times [0, 1]$, an orientation is defined by a smooth field of 2-dimensional frames on the annulus with the property that the first vector of the frame at any point $(x, y)$ is tangent to $S^1 \times \{y\}$.

Consider the free $\mathbb{C}[t, t^{-1}]$-module with basis the set of isotopy classes of framed oriented links in $M$, including the empty link $\emptyset$. Factor this module by all equalities of the form depicted in Figure 2. In each of these diagrams, the two links are identical except for an embedded ball in $M$, inside of which they look as shown. This means that in each link we are allowed to smoothen a crossing provided that we add a coefficient of $t$ or $t^{-1}$, and any trivial link component can be ignored. These are called *skein relations*. The skein relations are considered for all possible embeddings of a ball. We call the result of the factorization the *linking number skein module of* $M$ and denote it by $\mathcal{L}_t(M)$. If $M$ is a 3-dimensional sphere, then each link $L$ is, as an element of $\mathcal{L}_t(S^3)$, equivalent to the empty link with the coefficient equal to $t$ raised to the sum of the linking numbers of ordered pairs of components and the writhes of the components, hence the name.

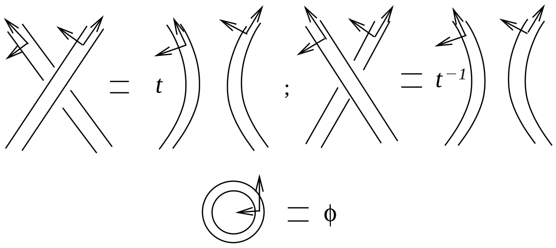


Figure 2

These skein modules were first introduced by Przytycki in [33] with slightly different, but equivalent skein relations. Przytycki pointed out that they represent one-parameter deformations of the group ring of $H_1(M,\mathbb{Z})$ and computed them for all 3-dimensional manifolds.

If $M = \mathbb{T}^2\times[0,1]$, the topological operation of gluing one cylinder on top of another induces a multiplication in $\mathcal{L}_t(\mathbb{T}^2\times[0,1])$ which turns $\mathcal{L}_t(\mathbb{T}^2\times[0,1])$ into an algebra, the *linking number skein algebra* of the cylinder over the torus. We want to explicate its structure.

For $p$ and $q$ coprime integers, orient the curve $(p,q)$ by the vector that joins the origin to the point $(p,q)$, and frame it so that the annulus is parallel to the torus. Call this the zero framing. Any framing of the curve $(p,q)$ can be represented by an integer $k$, where $|k|$ is the number of full twists that are inserted on this curve, with $k$ positive if the twists are positive, and $k$ negative if the twists are negative. Note that in $\mathcal{L}_t(\mathbb{T}^2\times[0,1])$, $(p,q)$ with framing $k$ is equivalent to $t^k(p,q)$.

If $p$ and $q$ are not coprime and $n$ is their greatest common divisor, we denote by $(p,q)$ the framed link consisting of $n$ parallel copies of $(p/n,q/n)$. It is standard to call $n$ parallel copies of a (framed) curve the *nth power of that curve*, so $(p,q)=(p/n,q/n)^n$. Finally, $\emptyset=(0,0)$ is the empty link, which is the multiplicative identity of $\mathcal{L}_t(\mathbb{T}^2\times[0,1])$.

We define the *reduced linking number skein algebra*, $\widetilde{\mathcal{L}}_t(\mathbb{T}^2\times[0,1])$, as the algebra obtained by factoring $\mathcal{L}_t(\mathbb{T}^2\times[0,1])$ by the relations $(Np,Nq)=(0,0)$ for every $p,q\in\mathbb{Z}$, and by setting $t=e^{\frac{\pi i}{N}}$. As a rule followed throughout the paper, in a skein module $t$ is a free variable while in a reduced skein module it is a root of unity.

**Theorem 3.8.** *The algebra $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ is isomorphic to the group algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$, with the isomorphism induced by*

$$t^k(p,q) \to (p,q,k).$$

*This map descends to an isomorphism between $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ and the algebra $\mathcal{A}_N$ of Weyl quantizations of trigonometric polynomials.*

*Proof.* Consider the free $\mathbb{C}[t,t^{-1}]$-module with basis the isotopy classes of links in $\mathbb{T}^2 \times [0,1]$. Any element of this module of the form $t^m L$, where $L$ is a framed oriented link, can be transformed using the skein relations into a framed oriented link whose projection onto the torus has no crossings, hence into a link of the form $t^k(p,q)$. The pair $(p,q)$ is the homology class of $L$ in $H_1(\mathbb{T}^2 \times [0,1]) = H^1(\mathbb{T}^2) = \mathbb{Z} \oplus \mathbb{Z}$ in the basis given by the curves $(1,0)$ and $(0,1)$, hence is uniquely determined by the link. In particular $(p,q)$ is the same for all elements equivalent to $L$ modulo the skein relation. It is not hard to see that $k$ equals $m$ plus the number of positive crossings minus the number of negative crossings in a projection of $L$ onto a framed link on the torus with framing parallel to the torus. This number is an isotopy invariant and also invariant under the skein relations.

By mapping $t^k(p,q)$ to $(p,q,k)$ and extending linearly we obtain an isomorphism of vector spaces between $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ and $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$. An easy induction shows that the multiplication in $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ is given by

$$(p,q)\cdot(p',q') = t^{\left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|}(p+p',q+q').$$

Therefore $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ and the group algebra of $\mathbf{H}(\mathbb{Z})$ are isomorphic. The skein algebra $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ was defined so that this isomorphism descends to an isomorphism between this algebra and $\mathcal{A}_N$. □

*Remark* 3.9. The determinant is the sum of the algebraic intersection numbers of the curves in $(p,q,k)$ with the curves in $(p',q',k')$, so the multiplication rule of the Heisenberg group is defined using the algebraic intersection number of curves on the torus.

Identifying the group algebra of the Heisenberg group with integer entries with $\mathbb{C}_t[U^{\pm 1}, V^{\pm 1}]$, we obtain the following

**Corollary 3.10.** *The linking number skein algebra of the cylinder over the torus is isomorphic to the ring of trigonometric polynomials in the noncommutative torus.*

Now let us turn to the skein module of the solid torus $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$. Let $\alpha$ be the curve that is the core of the solid torus, with a certain choice of orientation and framing. We consider the *reduced linking number skein module* obtained by setting $\alpha^{j+N} = \alpha^j$ and $t = e^{\frac{\pi i}{N}}$, and denote it by $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$.

Let $h_0$ be a homeomorphism of the torus to the boundary of the solid torus that maps $(1,0)$ to a framed curve isotopic to $\alpha$ and $(0,1)$ to the curve on the boundary of the solid torus that has blackboard framing and bounds a disk in the solid torus. The curves $(1,0)$ and $(0,1)$ on the boundary of the solid torus are called the *longitude* respectively the *meridian*. The operation of gluing $\mathbb{T}^2 \times [0,1]$ to the boundary of $S^1 \times \mathbb{D}^2$ via $h_0$ induces a left action of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ onto $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$. This descends to a left action of $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ onto $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$.

It is important to observe that $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$ and $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ are quotients of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ respectively $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$, with two framed curves equivalent on the torus if they are isotopic in the solid torus.

**Theorem 3.11.** *There is an isomorphism that intertwines the action of the algebra of Weyl quantizations of trigonometric polynomials on the space of theta functions and the representation of $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ onto $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$, and which maps the theta series $\theta_j^\tau(z)$ to $\alpha^j$ for all $j = 0, 1, \ldots, N-1$.*

*Proof.* The argument of the previous theorem can be repeated mutatis mutandis to show that module $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$ is free with basis $\alpha^j$, $j \in \mathbb{Z}$, where $\alpha^j$ stands for $|j|$ parallel copies of $\alpha$, with the same orientation as $\alpha$ if $j > 0$ and with opposite orientation if $j < 0$, and $\alpha^0 = \emptyset$. Consequently, $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is an $N$-dimensional vector space with basis $\alpha^j$, $j = 0, 1, \ldots, N-1$. Then, one can observe either that the equivalence relation defined in Proposition 3.7 is exactly the condition that the framed links are isotopic inside the solid torus, or that the left action of $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ onto $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is given by

$$(p,q) \cdot \alpha^j = t^{-qj} \alpha^{j+p}.$$

with the convention $\alpha^{j+N} = \alpha^j$ for all $j$. □

*Remark* 3.12. The number

$$-qj = \begin{vmatrix} p & q \\ j & 0 \end{vmatrix}$$

is the sum of the linking numbers of the curves in the system $(p,q)$ and those in the system $\alpha^j$. So the Schrödinger representation is defined in terms of the linking number of curves.

*Remark* 3.13. The oriented rigid structure completely determines the homeomorphism $h_0$, allowing us to identify the Hilbert space of the quantization with the vector space with basis $\alpha^0 = \emptyset, \alpha, \ldots, \alpha^{N-1}$. As we have seen above, these basis elements are the theta series. We see again how the rigid structure determines the theta series.

To obtain the more general Schrödinger representation defined by a simple closed curve $(p,q)$ on the torus as in §3.2, one should replace $h_0$ by a homeomorphism that maps $(p,q)$ to the meridian of the torus.

3.4. **The discrete Fourier transform for classical theta functions from a topological perspective.** The symmetries of classical theta functions are another instance of the Fourier transform, namely the discrete Fourier transform. We will give them a topological perspective.

An element

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \tag{3.5}$$

of the mapping class group of the torus induces the biholomorpic mapping between the complex torus with rigid structure specified by $\tau$ and the complex torus with rigid structure specified by $\tau' = \frac{a\tau+b}{c\tau+d}$. The mapping is $z' = \frac{z}{c\tau+d}$. Identifying the two tori via this mapping, we deduce that $h$ is a symplectomorphism of a complex torus that preserves the complex structure.

The map $h$ induces an action of the mapping class group on the Weyl quantizations of exponentials given by

$$h \cdot \exp(pP + qQ + kE) = \exp[(ap + bq)P + (cp + dq)Q + kE].$$

This action is easy to describe in the skein theoretical setting, it just maps every framed link $\gamma$ on the torus to $h(\gamma)$.

**Theorem 3.14.** *There is a projective representation $\rho$ of the mapping class group of the torus on the space of theta functions that satisfies the exact Egorov identity*

$$h \cdot \exp(pP + qQ + kE) = \rho(h) \exp(pP + qQ + kE)\rho(h)^{-1}$$

*Moreover, for every $h$, $\rho(h)$ is unique up to multiplication by a constant.*

*Proof.* We will give two proofs of this result, to which we will refer when discussing non-abelian Chern-Simons theory.
*Proof 1:* The map that associates to $\exp(pP + qQ + kE)$ the operator that acts on theta functions as

$$\theta_j^\tau \to \exp[(ap + bq)P + (cp + dq)Q + kE]\theta_j^\tau$$

is also a unitary irreducible representation of the finite Heisenberg group which maps $\exp(kE)$ to multiplication by $e^{\frac{i\pi}{N}}$. By the Stone-von Neumann theorem, this representation is unitary equivalent to the Schrödinger representation. This proves the existence of $\rho(h)$ satisfying the exact Egorov identity. By Schur's lemma, the map $\rho(h)$ is unique up to multiplication by a constant. Hence, if $h$ and $h'$ are two elements of the mapping class group, then $\rho(h' \circ h)$ is a constant multiple of $\rho(h')\rho(h)$. It follows that $\rho$ defines a projective representation.
*Proof 2:* The map $\exp(pQ + qQ + kE) \to h \cdot \exp(pP + qQ + kE)$ extends to an automorphism of the algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$. Because the ideal by which we factor to obtain $\mathcal{A}_N$ is invariant under the action of the mapping class group, this automorphism induces an automorphism

$$\Phi : \mathcal{A}_N \to \mathcal{A}_N,$$

which maps each scalar multiple of the identity to itself. Since, by Proposition 3.5, $\mathcal{A}_N$ is the algebra of all linear operators on the $N$-dimensional space of theta functions, $\Phi$ is inner [42], meaning that there is $\rho(h) : \mathcal{A}_N \to \mathcal{A}_N$ such that $\Phi(x) = \rho(h)x\rho(h)^{-1}$. In particular

$$h \cdot \exp(pP + qQ + kE) = \rho(h)\exp(pP + qQ + kE)\rho(h)^{-1}.$$

The Schrödinger representation of $\mathcal{A}_N$ is obviously irreducible, so again we apply Schur's lemma and conclude that $\rho(h)$ is unique up to multiplication by a constant and $h \to \rho(h)$ is a projective representation. $\square$

The representation $\rho$ is the well-known Hermite-Jacobi action given by discrete Fourier transforms.

As a consequence of Proposition 3.5, for any element $h$ of the mapping class group, the linear map $\rho(h)$ is in $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$, hence it can be represented by a skein $\mathcal{F}(h)$. This skein satisfies

$$h(\sigma)\mathcal{F}(h) = \mathcal{F}(h)\sigma$$

for all $\sigma \in \widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$. Moreover $\mathcal{F}(h)$ is unique up to multiplication by a constant. We want to find a formula for $\mathcal{F}(h)$.

We consider first the case where

$$h = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

the positive twist. Then $h((0,j)) = (0,j)$ for all $j$, and $h((1,0)) = (1,1)$. The equality $\mathcal{F}(T)(0,j) = (0,j)\mathcal{F}(T)$ for all $j$ implies that we can write $\mathcal{F}(T) = \sum_{j=0}^{N-1} c_j(0,j)$ for some coefficients $c_j$. The equality

$$(1,1)\sum_{j=0}^{N-1} c_j(0,j) = \sum_{j=0}^{N-1} c_j(0,j)(1,0)$$

yields

$$\sum_{j=0}^{N-1} t^j c_j(1,j+1) = \sum_{j=0}^{N-1} t^{-j} c_j(1,j).$$

It follows that $t^j c_j = t^{-j-1}c_{j+1}$ for all $j = 0, 1, \ldots, N-2$. Normalizing so that $c_0 > 0$ and $\mathcal{F}(T)$ defines a unitary map, we obtain $c_j = N^{-1/2}t^{1+3+\cdots+(2j-1)} = N^{-1/2}t^{j^2}$. We conclude that

$$\mathcal{F}(T) = N^{-1/2}\sum_{j=0}^{N-1} t^{j^2}(0,j).$$

To better understand this formula, and to deduce the formula for an arbitrary $h$, we recall a few basic facts in low dimensional topology, which can be found in more detail in [37].

Every 3-dimensional manifold is the boundary of a 4-dimensional manifold obtained by adding 2-handles $\mathbb{D}^2 \times \mathbb{D}^2$ to a 4-dimensional ball along the solid tori $\mathbb{D}^2 \times S^1$. On the boundary, $S^3$, of the ball, the operation of adding handles gives rise to surgery on a framed link. Thus any 3-dimensional manifold can be obtained as follows. Start with framed link $L \subset S^3$. Take a regular neighborhood of $L$ made out of disjoint solid tori, each with a framing curve on the boundary such that the core of the solid torus and this curve determine the framing of the corresponding link component. Remove these tori, then glue them back in so that the meridians are glued to the framing curves. The result is the desired 3-dimensional manifold.

The operation of sliding one 2-handle over another corresponds to sliding one link component along another using a Kirby band-sum move [23]. A *slide* of $K_1$ along $K$, denoted by $K_1\#K$, is obtained as by the following algorithm. Let $K_2$ be a copy of $K$ obtained by pushing $K$ in the direction of its framing. Identify $K_i$ with $S^1\times[-\epsilon,\epsilon]$ and take $D_i\in K_i$ to be rectangles corresponding to $[-\delta,\delta]$ for some small $\delta$, $i=1,2$. Embed a rectangle $D=[-\delta,\delta]\times[1,2]$ in $S^3\backslash(K\cup K_1\cup K_2)$ such that $[-\delta,\delta]\times\{i\}$ is identified with $[-\delta,\delta]\times\{(-1)^{i-1}\epsilon\}$ in $D_i$. Let $D'=[-\delta,\delta]\times[1-\epsilon,2+\epsilon]=D\cup D_1\cup D_2$. The slide of $K_1$ along $K$ is obtained by taking $K_1\cup K_2\cup D$ and removing $[-\delta/2,\delta/2]\times[1-\epsilon,2+\epsilon]$. The result of sliding a trefoil knot along a figure-eight knot, both with the blackboard framing, is shown in Figure 3.

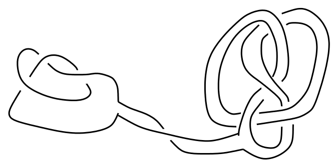

Figure 3

An element of the mapping class group of the torus can also be described by surgery. For example, the twist $T$ can be obtained by doing surgery on the curve $(0,1)$ with framing 1. Explicitly, the mapping cylinder of $T$ is obtained from $\mathbb{T}^2\times[0,1]$ by removing a solid torus that is a regular neighborhood of $(0,1)\times\{\frac{1}{2}\}$ and gluing it back such that its meridian (the homologically trivial curve on the boundary) is mapped to the curve which gives the framing. The result of the surgery is homeomorphic to $\mathbb{T}^2\times[0,1]$, such that the restriction of the homeomorphism to $\mathbb{T}^2\times\{0\}$ is the identity map and the restriction of the homeomorphism to $\mathbb{T}^2\times\{1\}$ is $T$.

We introduce the element

$$\Omega_{U(1)}=N^{-1/2}\sum_{j=0}^{N-1}\alpha^j\in\widetilde{\mathcal{L}}_t(S^1\times\mathbb{D}^2),$$

Here $U(1)$ stands for the gauge group $U(1)$ (see §4.1), which is related to classical theta functions by Chern-Simons theory. We will have an analogous element for the group $SU(2)$.

The skein $\mathcal{F}(T)$ is obtained by coloring the framed surgery curve of the twist $T$ by $\Omega_{U(1)}$. This means that we replace the surgery curve by the skein $\Omega_{U(1)}$ such that the curves $\alpha^j$ in the solid torus are parallel to the framing. In general, for a framed link $L$ we denote by $\Omega_{U(1)}(L)$ the skein obtained by replacing every link component by $\Omega_{U(1)}$ such that $\alpha$ becomes the framing.

A general twist $T'$ along an arbitrary simple closed curve on the torus is also given by surgery on a curve, and $\mathcal{F}(T')$ is obtained by coloring the surgery curve by $\Omega_{U(1)}$. The Lickorish twist theorem [26] states that any element of the mapping class group of a surface is a product of twists. Hence any element of the mapping class group can be obtained by performing surgery on a link in the cylinder over that surface. This is true in particular for the torus. Using the gluing of $\mathbb{T}^2 \times [0,1]$ by the homeomorphism $h_0$ to the boundary of the solid torus, we obtain the following skein theoretic description of the discrete Fourier transform.

**Theorem 3.15.** *Let $h$ be an element of the mapping class group of the torus obtained by performing surgery on a framed link $L_h$ in $\mathbb{T}^2 \times [0,1]$. The discrete Fourier transform $\rho(h) : \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2) \to \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is given by*

$$\rho(h)\beta = \Omega_{U(1)}(L_h)\beta.$$

*Remark* 3.16. For a framed curve $\gamma$ on the torus, $h(\gamma)$ is the slide of $\gamma$ along the components of $L_h$. The Egorov identity for $\Omega_{U(1)}(L_h)$ means in topological language that we are allowed to perform slides in the cylinder over the torus along curves colored by $\Omega_{U(1)}$. This points to a surgery formula for $U(1)$-quantum invariants of 3-manifolds. The formula will be exhibited in [15].

*Remark* 3.17. In the framework of §3.2, if $\mathbf{L} \subset H_1(\mathbb{T}^2 \times [0,1], \mathbb{R})$ is a Lagrangian space generated by the curve $(p,q)$, then $h(\mathbf{L})$ is obtained by sliding the elements of $\mathbf{L}$ over $L_h$, and $\rho(h)$ interpolates between the model of the Schrödinger representation corresponding to $\mathbf{L}$ (given by a homeomorphism that maps $(p,q)$ to the meridian of the solid torus), and the Schrödinger representation corresponding to $h(\mathbf{L})$.

Let us recall the well-known description of the Hermite-Jacobi action. For $h$ as in (3.5),

$$\rho(h)\theta_j^\tau(z) = \exp\left(-\frac{\pi i N c z^2}{c\tau + d}\right)\theta_j^{\tau'}\left(\frac{z}{c\tau+d}\right).$$

The exponential factor is introduced to enforce the periodicity conditions of theta functions for the function on the right-hand side (see Capter I §7 in [30]). For those familiar with the subject, there are no parity restrictions on $a, b, c, d$ because $N$ being even, $SL_\theta(2,\mathbb{Z}) = SL(2,\mathbb{Z})$.

In particular, for the generators

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of $SL(2,\mathbb{Z})$ one has the Jacobi identities

$$\rho(S)\theta_j^\tau(z) = \left(\frac{-i\tau}{N}\right)^{1/2} \exp\left(\frac{z^2}{2\tau}\right)\theta_j^{-1/\tau}\left(\frac{z}{\tau}\right) = \left(\frac{i\tau}{N}\right)^{1/2}\sum_{k=0}^{N-1} e^{-\frac{2\pi i}{N}kj}\theta_k^\tau(z)$$

$$\rho(T)\theta_j^\tau(z) = \theta_j^{\tau+1}(z) = e^{\frac{2\pi i}{N}j^2}\theta_j^\tau(z).$$

We normalize $\rho(S)$ to a unitary operator by dividing it by $(-i\tau)^{1/2}$. It is important to observe that

$$\Omega_{U(1)} = S\emptyset \in \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2).$$

Alternatively, in the real polarization, $S$ and $T$ act on $L^2(\mathbb{Z}_N)$ by the discrete (or finite) Fourier transform

$$(Sf)(k) = N^{-1/2} \sum_{j \in \mathbb{Z}_N} f(j) e^{-\frac{2\pi i}{N} jk}$$

and by the multiplication by the exponential of a quadratic function

$$(Tf)(j) = e^{\frac{2\pi i}{N} j^2} f(j),$$

which is interpreted as a partial Fourier transform.

The matrices $\rho(S)$ and $\rho(T)$ can be computed using skeins as shown in Figure 4. The $j, k$-entry of the matrix of $S$ is the invariant in $S^3$ of the Hopf link with one components replaced by $j$ parallel strands and the other by $k$ parallel strands, divided by $N^{1/2}$. This invariant is the power of $t$ obtained by resolving all crossings using the skein relations from Figure 2. The matrix of $T$ is diagonal, its $(j, j)$-entry is computed by introducing a full positive twist on the trivial link with $j$ components as shown in the figure, then resolving all crossings.

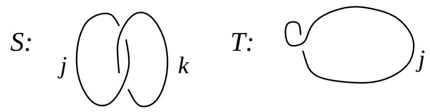


Figure 4

Like in the case of the metaplectic representation, the Hermite-Jacobi representation can be made into a true representation by passing to an extension of the mapping class group of the torus. While a $\mathbb{Z}_2$-extension would suffice, we will consider a $\mathbb{Z}$-extension instead, in order to exhibit the similarity with the non-abelian theta functions.

Fix a subspace $\mathbf{L}$ of $H_1(\mathbb{T}^2, \mathbb{R})$ spanned by a simple closed curve on the torus. Note that this subspace corresponds to a submodule of $H_1(\mathbb{T}^2, \mathbb{Z})$ spanned by the same curve. Define the $\mathbb{Z}$-extension of the mapping class group of the torus by the multiplication rule on $SL(2, \mathbb{Z}) \times \mathbb{Z}$,

$$(h', n') \circ (h, n) = (h' \circ h, n + n' + \boldsymbol{\tau}(\mathbf{L}, h(\mathbf{L}), h' \circ h(\mathbf{L})).$$

where $\boldsymbol{\tau}$ is the Maslov index. Standard results in the theory of theta functions show that the Hermite-Jacobi action lifts to a representation of this group.

We give this group itself an entirely topological description, following [43]. Let us consider two elements of the group $(h, n)$ and $(h', n')$. Let $h_1, h_2, h_3$ be the homeomorphisms of $\mathbb{T}^2$ onto the bondary of the solid torus such that $\mathbf{L} = ker(h_1 : H_1(\mathbb{T}^2) \to H_1(\mathbb{D}^2 \times S^1))$, $h(\mathbf{L}) = ker(h_2 : H_1(\mathbb{T}^2) \to$

$H_1(\mathbb{D}^2 \times S^1))$, and $h' \circ h(\mathbf{L}) = ker(h_3 : H_1(\mathbb{T}^2) \to H_1(\mathbb{D}^2 \times S^1))$. Consider the lens spaces

$$M = (\mathbb{D}^2 \times S^1) \cup_{h_1} (\mathbb{T}^2 \times [0,1]) \cup_{h_2} (\mathbb{D}^2 \times S^1)$$
$$M' = (\mathbb{D}^2 \times S^1) \cup_{h_2} (\mathbb{T}^2 \times [0,1]) \cup_{h_3} (\mathbb{D}^2 \times S^1).$$

Let also $W$ and $W'$ be 4-dimensional manifolds such that $\partial W = M$, $\text{sign}(W) = n$, $\partial W' = M'$, $\text{sign}(W') = n'$, where sign stands for the signature of the 4-dimensional manifold (i.e. the signature of the intersection form in 2-dimensional cohomology). With this construction we see that the elements of the group can be put in the form $(h, \text{sign}(W))$. We can also define

$$(h', \text{sign}(W')) \circ (h, \text{sign}(W)) = (h' \circ h, \text{sign}(W' \cup W)),$$

where $W' \cup W$ is obtained by gluing the first copy of the solid torus in $M'$ to the second copy of the solid torus in $M$ via the identity map. That this is the same multiplication as the one defined above using the Maslov index follows from Wall's non-additivity formula for the signature (see [44]):

$$\text{sign}(W' \cup W) = \text{sign}(W) + \text{sign}(W') + \boldsymbol{\tau}(L, h(L), h' \circ h(L)).$$

## 4. Non-abelian theta functions from geometric considerations

4.1. **Non-abelian theta functions from geometric quantization.** Let $G$ be a compact simple Lie group and $\mathfrak{g}$ its Lie algebra. Let also $\Sigma_g$ be a closed oriented surface of genus $g \geq 1$. The *non-abelian theta functions* are the elements of the Hilbert space of a quantization of the space of all fields on $\Sigma_g$ with symmetry group $G$ modulo gauge transformations. For $G = U(1)$ these are the same as the classical theta functions. Let us recall this theory in some detail.

Consider the moduli space of $\mathfrak{g}$-connections on $\Sigma_g$, which is the quotient of the affine space of all $\mathfrak{g}$-connections on $\Sigma_g$ (or rather on the trivial principal $G$-bundle $P$ on $\Sigma_g$) by the group $\mathcal{G}$ of gauge transformations $A \to \phi^{-1}A\phi + \phi^{-1}d\phi$, with $\phi : \Sigma_g \to G$ a smooth function. The space of all connections has a symplectic 2-form given by

$$\omega(A, B) = -\int_{\Sigma_g} \text{tr}\,(A \wedge B),$$

where $A$ and $B$ are connection forms in its tangent space. According to [4], this induces a symplectic form, denoted also by $\omega$, on the moduli space, which further defines a Poisson bracket. The group of gauge transformations acts on the space of all connections in a Hamiltonian fashion, with moment map the curvature. Thus the moduli space of *flat* $\mathfrak{g}$-connections

$$\mathcal{M}_g = \{A \,|\, A : \text{ flat } \mathfrak{g} - \text{connection}\}/\mathcal{G}.$$

arises as the symplectic reduction of the space of all connections by the group of gauge transformations. This space is the same as the moduli space of semi-stable $G$-bundles on $\Sigma_g$, and also the character variety of $G$-representations of the fundamental group of $\Sigma_g$. It is an affine algebraic set over the real numbers.

Each curve $\gamma$ on the surface and each irreducible representation $V^n$ of $SU(2)$ define a classical observable on this space

$$W_{\gamma,n} : A \to \mathrm{tr}_{V^n} \mathrm{hol}_\gamma(A),$$

called Wilson line, by taking the trace of the holonomy of the connection in the $n$-dimensional irreducible representation of $SU(2)$. Wilson lines are regular functions on the moduli space. When $n = 2$, we simply denote $W_{\gamma,2}$ by $W_\gamma$. The $W_\gamma$'s span the algebra of regular functions on $\mathcal{M}_g$.

The form $\omega$ induces a Poisson bracket, which in the case of the gauge group $SU(2)$ was found by Goldman [17] to be

$$\{W_\alpha, W_\beta\} = \frac{1}{2} \sum_{x \in \alpha \cap \beta} \mathrm{sgn}(x)(W_{\alpha\beta_x^{-1}} - W_{\alpha\beta_x})$$

where $\alpha\beta_x$ and $\alpha\beta_x^{-1}$ are computed as elements of the fundamental group with base point $x$ (see Figure 5), and $\mathrm{sgn}(x)$ is the signature of the crossing $x$; it is positive if the frame given by the tangent vectors to $\alpha$ respectively $\beta$ is positively oriented (with respect to the orientation of $\Sigma_g$), and negative otherwise.

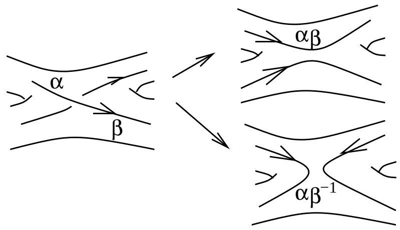


Figure 5

The moduli space $\mathcal{M}_g$, or rather the smooth part of it, can be quantized in the direction of Goldman's Poisson bracket as follows. First, set Planck's constant $\hbar = \frac{1}{N}$, where $N$ is a positive integer.

The Hilbert space can be obtained using the method of geometric quantization as the space of sections of a line bundle over $\mathcal{M}_g$. The general procedure is to obtain the line bundle as the tensor product of a line bundle $\mathcal{L}$ with curvature $N\omega$ and a half-density [39]. In our case the half-density is the square root of the canonical line bundle. Because the moduli space has a natural complex structure, it is customary to work in the complex polarization, in which case the Hilbert space consists of the holomorphic sections of the line bundle.

Let us briefly recall how each complex structure on the surface induces a complex structure on the moduli space. The tangent space to $\mathcal{M}_g$ at a nonsingular point $A$ is the first cohomology group $H^1_A(\Sigma_g, \mathrm{ad}\ P)$ of the complex of $\mathfrak{g}$-valued forms

$$\Omega^0(\Sigma_g, \text{ad } P) \stackrel{d_A}{\to} \Omega^1(\Sigma_g, \text{ad } P) \stackrel{d_A}{\to} \Omega^2(\Sigma_g, \text{ad } P).$$

Here $P$ denotes the trivial principal $G$-bundle over $\Sigma_g$. Each complex structure on $\Sigma_g$ induces a Hodge $*$-operator on the space of connections on $\Sigma_g$, hence a $*$-operator on $H^1_A(\Sigma_g, \text{ad } P)$. The complex structure on $\mathcal{M}_g$ is $I : H^1_A(\Sigma_g, \text{ad } P) \to H^1_A(\Sigma_g, \text{ad } P)$, $IB = - * B$. For more details we refer the reader to [20]. This complex structure turns the smooth part of $\mathcal{M}_g$ into a complex manifold. It is important to point out that the complex structure is compatible with the symplectic form $\omega$, in the sense that $\omega(B, IB) \geq 0$ for all $B$.

As said, the Hilbert space consists of the holomorphic sections of the line bundle of the quantization. These are the *non-abelian theta functions*.

The analogue of the group algebra of the finite Heisenberg group is the algebra of operators that are the quantizations of Wilson lines. They arise in the theory of the Jones polynomial [18] as outlined by Witten [47], being defined heuristically in the framework of quantum field theory. They are integral operators with kernel

$$< A_1|\text{Op}(W_{\gamma,n})|A_2 >= \int_{\mathcal{M}_{A_1A_2}} e^{iNL(A)} W_{\gamma,n}(A)\mathcal{D}A,$$

where $A_1, A_2$ are conjugacy classes of flat connections on $\Sigma_g$ modulo the gauge group, $A$ is a conjugacy class under the action of the gauge group on $\Sigma_g \times [0,1]$ such that $A_{\Sigma_g \times \{0\}} = A_1$ and $A_{\Sigma_g \times \{1\}} = A_2$, and

$$L(A) = \frac{1}{4\pi} \int_{\Sigma_g \times [0,1]} \text{tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

is the Chern-Simons Lagrangian. The operator quantizing a Wilson line is defined by a Feynman path integral, which does not have a rigorous mathematical formulation. It is thought as an average of the Wilson line computed over all connections that interpolate between $A_1$ and $A_2$.

The skein theoretic approach to classical theta functions outlined in §3.3 can be motivated by the Chern-Simons-Witten field theory point of view. Let us explain.

The moduli space of flat $U(1)$-connections on the torus is itself a torus, the Jacobian. The moduli space is canonically homeomorphic to the original torus in the following way. Let $p_0$ be the point on the original torus that is the image of the origin under the quotient map $\mathbb{C} \to \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$. Thinking of the Jacobian as the moduli space of holomorphic topologically trivial line bundles, the homeomorphism in question maps a point $p$ on the original torus to the line bundle $\mathcal{O}(p - p_0)$. Alternately, when viewing the Jacobian as a character variety, this homeomorphism maps the point $(x, y)$ on the original torus to the $U(1)$-representation of the fundamental group that maps $(1, 0)$ to $e^{2\pi ix}$ and $(0, 1)$ to $e^{2\pi iy}$.

The tangent space to the Jacobian torus is two dimensional; viewing it as given by cohomology classes of $u(1)$-valued 1-forms, it has a basis consisting of $dx$ and $dy$, which can be identified with the tangent vectors to the torus $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. Under this identification, a complex structure on the original torus induces exactly the same complex structure on the moduli space.

An oriented rigid structure on the original torus gives rise to a decomposition of the tangent space to the moduli space into a sum of two Lagrangian subspaces. To fit in the framework of §3.2, we can exponentiate these Lagrangian subspaces to obtain an oriented rigid structure on the moduli space, which happens to be the same as the oriented rigid structure on the original torus.

The Chern-Simons line bundle over the moduli space is the line bundle from §3.1, so in this case the theta functions are the classical theta functions. The Wilson lines can be quantized either by using one of the classical methods for quantizing the torus, or they can be quantized using the Feynman path integrals as above. The Feyman path integral approach allows localizations of computations to small balls, in which a single crossing shows up. Witten [47] has explained that in each such ball skein relations hold, in this case the skein relations from Figure 2, which compute the linking number. As such the path integral quantization gives rise to the skein theoretic model.

On the other hand, Witten's quantization is symmetric under the action of the mapping class group of the torus, a property shared by Weyl quantization. And indeed, we have seen in §3.3 that Weyl quantization and the skein theoretic quantization are the same. The relevance of Weyl quantization to Chern-Simons theory was first pointed out in [16] for the gauge group $SU(2)$. For the gauge group $U(1)$, it was noticed in [3]. The abelian Chern-Simons theory from the skein theoretic point of view will be described in detail in [15].

4.2. **The Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus.** The moduli space $\mathcal{M}_1$ of flat $SU(2)$-connections on the torus is the same as the character variety of $SU(2)$-representations of the fundamental group of the torus. It is, therefore, parametrized by the set of pairs of matrices $(A, B) \in SU(2) \times SU(2)$ satisfying $AB = BA$, modulo conjugation. Since commuting matrices can be diagonalized simultaneously, and the two diagonal entries can be permuted simultaneously, the moduli space can be identified with the quotient of the torus $\{(e^{2\pi i x}, e^{2\pi i y}),\, x, y \in \mathbb{R}\}$ by the "antipodal" map $x \to -y$, $y \to -y$. This space is called the *pillow case*.

The pillow case is the quotient of $\mathbb{R}^2$ by horizontal and vertical integer translations and by the symmetry $\sigma$ with respect to the origin. Alternatively, it is the algebraic set in $\mathbb{R}^3$ defined by the equation

$$X^2 + Y^2 + Z^2 - XYZ = 4.$$

Except for four singularities, $\mathcal{M}_1$ is a symplectic manifold, with symplectic form $\omega = 2\pi i dx \wedge dy$. The associated Poisson bracket is

$$\{f, g\} = \frac{1}{2\pi i}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$$

The Weyl quantization of $\mathcal{M}_1$ in the complex polarization has been described in [16] for one particular complex structure. We do it now in general. Again Planck's constant is the reciprocal of an even integer $\hbar = \frac{1}{N} = \frac{1}{2r}$.

The tangent space at an arbitrary point on the pillow case is spanned by the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. In the formalism of §4.1, these vectors are identified respectively with the cohomology classes of the $su(2)$-valued 1-forms

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dy.$$

It follows that a complex structure on the original torus induces exactly the same complex structure on the pillow case. So we can think of the pillow case as the quotient of the complex plane by the group generated by $\mathbb{Z} + \mathbb{Z}\tau$ and the symmetry $\sigma$ with respect to the origin, where $\tau = a + bi$ with $b > 0$. As before, we denote by $(x, y)$ the coordinates in the basis $(1, \tau)$ and by $z = x + \tau y$ the complex variable. A fundamental domain for the group action in the $(x, y)$-coordinates is $\mathcal{D} = [0, \frac{1}{2}] \times [0, 1]$.

As seen in [16], a holomorphic line bundle $\mathcal{L}_1$ with curvature $4\pi i r dx \wedge dy$ on the pillow case is defined by the cocycle $\Lambda_1 : \mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*$,

$$\begin{aligned} &\Lambda_1((x, y), (m, n)) = e^{4\pi i r(\frac{\tau}{2}n^2 - 2n(x + \tau y))} = e^{4\pi i r(\frac{\tau}{2}n^2 - 2nz)} \\ &\Lambda_1((x, y), \sigma) = 1. \end{aligned}$$

The square root of the canonical form is no longer the trivial line bundle, since we can see that the form $dx$ is not defined globally on the pillow case. The obstruction for $dx$ to be globally defined given by the map $\sigma$ can be incorporated in a line bundle $\mathcal{L}_2$ defined by the cocycle $\Lambda_2 : \mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*$,

$$\Lambda_2((x, y), (m, n)) = 1, \quad \Lambda_2((x, y), \sigma) = -1.$$

This line bundle can then be taken as the half-density.

The line bundle of the quantization is therefore $\mathcal{L}_1 \otimes \mathcal{L}_2$, defined by the cocycle $\Lambda_1\Lambda_2$. The Hilbert space $\mathcal{H}_r(\mathbb{T}^2)$ of non-abelian theta functions on the torus consists of the holomorphic sections of this line bundle. Hence the Hilbert space consists of the odd theta functions (this was discovered in [5]).

Because Weyl quantization of the pillow case is equivariant Weyl quantization of the torus, to specify a basis of $\mathcal{H}_r(\mathbb{T}^2)$ we need a pair of generators of the fundamental group of the torus, hence an oriented rigid structure on the torus (the orientation of the torus and the orientation of one of the generators of the fundamental group determine the orientation of the other generator of the fundamental group). This complex torus with oriented rigid structure determines a point in the Teichmüller space of the torus, specified by the complex number $\tau$ mentioned before. The orthonormal basis of the Hilbert space is

$$\zeta_j^\tau(z) = (\theta_j^\tau(z) - \theta_{-j}^\tau(z)), \quad j = 1, 2, \ldots, r-1,$$

where $\theta_j^\tau(z)$ are the theta series from §3.1. The definition of $\zeta_j^\tau(z)$ can be extended to all indices by the conditions $\zeta_{j+2r}^\tau(z) = \zeta_j^\tau(z)$, $\zeta_0^\tau(z) = 0$, and $\zeta_{r-j}^\tau(z) = -\zeta_{r+j}^\tau(z)$.

The space $\mathcal{H}_r(\mathbb{T}^2)$ can be embedded isometrically into $L^2(\mathcal{D})$, with the inner product

$$\langle f, g\rangle = (-i\pi r(\tau - \bar{\tau}))^{-1/2} \iint_{\mathcal{D}} f(x,y)\overline{g(x,y)} e^{-2\pi i r(\tau-\bar{\tau})y^2} dxdy$$

The Laplacian is given by the formula (3.2) (with $N = 2r$).

Let us turn our attention to the Weyl quantization of Wilson lines. If $p$ and $q$ are coprime integers, then the Wilson line of the curve $(p,q)$ of slope $p/q$ on the torus for the 2-dimensional irreducible representation is

$$W_{(p,q)}(x,y) = \frac{\sin 4\pi(px+qy)}{\sin 2\pi(px+qy)} = 2\cos 2\pi(px+qy),$$

when viewing the pillow case as a quotient of the plane. This is because the character of the 2-dimensional irreducible representation is $\sin 2x/\sin x$. In general, if $p$ and $q$ are arbitrary integers, the function

$$f(x,y) = 2\cos 2\pi(px+qy)$$

is a linear combination of Wilson lines. Indeed, if $n = \gcd(p,q)$ then

$$2\cos 2\pi(px+qy) = \frac{\sin[2\pi(n+1)(\frac{p}{n}x + \frac{q}{n}y)]}{\sin 2\pi(\frac{p}{n}x + \frac{q}{n}y)} - \frac{\sin[2\pi(n-1)(\frac{p}{n}x + \frac{q}{n}y)]}{\sin 2\pi(\frac{p}{n}x + \frac{q}{n}y)},$$

so $2\cos 2\pi(px+qy) = W_{\gamma,n+1} - W_{\gamma,n-1}$ where $\gamma$ is the curve of slope $p/q$ on the torus. This formula also shows that Wilson lines are linear combinations of cosines, so it suffices to understand the quantization of the cosines.

Because

$$2\cos 2\pi(px+qy) = e^{2\pi i(px+qy)} + e^{-2\pi i(px+qy)},$$

the Weyl quantization of cosines can be obtained by taking the Schrödinger representation of the quantum observables that are invariant under the map $\exp P \to \exp(-P)$ and $\exp Q \to \exp(-Q)$, and restrict it to odd theta functions. We obtain the formula

$$\mathrm{Op}(2\cos 2\pi(px+qy))\zeta_j^\tau(z) = e^{-\frac{\pi i}{2r}pq}\left(e^{\frac{\pi i}{r}qj}\zeta_{j-p}^\tau(z) + e^{-\frac{\pi i}{r}qj}\zeta_{j+p}^\tau(z)\right).$$

In particular the $\zeta_j^\tau$'s are the eigenvectors of $\mathrm{Op}(2\cos 2\pi y)$, corresponding to the holonomy along the curve which cuts the torus into an annulus. This shows that they are correctly identified as the analogues of the theta series.

## 5. Non-abelian theta functions from quantum groups

5.1. **A review of the quantum group $U_\hbar(sl(2,\mathbb{C}))$.** For the gauge group $SU(2)$, Reshetikhin and Turaev [35] constructed rigorously, by using quantum groups, a topological quantum field theory that fulfills Witten's programme. Within this theory, for each surface there is a vector space, an algebra of quantized Wilson lines, and a projective finite-dimensional representation of the mapping class group, namely the Reshetikhin-Turaev representation. The quantum group quantization has the advantage over geometric quantization that it does not depend on any additional structure, such as a polarization.

We restrict again Planck's constant to the reciprocal of an *even* positive integer $\hbar = \frac{1}{N} = \frac{1}{2r}$, and furthermore $r > 1$. Set $t = e^{\frac{i\pi}{2r}}$ and, for an integer $n$, let $[n] = \frac{t^{2n}-t^{-2n}}{t^2-t^{-2}} = \sin\frac{n\pi}{r}/\sin\frac{\pi}{r}$, called the quantized integer.

The quantum group associated to $SU(2)$, denoted $U_\hbar(sl(2,\mathbb{C}))$, is obtained by passing to the complexification $SL(2,\mathbb{C})$ of $SU(2)$, taking the universal enveloping algebra of its Lie algebra, then deforming this algebra with respect to $\hbar$. It is a finite dimensional Hopf algebra over $\mathbb{C}$ with generators $X, Y, K$ subject to the relations

$$KX = t^2XK, \quad KY = t^{-2}YK, \quad XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}},$$
$$X^r = Y^r = 0, \quad K^{4r} = 1.$$

As opposed to $SU(2)$, $U_\hbar(sl(2,\mathbb{C}))$ has only finitely many irreducible representations, among which we distinguish a certain family $V^1$, $V^2$, ..., $V^{r-1}$ (for more details see [35] or [24]). For each $k$, the space $V^k$ has basis $e_j$, $j = -k_0, \ldots, k_0-1, k_0$, where $k_0 = \frac{k-1}{2}$, and the quantum group acts on it by

$$Xe_j = [k_0 + j + 1]e_{j+1}, \quad Ye_j = [k_0 - j + 1]e_{j-1}, \quad Ke_j = t^{2j}e_j.$$

The heighest weight vector of this representation is $e_{k_0}$; it spans the kernel of $X$, is a cyclic vector for $Y$, and an eigenvector of $K$.

The Hopf algebra structure of $U_\hbar(sl(2,\mathbb{C}))$ makes duals and tensor products of representations be representations themselves. The quantum group acts on the dual $V^{k*}$ of $V^k$ by

$$Xe^j = -t^2[k_0 + j]e^{j-1}, \quad Ye^j = -t^{-2}[k_0 - j]e^{j+1}, \quad Ke^j = t^{-2j}e^j,$$

where $(e^j)_j$ is the basis dual to $(e_j)_j$. There is a (non-natural) isomorphism $D : V^{k*} \to V^k$,

$$D(e^j) = \frac{[k_0 - j][k_0 - j - 1]\cdots[1]}{[2k_0][2k_0 - 1]\cdots[k_0 - j + 1]}(-t^2)^j e_{-j}.$$

A Clebsch-Gordan theorem holds,

$$V^m \otimes V^n = \bigoplus_p V^p \oplus B,$$

where $p$ runs among all indices that satisfy $m+n+p$ odd and $|m-n|+1 \leq p \leq \min(m+n-1, 2r-1-m-n)$ and $B$ is a representation that is ignored because it has no effect on computations.

A corollary of the Clebsch-Gordan theorem is the following formula

$$V^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} (V^2)^{n-2j} = S_{n-1}(V^2), \quad \text{for } n = 1, 2, \cdots, r-1$$

where $S_n(x)$ is, as seen, the Chebyshev polynomial of second kind defined recursively by

$$S_{n+1}(x) = xS_n(x) - S_{n-1}(x), \quad S_0(x) = 1, S_1(x) = x.$$

The polynomial $S_n(x)$ will play an important role in our discussion.

We define the representation ring $R(U_\hbar(sl(2,\mathbb{C})))$ as the ring generated by $V^j$, $j = 1, 2, \ldots, r-1$ with multiplication $V^m \otimes V^n = \sum_p V^p$, where the sum is taken over all indices $p$ that satisfy the conditions from the Clebsch-Gordan theorem.

**Proposition 5.1.** *The representation ring $R(U_\hbar(sl(2,\mathbb{C})))$ is isomorphic to $\mathbb{C}[V^2]/S_{r-1}(V^2)$. If we define $V^n = S_{n-1}(V^2)$ in this ring for all $n \geq 0$, then $V^{r+n} = -V^{r-n}$, $V^r = 0$, and $V^{n+2r} = V^n$ for all $n > 0$.*

5.2. **The quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface of genus greater than 1.** The definition of the quantization of the moduli space $\mathcal{M}_g$ of flat $SU(2)$-connections on a genus $g$ surface $\Sigma_g$ uses *ribbon graphs and framed links* embedded in 3-dimensional manifolds. A ribbon graph consists of the embeddings in the 3-dimensional manifold of finitely many connected components, each of which is homeomorphic to either an annulus or a tubular $\epsilon$-neighborhood in the plane of a planar trivalent graph with small $\epsilon > 0$. As such, while the edges of a classical graph are 1-dimensional, those of a ribbon graph are 2-dimensional, an edge being homeomorphic to either a rectangle, or an annulus. Intuitively one should think the edges are made out of ribbons, hence the name. When embedding the ribbon graph in a 3-dimensional manifold, the framings keep track of the twistings of edges. A framed link is a particular case of a ribbon graph. The link components and the edges of ribbon graphs should be oriented. Viewing an edge as a product of two intervals, the orientation is given by a smooth field of 2-dimensional frames whose first vector is always tangent to the first interval.

Unless otherwise specified, all ribbon graphs and framed links used depicted below are taken with the "blackboard framing", meaning that the $\epsilon$-neighborhood is in the plane of the paper. From the frames that define orientation we draw only the first vector, tacitly understanding that the second vector is chosen so that the orientation agrees with that of the plane of the paper in the usual convention of the $x$- and $y$-axes.

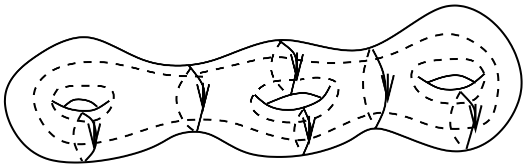

Figure 6

With these conventions at hand, let us quantize the moduli space $\mathcal{M}_g$. The Hilbert space $\mathcal{H}_r(\Sigma_g)$ is defined by specifying a basis, the analogue of the theta series. That needs an *oriented rigid structure* on the surface $\Sigma_g$.

A rigid structure on a surface is a collection of simple closed curves that decompose it into pairs of pants, together with "seams" that keep track of the twistings. The seams are simple closed curves that, when restricted to any pair of pants, give 3 nonintersecting arcs that connect pairwise the boundary components. An oriented rigid structure is one in which the curves decomposing the surface into pairs of pants are oriented. An example is shown in Figure 6, with the decomposing curves drawn with continuous line, and the seams drawn with dotted line.

Map the surface $\Sigma_g$ to the boundary of a handlebody $H_g$ such that the decomposition curves bound disks in $H_g$. The disks cut $H_g$ into balls. Consider the trivalent graph that is the core of $H_g$, with a vertex at the center of each ball and an edge drawn for each disk. The framing of edges should be parallel to the seams (more precisely, to the region of the surface that lies between the seems). The disks are oriented by the decomposition curves on the boundary, and the orientation of the edges of the graph should agree with that of the disks.

The vectors forming an orthonormal basis of $\mathcal{H}_r(\Sigma_g)$ consist of all possible colorings of this framed oriented trivalent graph by $V^j$'s such that at each vertex the three indices satisfy the conditions from the Clebsch-Gordan theorem (note that the double inequality is invariant under permutations of $m, n, p$). Such a coloring is called *admissible*. In genus 3 and for the rigid structure from Figure 6, a basis element is shown in Figure 7. The inner product $\langle \cdot, \cdot \rangle$ is defined so that these basis elements are orthonormal.

This is a nice combinatorial description of the non-abelian theta functions for the Lie group $SU(2)$, which obscures their geometric properties and the origin of the name. The possibility to represent non-abelian theta functions as such graphs follows from the relationship found by Witten between theta functions and conformal field theory. The basis elements associated to an oriented rigid structure on the surface are the analogues of the theta series associated to an element of the Teichmüller space of the torus.

The matrix of the operator $\mathrm{Op}(W_{\gamma,n})$ associated to the Wilson line

$$W_{\gamma,n} : A \to \mathrm{tr}_{V^n} \mathrm{hol}_\gamma(A)$$

is computed as follows. First, let $1 \leq n \leq r-1$. Place the surface $\Sigma_g$ in standard position in the 3-dimensional sphere so that it bounds a genus

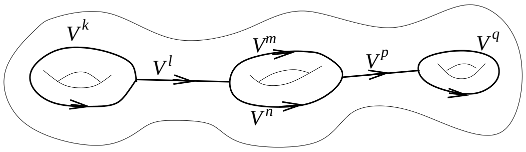


Figure 7

$g$ handlebody on each side. Draw the curve $\gamma$ on the surface, give it the framing parallel to the surface, then color it by the representation $V^n$ of $U_\hbar(sl(2,\mathbb{C}))$. Add two basis elements $e_p$ and $e_q$, viewed as admissible colorings by irreducible representation of the cores of the interior, respectively exterior handlebodies (see Figure 8). The interior and exterior handlebodies should be copies of the same handlebody with the same oriented rigid structure on the boundary (thus giving rise to the same non-abelian theta series), and these copies are glued along $\Sigma_g$ by an orientation reversing homeomorphism as to give rise to $S^3$.

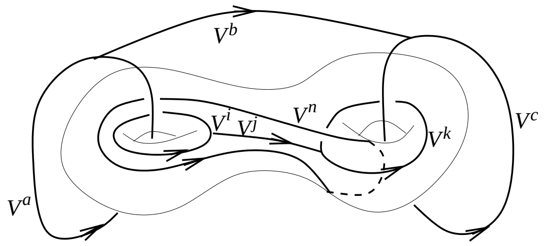


Figure 8

Erase the surface to obtain an oriented tangled ribbon graph in $S^3$ whose edges are decorated by irreducible representations of $U_\hbar(sl(2,\mathbb{C})$ (Figure 9). Project this graph onto a plane while keeping track of the crossings. The Reshetikhin-Turaev theory [35] gives a way of associating a number to this ribbon graph, which is independent of the particular projection and is called the Reshetikhin-Turaev invariant of the ribbon graph.

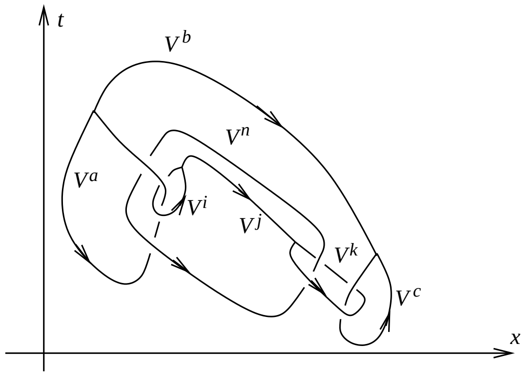


Figure 9

In short, the Reshetikhin-Turaev invariant is computed as follows. The ribbon graph should be mapped by an isotopy to one whose projection can be cut by finitely many horizontal lines into slices, each of which containing one of the phenomena from Figure 10 and some vertical strands. To each horizontal line that slices the graph one associates the tensor product of the representations that color the crossing strands, when pointing downwards, or their duals, when pointing upwards. To the phenomena from Figure 10 one associates, in order, the following operators:

- the flipped quasitriangular $R$-matrix $\check{R} : V^m \otimes V^n \to V^n \otimes V^m$ (obtained by composing the quasitriangular $R$-matrix with the flip $v \otimes w \to w \otimes v$),
- the inverse $\check{R}^{-1}$ of $\check{R}$,
- the projection operator $\beta_p^{mn} : V^m \otimes V^n \to V^p$, whose existence and uniqueness is guaranteed by the Clebsch-Gordan theorem,
- the inclusion $\beta_{mn}^p : V^p \to V^m \otimes V^n$,
- the contraction $E : V^{n*} \otimes V^n \to \mathbb{C}$, $E(f \otimes x) = f(x)$
- its dual $N : \mathbb{C} \to V^n \otimes V^{n*}$, $N(1) = \sum_j e_j \otimes e^j$,
- the isomorphism $D : V^{n*} \to V^n$,
- and its dual $D^* : V^n \to V^{n**} = V^n$ (see Lemma 3.18 in [24] for the precise identification of $V^{n**}$ with $V^n$).

One then composes these operators from the bottom to the top of the diagram, to obtain a linear map from $\mathbb{C}$ to $\mathbb{C}$, which must be of the form $z \to \lambda z$. The number $\lambda$ is the Reshetikhin-Turaev invariant of the ribbon graph. The blank coupons, i.e. the maps $D$, might be required in order to change the orientations of the three edges that meet at a vertex, in order to make them look as depicted in Figure 10. For example the map $V^{p*} \to V^m \otimes V^n$ is defined by identifying $V^{p*}$ with $V^p$ by the isomorphism $D$.

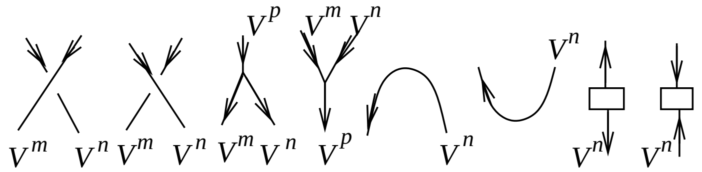


Figure 10

Returning to the quantization of Wilson lines, the Reshetikhin-Turaev invariant of the graph is equal to $[\mathrm{Op}(W_{\gamma,n})e_p, e_q]$, where $[\cdot,\cdot]$ is the nondegenerate bilinear pairing on $\mathcal{H}_r(\Sigma_g)$ defined in Figure 11. One can think of this as being the $p,q$-entry of the matrix of the operator, although this is not quite true because the bilinear pairing is not the inner product. But because the pairing is nondegenerate, the above formula completely determines the operator associated to the Wilson line.

In view of Proposition 5.1, this definition of quantized Wilson lines is extended to arbitrary $n$ by the conventions

$$\mathrm{Op}(W_{\gamma,r}) = 0, \quad \mathrm{Op}(W_{\gamma,r+n}) = -\mathrm{Op}(W_{\gamma,r-n}), \quad \mathrm{Op}(W_{\gamma,n+2r}) = -\mathrm{Op}(W_{\gamma,n})$$

It can be shown that this quantization is in the direction of Goldman's Poisson bracket [1].

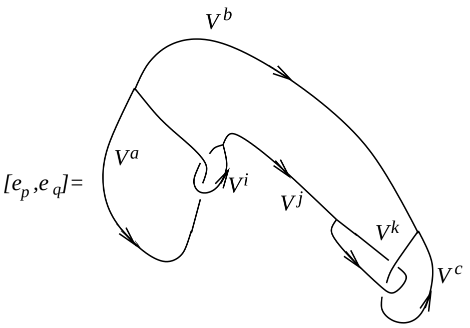


Figure 11

5.3. **Non-abelian theta functions from skein modules.** We will rephrase the construction from §5.2 in the language of skein modules. Our goal is to express the quantum group quantization of Wilson lines as the left representation of a skein algebra on a quotient of itself, in the same way that the Schrödinger representation was described in §3.3 as the left representation of the reduced linking number skein algebra of the cylinder over the torus on a quotient of itself.

One usually associates to $SU(2)$ Chern-Simons theory the skein modules of the Kauffman bracket. The Reshetikhin-Turaev topological quantum field theory has a Kauffman bracket analogue constructed in [6]. However, the Kauffman bracket skein relations introduce sign discrepancies in the computation of the desired left action. And since Theorem 5.10 in §5.4 brings evidence that the quantum group quantization is the non-abelian analogue of Weyl quantization we will define our modules by the skein relations found by Kirby and Melvin in [24] for the Reshetikhin-Turaev version of the Jones polynomial.

We first replace the oriented framed ribbon graphs and links colored by irreducible representations of $U_\hbar(sl(2,\mathbb{C}))$ by formal sums of oriented framed links colored by the 2-dimensional irreducible representation. Two technical results are required.

**Lemma 5.2.** *For all $n = 3, 4, \ldots, r-1$ the identity from Figure 12 holds.*

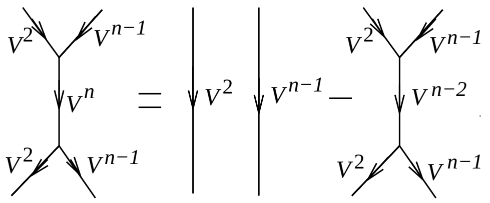


Figure 12

*Proof.* This is a corollary of the particular case of the Clebsch-Gordan theorem $V^n = V^2 \otimes V^{n-1} - V^{n-2}$. $\square$

**Lemma 5.3.** *For any integers $m, n, p$ satisfying $m+n+p$ odd and $|m-n|+1 \leq p \leq \min(m+n-1, 2r-1-m-n)$, the identity from Figure 13 holds.*

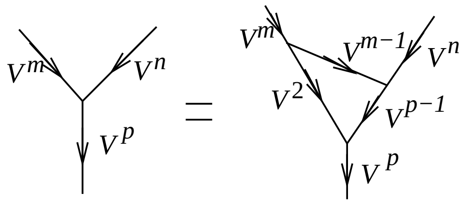


Figure 13

*Proof.* We assume familiarity with the proof of the quantum Clebsch-Gordan theorem, which can be found in [24] or in more detail in [22] (it is true that this second source does the proof when the quantum group is *not* at a root of unity, but since we are only concerned with the situation where $V^p$ is a summand of $V^m \otimes V^n$, all necessary facts apply mutatis mutandis to our situation).

Set $m_0 = \frac{m-1}{2}$, $p_0 = \frac{p-1}{2}$. The morphism described by the diagram on the right is the composition of maps

$$\begin{aligned} V^p &\xrightarrow{\beta^p_{2,p-1}} V^2 \otimes V^{p-1} \xrightarrow{1\otimes\beta^{p-1}_{mn}} V^2 \otimes \left(V^{m-1} \otimes V^n\right) \\ &= \left(V^2 \otimes V^{m-1}\right) \otimes V^n \xrightarrow{\beta^{2,m-1}_m \otimes 1} V^m \otimes V^n. \end{aligned}$$

Because of Schur's lemma and the quantum Clebsch-Gordan theorem, this composition is either the zero or the identity map. To show that is not the zero map, we look how the vector $e_{p_0}$ of heighest weight in $V^p$ transforms under it. We have

$$\begin{aligned} e_{p_0} &\to e_{\frac{1}{2}} \otimes e_{p_0-\frac{1}{2}} \to e_{\frac{1}{2}} \otimes \sum_{i+j=p_0} c_{ij} e_i \otimes e_j \\ &= \sum_{i+j=p_0} c_{ij} e_{\frac{1}{2}} \otimes e_i \otimes e_j \in V^2 \otimes V^{m-1} \otimes V^n. \end{aligned}$$

The Clebsch-Gordan coefficients $c_{ij}$ are nonzero, and in the sum there is a term $c_{m_0-\frac{1}{2},j} e_{\frac{1}{2}} \otimes e_{m_0-\frac{1}{2}} \otimes e_j$.

On the other hand, the inclusion $\beta^m_{2,m-1} : V^m \to V^2 \otimes V^{m-1}$ maps the vector $e_{m_0}$ of heighest weight in $V^m$ to $e_{\frac{1}{2}} \otimes e_{m_0-\frac{1}{2}}$, which is the product of the vectors of heighest weights in $V^2$ and $V^{m-1}$. Hence if the $\frac{1}{2}, m_0 - \frac{1}{2}$-component of a vector $v$ written in the basis $e_i \otimes e_j$ of $V^2 \otimes V^{m-1}$ is nonzero, then $\beta^m_{2,m-1} v$ is nonzero in $V^m$.

In particular, the above sum maps to a nonzero vector in $V^m \otimes V^n$. It follows that the diagram on the right of Figure 13 equals the inclusion $\beta^p_{mn} : V^p \to V^m \otimes V^n$, and the identity is proved. □

**Proposition 5.4.** *There is an algorithm for replacing each connected ribbon graph $\Gamma$ colored by irreducible representations of $U_\hbar(sl(2,\mathbb{C}))$ by a sum of oriented framed links colored by $V^2$ that lie in an $\epsilon$-neighborhood of the graph, such that if in any ribbon graph $\Gamma'$ that has $\Gamma$ as a connected component we replace $\Gamma$ by this sum of links, we obtain a ribbon graph with the same Reshetikhin-Turaev invariant as $\Gamma'$.*

*Proof.* For framed knots, the property follows from the cabling formula given in Theorem 4.15 in [24]; a knot colored by $V^n$ is replaced by $S_{n-1}(V^2)$.

If the connected ribbon graph has vertices, note first that by using the isomorphism $D$ to identify irreducible representations of $U_\hbar(sl(2,\mathbb{C})$ with their duals, we can obtain the identity from Figure 13 with the arrows reversed. Also, by taking the adjoint of the map described by the diagram, we can turn it upside down, meaning that we can write a similar identity for $\beta_p^{mn}$.

Based on the two lemmas, the algorithm works as follows. First, use the identities from Figure 14 to remove all edges colored by $V^1$. Then apply repeatedly Lemma 5.3 until at each vertex of the newly obtained ribbon graph at least one of the three edges is colored by $V^2$. Then use Lemma 5.2 to obtain a sum of graphs with the property that, at each vertex, two of the three edges are colored by $V^2$ and one is colored by $V^1$. Next, use the identities from Figure 14 for $n = 2$ to transform everything into a sum of framed links whose edges are colored by $V^2$. Each of the links in the sum has and even number of blank coupons (representing the isomorphism $D$ or its dual) on each component. Cancel the coupons on each link component in pairs, adding a factor of $-1$ each time the two coupons are separated by an odd number of maxima on the link component. The result is a formal sum of framed links with components colored by $V^2$. □

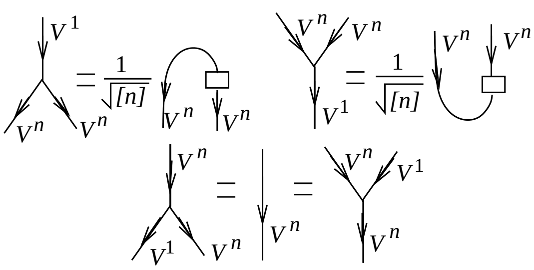


Figure 14

Theorem 4.3 in [24] allows us to compute the Reshetikhin-Turaev invariant of a framed link whose components are colored by $V^2$ using skein relations. First, forget about the orientation of links. Next, if three framed links

$L, H, V$ in $S^3$ colored by $V^2$ coincide except in a ball where they look like in Figure 15, then their Reshetikhin-Turaev invariants, denoted by $J_L, J_H$, and $J_V$ satisfy

$$J_L = tJ_H + t^{-1}J_V$$

if the two crossing strands come from different components, and

$$J_L = \epsilon(tJ_H - t^{-1}J_V)$$

if the crossing strands come from the same component.

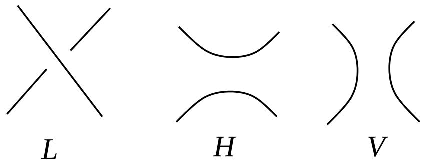


Figure 15

Here $\epsilon$ is the sign of the crossing, obtained after orienting that link component (either orientation produces the same sign). Specifically, if the crossing looks like in Figure 16 a) then the sign is plus, if it looks like in Figure 16 b) the sign is minus.

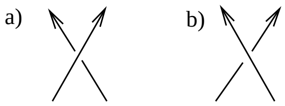


Figure 16

Additionally if a link component bounds a disk inside a ball disjoint from the rest of the link, then it is replaced by a factor of $t^2 + t^{-2}$.

When $t = 1$, namely when $\hbar = 0$, it no longer matters whether one has an undercrossing or overcrossing, and both skein relations express the trace identity of $SU(2)$,

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B),$$

for Wilson lines

$$W_\alpha W_\beta = W_{\alpha\beta} + W_{\alpha^{-1}\beta}.$$

For arbitrary $t$, the skein relations are the trace identity for the quantum group $U_\hbar(sl(2, \mathbb{C}))$,

$$t\mathrm{tr}(AB) + t^{-1}\mathrm{tr}(S(A)B) = \sum_i \mathrm{tr}(s_iA)\mathrm{tr}(t_iB)$$

where $\sum_i s_i \otimes t_i$ is the quasitriangular $R$-matrix of $U_h(sl(2, \mathbb{C}))$ (see [7]). One should observe that these skein relations correspond to the trace identity, while the Kauffman bracket skein relations correspond to the trace identity for the negative of the trace.

This prompts us to introduce skein modules defined by these skein relations. Let now $t$ be an abstract variable, rather than the root of unity chosen at the begining of §5.2. For an orientable 3-dimensional manifold $M$, consider the free $\mathbb{C}[t, t^{-1}]$-module with basis the isotopy classes of framed links in $M$ including the empty link. Factor this module by the skein relations

$$L = tH + t^{-1}V$$

if the two crossing strands come from different components, and

$$L = \epsilon(tH - t^{-1}V)$$

if the crossing strands come from the same component where the links $L, H, V$ are the same except in an embedded ball in $M$, inside of which they look as depicted in Figure 15. Here use the same convention for $\epsilon$, with the orientation of the crossing decided inside the ball. Additionally, replace any trivial link component that lies inside a ball disjoint from the rest of the link by a factor of $t^2 + t^{-2}$. We will call the result of the factorization the *Reshetikhin-Turaev skein module* and denote it by $RT_t(M)$. One can show that $RT_t(M)$ is isomorphic to the Kauffman bracket skein module of $M$.

If $M = \Sigma_g \times [0,1]$ then the homeomorphism

$$\Sigma_g \times [0,1] \cup_{\Sigma_g} \Sigma_g \times [0,1] \approx \Sigma_g \times [0,1]$$

induces a multiplication on $R_t(\Sigma_g \times [0,1])$, turning it into an algebra, the *Reshetikhin-Turaev skein algebra*. This algebra is not isomorphic to the Kauffman bracket skein algebra except in genus one. In higher genus the multiplication rules are different, as can be seen by examining the product of a separating and a nonseparating curve that intersect.

The operation of gluing $\Sigma_g \times [0,1]$ to the boundary of a genus $g$ handlebody $H_g$ by a homeomorphism of the surface induces an $RT_t(\Sigma_g \times [0,1])$-module structure on $RT_t(H_g)$. Moreover, by gluing $H_g$ with the empty skein inside to $\Sigma_g \times [0,1]$ we see that $RT_t(H_g)$ is the quotient of $RT_t(\Sigma_g \times [0,1])$ obtained by identifying the skeins in $\Sigma_g \times [0,1])$ that are isotopic in $H_g$.

In view of Lemma 5.2 and the identities from Figure 14, the irreducible representations $V^n$ can be represented by the skeins. Explicitly, $V^n = S_{n-1}(V^2) = f^{n-1}$, where $f^n$ are defined recursively in Figure 17. These are the well-known Jones-Wenzl idempotents [19], [46].

The condition $S_{r-1}(V^2) = 0$ translates to the condition $f^{r-1} = 0$. This prompts us to define the *reduced Reshetikhin-Turaev skein module* $\widetilde{RT}_t(M)$ by factoring $RT_t(M)$ by $t = e^{\frac{i\pi}{2r}}$ and by the skein relation $f^{r-1} = 0$, taken in every possible embedded ball. This reduction is compatible with the multiplicative structure of $RT_t(\Sigma_g \times [0,1])$ and with its action on $RT_t(H_g)$.

**Proposition 5.5.** *The quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface $\Sigma_g$ can be represented as the left multiplication of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ on $\widetilde{RT}_t(H_g)$.*

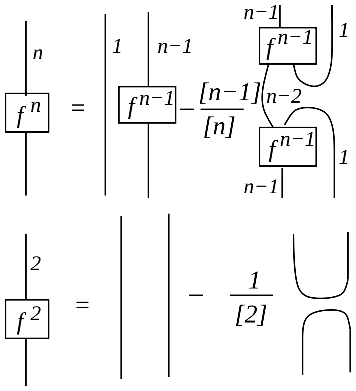


Figure 17

*Proof.* The proof is based on Proposition 5.4 and Proposition 5.1. Because each $f^n$ involves $n$ parallel strands, $RT_t(H_g)$ is a free $C[t,t^{-1}]$-module with basis the skeins obtained by

- replacing each edge of the core of $H_g$ by a Jones-Wenzl idempotent in such a way that, if $f^m, f^n, f^p$ meet at a vertex, then $m+n+p$ is even, $m+n\leq p$, $m+p\leq n$, $n+p\leq m$, and
- replacing the vertices by the unique collection of strands that lie in a disk neighborhood of the vertex and join the ends of the three Jones-Wenzl idempotents meeting there in such a way that there are no crossings.

Because of the Clebsch-Gordan theorem and Proposition 5.1, in the reduced skein module $\widetilde{RT}_t(H_g)$, only edges colored by $f^n$ with $n\leq r-2$ need to be considered, and also if $f^m, f^n, f^p$ meet at a vertex, then $m+1, n+1, p+1$ and their cyclic permutations should satisfy the double inequality from the Clebsch-Gordan theorem. Each element of this form comes from a basis element in the quantum group quantization. A more detailed explanation of this can be found, at least for the Kauffman bracket skein modules, in [28].

The computation from Figure 18, perfomed in the dotted annulus, shows that for a simple closed curve $\gamma$ on the torus, $\mathrm{Op}(W_{\gamma,n})$ can be identified with the skein $S_{n-1}(\gamma)\in\widetilde{RT}_t(\Sigma_g\times[0,1])$. We conclude that the action of quantum observables on the Hilbert space is modeled by the action of $\widetilde{RT}_t(\Sigma_g\times[0,1])$ on $\widetilde{RT}_t(H_g)$.

To identify the two quantization models, we also have to prove that the skeins associated to admissible colorings of the core of the handlebody form a basis, namely that they are linearly independent in $\widetilde{RT}_t(H_g)$.

The smooth part of $\mathcal{M}_g$ has real dimension $6g-6$. This smooth part is a completely integrable manifold in the Liouville sense. Indeed, the Wilson lines $W_{\alpha_i}$, where $\alpha_i$, $i=1,2,\ldots,3g-3$, are the curves in Figure 19, form a maximal set of Poisson commuting functions (meaning that $\{W_{\alpha_i},W_{\alpha_j}\}=0$). The quantum group quantization of the moduli space

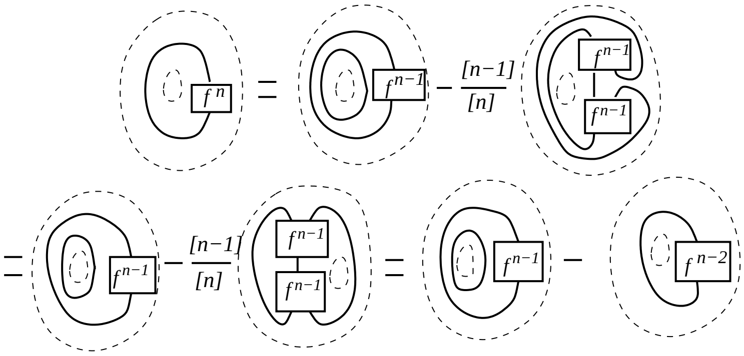


Figure 18

of flat $SU(2)$-connections is thus a quantum integrable system, with the operators $\mathrm{Op}(W_{\alpha_1})$, $\mathrm{Op}(W_{\alpha_2})$, ..., $\mathrm{Op}(W_{\alpha_{3g-3}})$ satisfying the integrability condition.

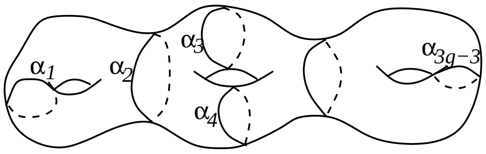


Figure 19

The identity from Figure 20, which holds for any choice of orientation of the strands, implies that the spectral decomposition of the commuting $(3g-3)$-tuple of self-adjoint operators

$$(\mathrm{Op}(W_{\alpha_1}), \mathrm{Op}(W_{\alpha_2}), ..., \mathrm{Op}(W_{\alpha_{3g-3}}))$$

has only 1-dimensional eigenspaces consisting precisely of the colorings of the edges following the specified rule. Indeed, the basis elements are as described in §5.2 for the case where the curves that cut the surface into pairs of pants are $\alpha_1, \alpha_2, \cdots, \alpha_{3g-3}$, and the identity from Figure 20 shows that the eigenvalues of an $e_j$ with respect to the $3g-3$ quantized Wilson lines completely determine the colors of its edges. This concludes the proof. $\square$

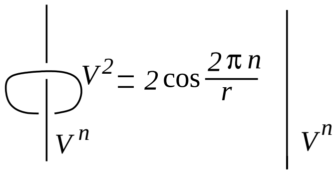


Figure 20

*Remark* 5.6. Proposition 5.5 should be compared to Theorem 3.11. Again the algebra of quantized observables is a skein algebra, the space of non-abelian theta functions is a quotient of this algebra, and the factorization relation is of topological nature; it is defined by gluing the cylinder over the surface to a handlebody via a homeomorphism. The skein modules $RT_t(\Sigma_g \times [0,1])$ and $\widetilde{RT}_t(\Sigma_g \times [0,1])$ are the analogues, for the gauge group $SU(2)$, of the algebras $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$ and $\mathcal{A}_N$.

Since we have not shown yet that the pairing $[\,,\,]$ defined in §5.2 is nondegenerate, we will take for the moment this representation of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ to actually be the quantum group quantization of the moduli space of flat $SU(2)$-connections on $\Sigma_g$. We will prove the nondegeneracy in §6.3.

*Remark* 5.7. The quantum group quantization is more natural than it seems. Quantum groups were introduced by Drinfeld to solve vertex models, as means of finding operators that satisfy the Yang-Baxter equation. They lead to the deformation quantization model for the quantization of $\mathcal{M}_g$ in [1]. This gives rise to the skein algebra of the surface, and by analogy with §3.3 we are led to consider the skein module of the handlebody. The basis consisting of admissible colorings of the core of the handlebody shows up when looking at the spectral decomposition of the system of commuting operators used in the proof of Proposition 5.5.

5.4. **The quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus.** The quantum group quantization of $\mathcal{M}_1$ is a particular case of the construction in §5.2. To find a basis for the Hilbert space of the quantization we take a rigid structure on the torus. By mapping the torus to the boundary of the solid torus such that the rigid structure becomes the one from Figure 1, we obtain an orthonormal basis consisting of the $r-1$ vectors $V^1(\alpha), V^2(\alpha), \ldots, V^{r-1}(\alpha)$, which are the colorings the core $\alpha$ of the solid torus by the irreducible representations $V^1, V^2, \ldots, V^{r-1}$ of $U_\hbar(sl(2,\mathbb{C}))$. These are the quantum group analogues of the $\zeta_j^\tau$'s. In this case, the orientation of the rigid structure, and hence of the core of the solid torus are irrelevant, reversing the orientation gives the same results in computations (orientation of link components is irrelevant by [47]).

The operator associated to the function $f(x,y) = 2\cos 2\pi(px+qy)$ is computed like for higher genus surfaces. The bilinear form on the Hilbert space comes from the Hopf link and is $[V^j(\alpha), V^k(\alpha)] = [jk]$, $j,k = 1,2,\ldots,r-1$. The value of $[\mathrm{Op}(2\cos 2\pi(px+qy))V^j(\alpha), V^k(\alpha)]$ is equal to the Reshetikhin-Turaev invariant of the three-component colored framed link consisting the curve of slope $p/q$ on the torus embedded in standard position in $S^3$, colored by $V^{n+1} - V^{n-1}$ where $n$ is the greatest common divisor of $p$ and $q$, the core of the solid torus that lies on one side of the torus colored by $V^j$, and the core of the solid torus that lies on the other side colored by $V^k$. Coloring the curve by $V^{n+1} - V^{n-1}$ is the same as coloring it by

$$T_n(V^2) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (V^2)^{n-2j},$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind defined recursively by $T_0(x) = 2$, $T_1(x) = x$, $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$, for $n \geq 1$. Again, the quantum group quantization can be modeled by the action of the reduced Reshetikhin-Turaev skein algebra of the torus on the reduced Reshetikhin-Turaev skein module of the solid torus.

It has been shown in [16] that the quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus is unitary equivalent to Weyl quantization. However, that proof makes use of the Reshetikhin-Turaev representation of the mapping class group of the torus, and does not serve our purpose to show how the Reshetikhin-Turaev representation arises from quantum mechanical considerations. For that reason we give below a different proof of this result using the structure of the Reshetikhin-Turaev skein algebra of the torus.

For a pair of integers $p, q$, let $n$ be their common divisor and define $(p, q)_T = T_n((p/n, q/n)) \in RT_t(\mathbb{T}^2 \times [0, 1])$.

**Theorem 5.8.** *For any integers $p, q, p', q'$ the following product-to-sum formula holds*

$$(p, q)_T (p', q')_T = t^{\left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|}(p + p', q + q')_T + t^{-\left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|}(p - p', q - q')_T.$$

*Proof.* This product-to-sum formula is identical to the one for the Kauffman bracket skein algebra of the torus in [12], and we adapt that proof to our situation.

We induct on the absolute value $D$ of the determinant $\left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|$. If $D = 0$, then $(p, q)_T$ and $(p', q')_T$ are Chebyshev polynomials of the same curve and the formula follows from the properties of Chebyshev polynomials. If $D = 1$, then $(p, q)_T$ and $(p', q')_T$ are simple closed curves on the torus, and the product-to-sum formula is just the skein relation.

If $D = 2$, then either $\gcd(p, q) = \gcd(p', q') = 1$, or one of the greatest common divisors is 1 and the other is 2. In the first situation, we can map by an orientation preserving homeomorphism of the torus $(p, q)_T$ to $(1, 0)_T$ and $(p', q')_T$ to $(1, \pm 2)_T$ and the formula can be checked again. In the other situation, the skein with the greatest common divisor 2 can be mapped to $(2, 0)_T$ and the other to $(1, \pm 1)$, and the formula follows.

Let $D = k \geq 3$, and assume that the formula holds for all values of $D$ less than $k$. If $\gcd(p, q) = \gcd(p', q') = 1$, then by applying an orientation preserving homeomorphism of the torus we may assume $q < p$ and $(p', q') = (0, 1)$. The following result was proved in [12].

**Lemma 5.9.** *Given $p \geq 3$, and $0 < q < p$ with $gcd(p, q) = 1$, there exist integers $u, v, w, z$ satisfying $u + w = p$, $v + z = q$, $\left|\begin{smallmatrix} u & v \\ w & z \end{smallmatrix}\right| = \pm 1$, $0 < w < p$, $0 < u < p - 1$, $0 < v < z$.*

Using the lemma and the induction hypothesis we can write

$$\begin{aligned}
&(p,q)(0,1) = t^{-|{uv \atop wz}|}(u,v)(w,z)(0,1) - t^{-2|{uv \atop wz}|}(u-w,v-z)(0,1)\\
&= t^{-1}(u,v)[t^{w}(w,z+1)_T + t^{-w}(w,z-1)_T]\\
&-t^{-2}[t^{u-w}(u-w,v-z+1)_T + t^{w-u}(u-w,v-z-1)_T]\\
&= t^{w+u}(u+w,v+z+1)_T + t^{-2+w-u}(u-w,v-z-1)_T\\
&+t^{-2-w+u}(u-w,v-z+1)_T + t^{-w-u}(u+w,v+z-1)_T\\
&-t^{-2+u-w}(u-w,v-z+1)_T + t^{-2-u+w}(u-w,v-z-1)_T\\
&= t^{u+w}(p,q+1)_T + t^{-u-w}(p,q-1)_T = t^{p}(p,q+1)_T + t^{-p}(p,q-1)_T.
\end{aligned}$$

If one of the greatest common divisors is greater than 1, say $n = \gcd(p,q)$, then we can write

$$\begin{aligned}
&(p,q)_T(p',q')_T = T_n(p/n,q/n)(p',q')_T\\
&= T_{n-1}(p/n,q/n)[(p/n,q/n)(p',q')_T] - T_{n-2}(p/n,q/n)(p',q')_T,
\end{aligned}$$

and obtain the desired formula by inducting on the greatest common divisor of $p$ and $q$. □

As we can see, the Reshetikhin-Turaev and the Kauffman bracket skein algebras of the torus are isomorphic.

**Theorem 5.10.** [16] *The Weyl quantization and the quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus are unitary equivalent.*

*Proof.* We rephrase the quantum group quantization in the language of skein modules. The Hilbert space of the quantization is $\widetilde{RT}_t(S^1 \times \mathbb{D}^2)$. Indeed, this skein module is spanned by the vectors $S_{j-1}(\alpha)$, $j = 1, 2, \ldots, r-1$, and these vectors are linearly independent because they are eigenvectors with different eigenvalues of the operator defined by $(0,1)$.

Considering the projection $\pi : RT_t(\mathbb{T}^2 \times [0,1]) \to RT_t(S^1 \times \mathbb{D}^2)$ defined by attaching the cylinder over the torus to the solid torus by the homeomorphism $h_0$ from §3.3, and using Theorem 5.8 we deduce the recursive formula

$$\pi((p+1,q)_T) = t^{-q}\alpha\pi((p,q)_T) - t^{-2q}\pi((p-1,q)_T).$$

Also $\pi((0,q)_T) = t^{2q} + t^{-2q}$, and $\pi((1,q)_T) = t^{-2q}\alpha$. Solving the recurrence we obtain

$$\pi((p,q)_T) = t^{-pq}(t^{2q}S_p(\alpha) - t^{-2q}S_{p-2}(\alpha)).$$

Using again Theorem 5.8 we have

$$\begin{aligned}
(p,q)_T T_j(\alpha) &= \pi[(p,q)_T(j,0)_T] = \pi[t^{-jq}(p+j,q)_T + t^{jq}(p-j,q)_T]\\
&= t^{-pq}[t^{-(2j-2)q}S_{p+j}(\alpha) + t^{(2j+2)q}S_{p-j}(\alpha) - t^{-(2j-2)q}S_{p+j-2}(\alpha)\\
&\qquad -t^{(2j-2)q}S_{p-j-2}(\alpha).
\end{aligned}$$

Since $T_n(x) = S_n(x) - S_{n-2}(x)$ for all $n$, we have

$$(p,q)_T S_{j-1}(\alpha) = t^{-pq}(t^{-2qj}S_{p+j-1}(\alpha) + t^{2qj}S_{p-j+1}(\alpha), \text{ for } j > 0.$$

Reducing to the relative skein modules and using the fact that $S_{j-1}(\alpha) = V^j(\alpha)$, we obtain

$$\begin{aligned}\mathrm{Op}(2\cos 2\pi(px+qy))V^j(\alpha) = e^{-\frac{\pi i}{2r}pq}\Big(&e^{\frac{\pi i}{r}qj}V^{j-p}(\alpha)\\ &+e^{-\frac{\pi i}{r}qj}V^{j+p}(\alpha)\Big),\end{aligned} \tag{5.1}$$

with the conventions $V^r(\alpha) = 0$, $V^{j+2r}(\alpha) = V^j(\alpha)$, $V^{r+j}(\alpha) = -V^{r-j}(\alpha)$ if the indices leave the range $1, 2, \dots, r-1$. This is the formula for the Weyl quantization of the pillow case given in §4.2, which proves the theorem. □

5.5. **A Stone–von Neumann theorem on the moduli space of flat $SU(2)$-connections on the torus.** Weyl quantization yields a representation of the reduced Reshetikhin-Turaev skein algebra of the cylinder over the torus, $\widetilde{RT}_t(\mathbb{T}\times[0,1])$, satisfying

1. $t$ acts as multiplication by $e^{\frac{i\pi}{2r}}$,
2. each simple closed curve on the torus $\gamma$ acts as a self-adjoint operator.

The algebra $\widetilde{RT}_t(\mathbb{T}\times[0,1])$ is a non-abelian analogue of the group algebra of the finite Heisenberg group. Like for the case of the finite Heisenberg group, a Stone-von Neumann theorem holds.

**Theorem 5.11.** *The representation of the reduced Reshetikhin-Turaev skein algebra of the torus defined by the Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus is the* unique *irreducible representation of this algebra that maps simple closed curves to self-adjoint operators and $t$ to multiplication by $e^{\frac{\pi i}{2r}}$. Moreover, quantized Wilson lines span the algebra of all linear operators on the Hilbert space of the quantization.*

*Proof.* We prove irreducibility by showing that any vector is cyclic. Because the eigenspaces of each quantized Wilson line are 1-dimensional, in particular those of $\mathrm{Op}(2\cos 2\pi y)$, it suffices to check this property for the eigenvectors of this operator, namely for $\zeta_j^\tau$, $j = 1, 2, \dots, r-1$. And because

$$\begin{aligned}\mathrm{Op}(2\cos 2\pi x)\zeta_j^\tau &= \zeta_{j-1}^\tau + \zeta_{j+1}^\tau\\ \mathrm{Op}(2\cos 2\pi(x+y))\zeta_j^\tau &= t^{-1}(t^2\zeta_{j-1}^\tau + t^{-2}\zeta_{j+1}^\tau),\end{aligned}$$

by taking linear combinations we see that from $\zeta_j^\tau$ we can generate both $\zeta_{j+1}^\tau$ and $\zeta_{j-1}^\tau$. Repeating, we can generate the entire basis. This shows that $\zeta_j^\tau$ is cyclic for each $j = 1, 2, \dots, r-1$, hence the representation is irreducible.

To prove uniqueness, consider an irreducible representation of $\widetilde{RT}_t(\mathbb{T}^2\times[0,1])$ with the required properties. Note first that the condition that $S_{r-1}(\gamma) = 0$ for any simple closed curve $\gamma$ on the torus implies, by the spectral mapping theorem, that the eigenvalues of the operator associated to $\gamma$ are among the numbers $2\cos\frac{k\pi}{r}$, $k = 1, 2, \dots, r-1$.

Using the presentation of the Kauffman bracket skein algebra of the torus found by Bullock and Przytycki in [8], we can write for the generators $X = (1,0)$, $Y = (0,1)$, and $Z = (1,1)$ of the algebra $\widetilde{RT}_t(\mathbb{T}^2 \times [0,1])$ the relations

$$\begin{aligned}
&tXY - t^{-1}YX = (t^2 - t^{-2})Z\\
&tYZ - t^{-1}ZY = (t^2 - t^{-2})X\\
&tZX - t^{-1}XZ = (t^2 - t^{-2})Y\\
&t^2X^2 + t^{-2}Y^2 + t^2Z^2 - tXYZ - 2t^2 - 2t^{-2} = 0.
\end{aligned}$$

In fact $\widetilde{RT}_t(\mathbb{T}^2 \times [0,1])$ is generated by just $X$ and $Y$, since we can substitute $Z$ from the first equation. This gives the equivalent presentation

$$\begin{aligned}
&(t^2 + t^{-2})YXY - (XY^2 + Y^2X) = (t^4 + t^{-4} - 2)X\\
&(t^2 + t^{-2})XYX - (YX^2 + X^2Y) = (t^4 + t^{-4} - 2)Y\\
&(t^6 + t^{-2} - 2t^2)X^2 + (t^{-6} + t^2 - 2t^{-2})Y^2 + XYXY + YXYX\\
&\quad -t^2YX^2Y - t^{-2}XY^2X = 2(t^6 + t^{-6} - t^2 - t^{-2}).
\end{aligned} \tag{5.2}$$

Setting $t = e^{\frac{i\pi}{2r}}$ the first equation in (5.2) becomes

$$2\cos\frac{\pi}{r}YXY - (XY^2 + Y^2X) = 4\sin^2\frac{\pi}{r}Y.$$

Let $v_k$ be an eigenvector of $Y$ with eigenvalue $2\cos\frac{k\pi}{r}$ for some $k \in \{1, 2, \ldots, r-1\}$. As for the finite Heisenberg group, we wish to generate a basis of the representation by acting repeatedly on $v_k$ by $X$. To this end, set $Xv_k = w$. The above relation yields

$$2\cos\frac{\pi}{r} \cdot 2\cos\frac{k\pi}{r}Yw - 4\cos^2\frac{k\pi}{r}w - Y^2w = 4\sin^2\frac{\pi}{r}w.$$

Rewrite this as

$$\left[Y^2 - 4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}Y - 4\left(\sin^2\frac{\pi}{r} + \cos^2\frac{k\pi}{r}\right)\right]w = 0.$$

It follows that either $w = 0$ or $w$ is in the kernel of the operator

$$Y^2 - 4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}Y - 4\left(\sin^2\frac{\pi}{r} + \cos^2\frac{k\pi}{r}\right)Id. \tag{5.3}$$

The second equation in (5.2) shows that if $Xv_k = w = 0$ then $Yv_k = 0$. This is impossible because of the third relation in (5.2). Hence $w \neq 0$, so $w$ lies in the kernel of the operator from (5.3). Note that if $\lambda$ is an eigenvalue of $Y$ which satisfies

$$\lambda^2 - 4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}\lambda - 4\left(\sin^2\frac{\pi}{r} + 4\cos^2\frac{k\pi}{r}\right) = 0,$$

then necessarily $\lambda = 2\cos\frac{(k\pm1)\pi}{r}$. It follows that

$$Xv_k = v_{k+1} + v_{k-1},$$

where $Yv_{k\pm1} = 2\cos\frac{(k\pm1)\pi}{4}v_{k\pm1}$, and $v_{k+1}$ and $v_{k-1}$ are not simultaneously equal to zero. We wish to enforce $v_k$, $v_{k+1}$, and $v_{k-1}$ to be elements of a basis. For that we need to check that $v_{k+1}$ and $v_{k-1}$ are nonzero, and we also need to understand the action of $X$ on them.

Set $Xv_{k+1} = \alpha v_k + v_{k+2}$ and $Xv_{k-1} = \beta v_k + v_{k-2}$, where $Yv_{k\pm 2} = 2\cos\frac{(k\pm 2)\pi}{r} v_{k\pm 2}$. It might be possible that the scalars $\alpha$ and $\beta$ are zero. The vectors $v_{k+2}$, $v_{k-2}$ might as well be zero; if they are not zero, then they are eigenvectors of $Y$, and their respective eigenvalues are as specified (which can be seen by repeating the above argument).

Applying both sides of the second equation in (5.2) to $v_k$ and comparing the $v_k$ coordinate of the results we obtain

$$\cos\frac{\pi}{r}\cos\frac{(k+1)\pi}{r}\alpha + \cos\frac{\pi}{r}\cos\frac{(k-1)\pi}{r}\beta - \cos\frac{k\pi}{r}(\alpha+\beta) \\ = \cos\frac{2\pi}{r}\cos\frac{k\pi}{r} - \cos\frac{k\pi}{r}.$$

This is equivalent to

$$\left(\cos\frac{(k+2)\pi}{r} + \cos\frac{k\pi}{r}\right)(\alpha-1) + \left(\cos\frac{(k-2)\pi}{r} + \cos\frac{k\pi}{r}\right)(\beta-1) = 0$$

that is

$$\sin\frac{(k+1)\pi}{r}(\alpha-1) + \sin\frac{(k-1)\pi}{r}(\beta-1) = 0.$$

For further use, we write this as

$$(t^{4k+4}-1)(\alpha-1) + (t^{4k}-t^4)(\beta-1) = 0. \tag{5.4}$$

Applying the two sides of the last equation in (5.2) to $v_k$ and comparing the $v_k$ coordinate of the results we obtain

$$(t^6+t^{-2}-2t^2)(\alpha+\beta) + (t^{-6}+t^2-2t^{-2})4\cos^2\frac{k\pi}{r} \\ +8\cos\frac{k\pi}{r}\cos\frac{(k+1)\pi}{r}\alpha + 8\cos\frac{k\pi}{r}\cos\frac{(k-1)\pi}{r}\beta - 4t^2\cos^2\frac{k\pi}{r}(\alpha+\beta) \\ -4t^{-2}\cos^2\frac{(k+1)\pi}{r}\alpha - 4t^{-2}\cos^2\frac{(k-1)\pi}{r}\beta = 2(t^6+t^{-6}-t^2-t^{-2}).$$

This can be rewritten as

$$(t^6+t^{-2}-2t^2)(\alpha+\beta) + 4\cos\frac{(2k+1)\pi}{r}\alpha + 4\cos\frac{\pi}{r}\alpha + 4\cos\frac{(2k-1)\pi}{r}\beta \\ +4\cos\frac{\pi}{r}\beta - 2t^2\cos\frac{2k\pi}{r}\alpha - 2t^2\cos\frac{2k\pi}{r}\beta - 2t^2\alpha - 2t^2\beta \\ -2t^{-2}\cos\frac{(2k+2)\pi}{r}\alpha - 2t^{-2}\alpha - 2t^{-2}\cos\frac{(2k-2)\pi}{r}\beta - 2t^{-2}\beta \\ = 2(t^6+t^{-6}-t^2-t^{-2}) - 2(t^{-6}+t^2-2t^{-2}) - 2(t^{-6}+t^2-2t^{-2})\cos\frac{2k\pi}{r}.$$

Using the fact that $t = \cos\frac{k\pi}{2r} + i\sin\frac{k\pi}{2r}$ we can transform this into

$$(t^{-4k-6}+t^{-4k+2}+2t^2-2t^{-4k-2}-t^6-t^{-2})(\alpha-1) \\ +(t^{4k-6}+t^{4k+2}+2t^2-2t^{4k-2}-t^6-t^{-2})(\beta-1) = 0.$$

Dividing through by $t^{-6}+t^2-2t^{-2}$ we obtain

$$(t^{-4k}-t^4)(\alpha-1)+(t^{4k}-t^4)(\beta-1)=0.$$

Combining this with (5.4), we obtain the system

$$(t^{4k+4}-1)u+(t^{4k}-t^4)v=0$$
$$(t^{-4k}-t^4)u+(t^{4k}-t^4)v=0$$

in the unknowns $u=\alpha-1$ and $v=\beta-1$. Recall that $t=e^{\frac{i\pi}{2r}}$.

The coefficient of $v$ equals zero if and only if $k=1$, in which case we are forced to have $\beta=0$, because 0 is not an eigenvalue of $Y$. The coefficient of $u$ in one of the equations is equal to zero if and only if $k=r-1$, in which case we are forced to have $\alpha=0$, because $-1$ is not an eigenvalue of $Y$.

In any other situation, by subtracting the equations we obtain

$$(t^4-t^{-4k})(t^{4k}+1)u=0.$$

This can happen only if $t^{4k}=-1$, namely if $2k=r$.

So, if $k\neq\frac{r}{2}$, then $Xv_k=v_{k+1}+v_{k-1}$ with $v_{k+1}$ and $v_{k-1}$ eigenvectors of $Y$ with eigenvalues $2\cos\frac{(k+1)\pi}{r}$ respectively $2\cos\frac{(k-1)\pi}{r}$, and $Xv_{k\pm1}=v_k+v_{k\pm2}$, where $v_{k\pm2}$ lie in the eigenspaces of $Y$ of the eigenvalues $2\cos\frac{(k\pm2)\pi}{r}$.

What if $k=\frac{r}{2}$? One of $v_{k+1}$ and $v_{k-1}$ is not zero, say $v_{k+1}$. Applying the above considerations to $v_{k+1}$ we have $Xv_{k+1}=\alpha v_k+v_{k+2}$ and $X\alpha v_k=v_{k+1}+v'_{k-1}$, for some $v'_{k-1}$ in the eigenspace of $Y$ of the eigenvalue $2\cos\frac{(k-1)\pi}{r}$. Then on the one hand $Xv_k=v_{k+1}+v_{k-1}$ and on the other $\alpha Xv_k=v_{k+1}+v'_{k-1}$. This shows that $\alpha=1$, and because $(\alpha-1)+(\beta-1)=0$, it follows that $\beta=1$ as well. A similar conclusion is reached if $v_{k-1}\neq0$.

Repeating the argument we conclude that the irreducible representation, which must be the span of $X^mY^nv_k$ for $m,n\geq0$, has the basis $v_1,v_2,\ldots,v_{r-1}$, and $X$ and $Y$ act on these vectors by

$$Xv_j=v_{j+1}+v_{j-1},\quad Yv_j=2\cos\frac{j\pi}{r},$$

with the convention $v_0=v_r=0$. And we recognize the representation defined by the Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus.

The fact that the algebra of all quantized Wilson lines is the algebra of all linear operators on the Hilbert space of the quantization is a corollary of Theorem 6.1 in [14]. $\square$

## 6. The Reshetikhin-Turaev representation as a Fourier transform for non-abelian theta functions

6.1. **The Reshetikhin-Turaev representation of the mapping class group of the torus.** In this section we deduce the existence of the Reshetikhin-Turaev projective representation of the mapping class group of the torus from quantum mechanical considerations, and we compute it explicitly.

There is an action of the mapping class group of the torus on the ring of functions on the pillow case, given by

$$h \cdot f(A) = f(h_*^{-1}A),$$

where $h_*A$ denotes the image of the connection $A$ through $h$. In particular the Wilson line of a curve $\gamma$ is mapped to the Wilson line of the curve $h(\gamma)$. The action of the mapping class group on functions on the pillow case induces an action on the quantum observables by

$$h \cdot \mathrm{Op}(f(A)) = \mathrm{Op}(f(h_*^{-1}A)),$$

which for Wilson lines is

$$h \cdot \mathrm{Op}(W_\gamma) = \mathrm{Op}(W_{h(\gamma)}).$$

**Theorem 6.1.** *There exists a projective representation of the mapping class group of the torus that satisfies the exact Egorov identity*

$$Op(W_{h(\gamma)}) = \rho(h)\, Op(W_\gamma)\rho(h)^{-1}$$

*with the quantum group quantization of Wilson lines. Moreover, $\rho(h)$ is unique up to multiplication by a constant.*

*Proof.* We follow the first proof to Theorem 3.14. The bijective map $L \to h(L)$ on the set of isotopy classes of framed links in the cylinder over the torus induces an automorphism of the free $\mathbb{C}[t,t^{-1}]$-module with basis these isotopy classes of links. Because this map leaves invariant the ideal defined by the skein relations (for crossings and for the $r-1$st Jones-Wenzl idempotent), it defines an automorphism $\Phi : \widetilde{RT}_t(\mathbb{T}^2\times[0,1]) \to \widetilde{RT}_t(\mathbb{T}^2\times[0,1])$. The representation of $RT_t(\mathbb{T}^2\times[0,1])$ given by $V^j(\alpha) \to \mathrm{Op}(W_{h(\gamma)})V^j(\alpha)$ is an irreducible representation of $\widetilde{RT}_t(\mathbb{T}^2\times[0,1])$ which still maps $t$ to multiplication by $e^{\frac{i\pi}{2r}}$ and simple closed curves to self-adjoint operators. In view of Theorem 5.11 this representation is equivalent to the standard representation. This proves the existence of the map $\rho(h)$ that satisfies the exact Egorov identity with quantizations of Wilson lines. Schur's lemma implies that $\rho(h)$ is unique up to multiplication by a constant and that $\rho$ is a projective representation. Let us mention that a computational proof of uniqueness was given in [14]. □

By Theorem 5.11, for every $h$ in the mapping class group, the map $\rho(h)$ can be represented as multiplication by a skein $\mathcal{F}(h) \in \widetilde{RT}_t(\mathbb{T}^2\times[0,1])$. We want to find $\mathcal{F}(h)$ explicitly. We consider first the case $h = T$, the positive twist along the curve $(0,1)$ on the torus. Because the twist leaves the curve $(0,1)$ invariant,

$$\mathcal{F}(T)(0,1)V^k(\alpha) = (0,1)\mathcal{F}(T)V^k(\alpha), \qquad \text{for all } k.$$

And because the eigenspaces of $\mathrm{Op}(W_{(0,1)})$ are 1-dimensional, the linear operator defined by $\mathcal{F}(T)$ on the Hilbert space is a polynomial in $\mathrm{Op}(W_{(0,1)})$.

The polynomials $S_j(x)$, $0 \leq j \leq r-1$ form a basis for $\mathbb{C}[x]/S_{r-1}(x)$, so

$$\mathcal{F}(T) = \sum_{j=1}^{r-1} c_j S_{j-1}((0,1)), \quad c_j \in \mathbb{C}.$$

On the other hand,

$$(1,1)\mathcal{F}(T)V^k(\alpha) = \mathcal{F}(T)(1,0)V^k(\alpha). \tag{6.1}$$

Using (5.1) and the fact that $S_{j-1}((0,1))V^k(\alpha) = \frac{[jk]}{[k]}V^k(\alpha)$ for all $j$ and $k$, we rewrite (6.1) as

$$\sum_j c_j \frac{[jk]}{k} t^{-1}(t^{-2k}V^{k+1}(\alpha) + t^{2k}V^{k-1}(\alpha)$$
$$= \sum_j c_j \left( \frac{[j(k+1)]}{[k+1]} V^{k+1}(\alpha) + \frac{[j(k-1)]}{[k-1]} V^{k-1}(\alpha) \right).$$

Setting the coefficients of $V^{k+1}$ on both sides equal yields

$$\sum_{j=1}^{r-1} c_j [j(k+1)] = \frac{[k+1]}{[k]} t^{-2k-1} \sum_{j=1}^{r-1} c_j [jk].$$

If we denote $\sum_j c_j[j] = t^{-1}u$, we obtain the system of equations in the unknowns $c_j$, $j = 1, 2, \ldots, r-1$,

$$\sum_{j=1}^{r-1} [kj]c_j = [k]t^{-k^2}u, \quad k = 1, 2, \ldots, r-1.$$

Because for every integer $n$, $[n] = \sin \frac{n\pi}{r} / \sin \frac{\pi}{r}$, the coefficient matrix is a multiple of the matrix of the discrete sine transform. The square of the discrete sine transform is the identity map, so there is a constant $C$ such that

$$c_j = C \sum_k [jk][k]t^{-k^2}.$$

Standard results in the theory of Gauss sums [25] show that $\sum_k [jk][k]t^{-k^2} = C'[j]t^{j^2}$ where $C'$ is a constant that does not depend on $j$. We conclude that $\mathcal{F}(T)$ is a multiple of $\sum_{j=1}^{r-1}[j]t^{j^2}S_{j-1}((0,1))$. We make $\mathcal{F}(T)$ unitary by mulitplying it by $\eta = \sqrt{\frac{r}{2}} \sin \frac{\pi}{r} = (\sum_{j=1}^{r-1}[j]^2)^{-1/2}$, and for reasons that will become apparent in a moment we also multiply it by $t^{-1}$.

In view of this formula for $\mathcal{F}(t)$, and by analogy with the construction in §3.4, we define

$$\Omega_{SU(2)} = \eta \sum_{j=1}^{r-1} [j]V^j(\alpha) = \eta \sum_{j=1}^{r-1} S_{r-1}(\alpha) \in \widetilde{RT}_t(S^1 \times \mathbb{D}^2).$$

We should note that the coefficients [j] have a special significance, they are the quantum dimensions of the representations $V^j$.

For a framed link $L$, let $\Omega_{SU(2)}(L)$ be the skein obtained by replacing each component of $L$ by $\Omega_{SU(2)}$ such that the curve $(1,0)$ on the boundary of the solid torus is mapped to the framing. Then $\mathcal{F}(T)$ is the coloring by $\Omega_{SU(2)}$ of the surgery curve of the twist. This is true for any twist, and because on the one hand any element of the mapping class group is a product of twists, and on the other $\rho(h)$ is unique, we obtain a description of the map $\rho(h)$ in topological terms, as an the analogue of Theorem 3.15.

**Theorem 6.2.** *Let $h$ be an element of the mapping class group of the torus obtained by peforming surgery on a framed link $L_h$ in $\mathbb{T}^2 \times [0,1]$. The map $\rho(h) : \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2) \to \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is given by*

$$\rho(h)\beta = \Omega_{SU(2)}(L_h)\beta.$$

*Remark* 6.3. Everyone familiar with the Witten-Reshetikhin-Turaev theory has recognized the element $\Omega_{SU(2)}$, which is a fundamental building block of that theory. The exact Egorov identity in Theorem 6.1 implies that $\gamma$ can be slid along $L_h$ colored by $\Omega_{SU(2)}$. Once this is observed, it is natural to try slides over knots in general, and to deduce the Reshetikhin-Turaev formula for 3-manifold invariants [35].

The projective representation of the mapping class group from Proposition 6.2 is the Reshetikhin-Turaev representation. There is an alternative representation, with a more geometric flavor. Since odd and even theta functions are invariant under the Hermite-Jacobi action given by the discrete Fourier transform, for an element $h$ of the mapping class group of the torus given by (3.5) we have the action on the $\zeta_j^\tau$'s:

$$\rho_{HJ}(h)\zeta_j^\tau(z) = \exp\left(-\frac{\pi i N c z^2}{c\tau + d}\right)\zeta_j^{\tau'}\left(\frac{z}{c\tau+d}\right)$$

where $\tau' = \frac{a\tau+b}{c\tau+d}$.

We compare this representations to the Reshetikhin-Turaev representation. Modulo a multiplication by a positive normalization constant,

$$\begin{aligned}
\rho_{HJ}(S)\zeta_j^\tau(z) &= \rho(S)(\theta_j^\tau(z) - \theta_{-j}^\tau(z)) = \sum_{k=0}^{2r-1}\left(e^{-\frac{\pi i}{r}jk}\theta_k^\tau(z) - e^{\frac{\pi i}{r}jk}\theta_{-k}^\tau(z)\right)\\
&= \sum_{k=1}^{r-1}\left(e^{-\frac{\pi i}{r}jk}\theta_k^\tau(z) - e^{\frac{\pi i}{r}jk}\theta_{-k}^\tau(z)\right) + \sum_{k=1}^{r-1}\left(e^{-\frac{\pi i}{r}j(2r-k)}\theta_{2r-k}^\tau(z)\right.\\
&\left.-e^{\frac{\pi i}{r}j(2r-k)}\theta_k^\tau(z)\right) = 2\sum_{k=1}^{r-1}\left(e^{-\frac{\pi i}{r}jk} - e^{\frac{\pi i}{r}jk}\right)(\theta_k^\tau(z) - \theta_{-k}^\tau(z))\\
&= -4i\sum_{k=1}^{r-1}\sin\frac{\pi jk}{r}\zeta_k^\tau(z),
\end{aligned}$$

which is a scalar multiple of the discrete sine transform. To make it unitary, we multiply it by the positive constant $\eta = (\sum_{j=1}^{r-1}[j]^2)^{-1/2}$.

Also

$$\begin{aligned}\rho_{HJ}(T)\zeta_j^\tau(z) &= \rho(T)(\theta_j^\tau - \theta_{-j}^\tau(z)) = e^{\frac{\pi i}{r}j^2}(\theta_j^\tau(z) - \theta_{-j}^\tau(z))\\ &= e^{\frac{\pi i}{r}j^2}\zeta_j^\tau(z).\end{aligned}$$

The matrix of $\rho(S)$ defined via quantum groups has the $j,k$-entry equal to the Reshetikhin-Turaev invariant of the Hopf link with components colored by $V^j$ respectively $V^k$, multiplied by $\eta$. The map $\rho(T)$ introduces a positive twist on each basis element. These two maps are described schematically in Figure 21. These should be compared with the diagrams from §3.4.

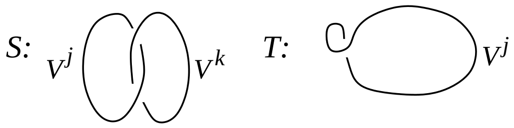


Figure 21

In both situations the $T$-matrices are unitary and we have

$$\rho(T) = e^{\frac{i\pi}{2r}}\rho_{HJ}(T).$$

This explains why for Weyl quantization one can pass to a $\mathbb{Z}_2$-extension of $SL(2,\mathbb{Z})$ to remove projectiveness and obtain a true representation, while in the Reshetikhin-Turaev theory one must take a full $\mathbb{Z}$-extension.

The $S$-matrices satisfy

$$\rho(S) = i\rho_{HJ}(S).$$

We have $\rho(S)^2 = Id$ but $\rho_{HJ}(S)^2 = -Id$. This is reflected in the fact that the map $z \to -z$ does not change the basis element $V^j(\alpha)$, while $\zeta_j(-z) = -\zeta_j(z)$. So for the Weyl quantization the rigid structure is oriented, while for the quantum group quantization it is not.

One should also note the equality $\Omega_{SU(2)} = \rho(S)\phi$.

The relationship with classical theta functions allows us to adapt a formula of Kač and Peterson [21] in order to obtain an explicit formula for the Reshetikhin-Turaev representation of the mapping class group of the torus.

**Theorem 6.4.** *Let*

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

*be an element of the mapping class group of the torus. Then there is a number* $c(2r,h) \in \mathbb{C}$ *such that*

$$\rho(h)\zeta_j^\tau(z) = c(2r,h)\sum_k e^{\frac{\pi i}{2r}(cdk^2+abj^2)}[bckj]\zeta_{aj+ck}^\tau(z)$$

*where the sum is taken over a family of $j \in \mathbb{Z}$ that give all representatives of the classes $cj$ modulo $2r\mathbb{Z}$ and the square brackets denote a quantized integer.*

*Proof.* Because $2r$ is an even integer, the group $SL_\theta(2,\mathbb{Z})$ is the whole $SL(2,\mathbb{Z})$. By Proposition 3.17 in [21], there is a constant $\nu(2r,h)$ such that

$$\theta_j^{\tau'}\left(\frac{z}{c\tau+d}\right) = \nu(2r,h)\exp\left(\frac{2\pi ircz^2}{c\tau+d}\right)\sum_k e^{\frac{i\pi}{2r}(cdk^2+2bckj+abj^2)}\theta^\tau_{aj+ck}(z)$$

with the same summation convention as in the statement of the theorem, and with $\tau' = \frac{a\tau+b}{c\tau+d}$. The map

$$\theta_j^\tau(z) \to \sum_k e^{\frac{i\pi}{2r}(cdk^2+2bckj+abj^2)}\theta^\tau_{aj+ck}(z)$$

is, up to multiplication by a constant, the unique map that satisfies the exact Egorov identity with the representation of the Heisenberg group. It follows that

$$\zeta_j^\tau(z) \to \sum_k e^{\frac{\pi i}{2r}(cdk^2+abj^2)}[bcjk]\zeta^\tau_{aj+ck}(z)$$

satisfies the exact Egorov identity with the Weyl quantization of the pillow case. There is a unique map with this property, up to multiplication by a constant. Hence the conclusion. □

6.2. **The structure of the reduced Reshetikhin-Turaev skein algebra of the cylinder over a surface.** As mentioned above, the element $\Omega_{SU(2)}$ is the fundamental building block in the construction of the Witten-Reshetikhin-Turaev quantum invariants of 3-manifolds. Here are some of its well known properties that will be used in the sequel.

**Proposition 6.5.** *If $M$ is an orientable compact 3-manifold, then in $\widetilde{RT}_t(M)$ the following hold:*

a) *if $O$ is the framed unknot in $M$, then $\Omega_{SU(2)}(O) = \eta^{-1}\emptyset$,*
b) *if $K$ and $K'$ are framed knots in $M$, then in $\widetilde{RT}_t(M)$,*

$$K \cup \Omega_{SU(2)}(K) = (K\#K') \cup \Omega_{SU(2)}(K'),$$

*(recall that $K\#K'$ denotes the slide of $K$ along $K'$, see §3.4),*

c) *for all skeins, the skein relation from Figure 22 holds.*

**Lemma 6.6.** *The quantum group quantizations of all Wilson lines on a surface generate the algebra of all linear operators on the Hilbert space of the quantization.*

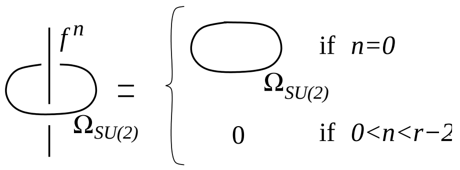


Figure 22

*Proof.* The case of the torus was addressed in Theorem 5.11. For a higher genus surface $\Sigma_g$, the conclusion follows if we show that every vector is cyclic for the algebra generated by quantized Wilson lines.

Recall the curves $\alpha_i$ from Figure 20. Given a knot in the handlebody $H_g$, we can talk about the linking number of this knot with one of the curves $\alpha_i$; just embed the handlebody in $S^3$ in standard position. We agree to take this with a positive sign. The linking number of a link $L$ in $H_g$ with the curve $\alpha_i$ is the sum of the linking numbers of the components. Associate to $L$ the number $d(L)$ obtained by summing these for all $i = 1, 2, \ldots, 3g-3$. Finally, for a skein $\sigma = \sum c_j L_j$, where $L_j$ are links and $c_j \in \mathbb{C}$, let $d(\sigma) = \max_j d(L_j)$. We claim that for each skein $\sigma$ that is not a multiple of the empty link, there is a skein $\sigma'$ such that $d(\sigma') < d(\sigma)$ and $\sigma'$ is in the cyclic representation generated by $\sigma$.

To this end write $\sigma$ in a basis of eigenvectors of the $3g-3$-tuple

$$(\mathrm{Op}(W_{\alpha_1}), \mathrm{Op}(W_{\alpha_2}), \ldots, \mathrm{Op}(W_{\alpha_{3g-3}}))$$

as $\sigma = \sum c_j e_j$. Because the spectral decomposition of this $3g-3$-tuple of operators has only 1-dimensional eigenspaces, each $e_j$ with nonzero coefficient is in the cyclic representation generated by $\sigma$. For each such $e_j$, $d(e_j) \leq d(\sigma)$. If one of these inequalities is sharp, then the claim is proved. If not, we show that if $e_k$ is not the empty link (i.e. the trivalent graph with all edges colored by $V^1$), then in the cyclic representation generated by $e_k$ there is a skein $\sigma'$ with $d(\sigma') < d(e_k)$.

After deleting all edges of $e_k$ colored by the trival representation $V^1$, the not necessarily connected graph obtained has an edge whose endpoints coincide, which is colored by some nontrivial representation $V^n$. Let $\beta$ be a framed simple closed curve on $\sigma_g$ that is isotopic to this edge and choose an $\alpha_i$ that intersects $\beta$ as shown in Figure 23.

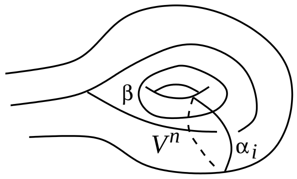


Figure 23

The recursive formula in Lemma 5.2 implies that $\mathrm{Op}(W_\beta)e_k$ is the sum of two skeins, $\sigma'$ that has the edge linking $\alpha_i$ colored by $V^{n-1}$ and $\sigma''$ that has the edge linking $\alpha_i$ colored by $V^{n+1}$. It is a standard fact that $\sigma'$ is an eigenvector of $\mathrm{Op}(W_{\alpha_i})$ with eigenvalue $[2n-2]$, while, if it is nonzero, then $\sigma''$ is an eigenvector of $\mathrm{Op}(W_{\alpha_i})$ with eigenvalue $[2n+2]$. We can therefore conclude that $\sigma'$ is in the cyclic representation generated by $e_k$, and therefore in the cyclic representation generated by $\sigma$.

Repeating, we eventually descend to the empty link. It remains to show that the empty link is cyclic. But this is obviously true, since each basis element can be represented as a framed link, hence is the image of a collection of nonintersecting framed simple closed curves on the boundary. This completes the proof. □

**Theorem 6.7.** *Given a genus $g$ surface $\Sigma_g$, $g \geq 1$, the representation of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ on $\widetilde{RT}_t(H_g)$ is faithful. Moreover, $\widetilde{RT}_t(\Sigma_g \times [0,1])$ is the algebra of all linear operators on $\widetilde{RT}_t(H_g)$.*

*Proof.* In view of Lemma 6.6, it suffices to show that the dimension of the vector space $\widetilde{RT}_t(\Sigma_g \times [0,1])$ equals the square of the dimension of $\widetilde{RT}_t(H_g)$. This fact, at least for the Kauffman bracket, goes back to unpublished work of J. Roberts. In the case of the Kauffman bracket, part of the proof can be found in [38] and the case where $r$ is an odd prime can be found in [13].

For two compact, orientable 3-dimensional manifolds $M$ and $N$, we denote by $M\#N$ their connected sum, obtained by removing a 3-dimensional ball from both $M$ and $N$ and then gluing the resulting manifolds along the newly obtained boundary spheres. In $M\#N$, the manifolds $M$ and $N$ are separated by a sphere $S_{sep}$. In particular, by turning the handlebody $H_g$ inside out, we see that $H_g\#H_g$ is an $S^3$ with two handlebodies removed.

**Lemma 6.8.** *Given the 3-dimensional manifolds $M$ and $N$, the map*

$$\widetilde{RT}_t(M) \otimes \widetilde{RT}_t(N) \to \widetilde{RT}_t(M\#N)$$

*defined by $(L, L') \to L \cup L'$, where $L$ and $L'$ are framed links in $M$ respectively $N$, is an isomorphism of vector spaces.*

*Proof.* The proof of this lemma was inspired by [34]. Any skein in $\widetilde{RT}_t(M\#N)$ can be written as $\sum_{j=1}^{k} c_j \sigma_j$, where $c_j$'s are complex coefficients and each $\sigma_j$ is a skein that intersects $S_{sep}$ along the $j$th Jones-Wenzl idempotent. Taking a trivial knot colored by $\Omega_{SU(2)}$ and sliding it over $S_{sep}$ we obtain, by using Proposition 6.5 c), the equality

$$\eta^{-1} \sum_{j=0}^{r-2} c_j \sigma_j = \eta^{-1} c_0.$$

Hence any skein is equal to a skein that is disjoint from $S_{sep}$. This proves that the map from the statement is an epimorphism. It is also a monomorphism because the skein module of a regular neighborhood of $S_{sep}$ is trivial. Hence it is an isomorphism. □

From here we continue as in [38]. The manifold $H_g \times [0,1]$ can be obtained from $H_g \# H_g$ by surgery on a $g$-component framed link $L_g$ as shown in Figure 24. Let $N_1 \subset H_g \# H_g$ be a regular neighborhood of $L_g$, which consists of $g$ solid tori. Let $N_2 \subset H_g \times [0,1]$ be the union of the $g$ surgery tori and $L'_g$ the framed link in $H_g \times [0,1]$ consisting of the cores of these tori. Then $H_g \# H_g$ can be obtained from $H_g \times [0,1]$ by performing surgery on $L'_g$.

Every skein in $H_g \# H_g$ respectively $H_g \times [0,1]$ can be isotoped as to miss $N_1$ respectively $N_2$. The homeomorphism $\phi : (H_g \# H_g)\backslash N_1 \to (H_g \times [0,1])\backslash N_2$ induces an isomorphism of skein modules

$$\phi : \widetilde{RT}_t((H_g \# H_g)\backslash N_1) \to \widetilde{RT}_t((H_g \times [0,1])\backslash N_2).$$

Unfortunately $\phi$ does not induce an isomorphism between $\widetilde{RT}_t(H_g \# H_g)$ and $\widetilde{RT}_t(H_g \times [0,1])$, it does not even give a well defined map because a skein in $H_g \# H_g$ can be pushed in many ways off $N_1$, while the images of these pushoffs through $\phi$ are not isotopic.

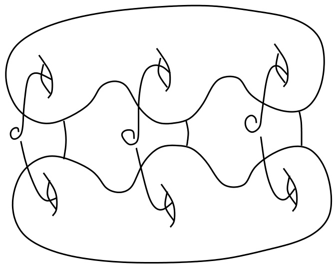

Figure 24

Sikora's idea was to define $F_1 : \widetilde{RT}_t(H_g \times [0,1]) \to \widetilde{RT}_t(H_g \# H_g)$ by $F_1(\sigma) = \phi(\sigma) \cup \Omega_{SU(2)}(L'_g)$. By Proposition 6.5 b), we are allowed to slide $\phi(\sigma)$ along $L'_g$, which implies that this map is well defined. Its inverse is $F_2(\sigma) = \phi^{-1}(\sigma) \cup \Omega_{SU(2)}(L_g)$. Indeed, to see that these maps are the inverse of each other, push $L'_g$ off $N_2$ in the direction of the framing of $L'_g$. Then each of the components of $\phi^{-1}(L'_g)$ is the meridian of the surgery torus, and it surrounds once the corresponding component of $L_g$. By Proposition 6.5 c), $\Omega_{SU(2)}(L_g) \cup \Omega_{SU(2)}(\phi^{-1}(L'_g)) = \emptyset \in \widetilde{RT}_t(H_g \# H_g)$. Hence $F_2 \circ F_1 = Id$. Similarly we check that $F_1 \circ F_2 = Id$, and the theorem is proved. $\square$

*Remark* 6.9. This result is the non-abelian analogue of Proposition 3.5.

6.3. **The quantization of Wilson lines determines the Reshetikhin-Turaev representation.** Like for the torus, there is an action of the mapping class group of a surface $\Sigma_g$ on the ring of regular functions on the moduli space $\mathcal{M}_g$ of flat $SU(2)$-connections on $\Sigma_g$ given by $h \cdot f(A) = f(h_*^{-1}A)$. The action of the mapping class group on regular functions on $\mathcal{M}_g$ induces an action on the quantum observables, which for Wilson lines is given by

$$h \cdot \mathrm{Op}(W_\gamma) = \mathrm{Op}(W_{h(\gamma)}).$$

**Theorem 6.10.** *There is a projective representation $\rho$ of the mapping class group of a closed surface that satisfies the exact Egorov identity*

$$Op(W_{h(\gamma)}) = \rho(h)\,Op(W_\gamma)\rho(h)^{-1}$$

*with the quantum group quantization of Wilson lines. Moreover, for every $h$, $\rho(h)$ is unique up to multiplication by a constant.*

*Proof.* We mimic the second proof to Theorem 3.14. The bijective map $L \to h(L)$ on the set of isotopy classes of framed links in the cylinder over the torus induces an automorphism of the free $\mathbb{C}[t, t^{-1}]$-module with basis these isotopy classes of links. Because the ideal defined by the skein relations is invariant under this map, the map defines an automorphism

$$\Phi : \widetilde{RT}_t(\Sigma_g \times [0,1]) \to \widetilde{RT}_t(\Sigma_g \times [0,1]).$$

By Proposition 6.7 the algebra $\widetilde{RT}_t(\Sigma_g \times [0,1])$ is the algebra of all linear operators on $\widetilde{RT}_t(H_g)$, so the automorphism $\Phi$ is inner [42]. This proves the existence of $\rho(h)$. The fact that $\rho$ is a representation and the uniqueness are consequences of Schur's lemma. $\square$

Each element of the mapping class group preserves the Atiyah-Bott symplectic form, so it induces a symplectomorphism of the $\mathcal{M}_g$. The theorem proves that the symplectomorphisms of $\mathcal{M}_g$ that arise from elements of the mapping class group can be quantized. Their quantization plays the role of the Fourier transform for non-abelian theta functions. Of course $\rho$ is (up to multiplication by constants) the Reshetikhin-Turaev representation. This theorem shows that all the information about the Reshetikhin-Turaev representation of the mapping class group is contained in the quantum group quantization of Wilson lines.

Once we know that the element $\Omega_{SU(2)}$ allows handle slides, discovering the formula for $\rho(h)$ is easy. However if we were able to prove a Stone-von Neumann theorem in higher genus, or to prove Theorem 6.7 without using links colored by $\Omega_{SU(2)}$, then the formula for $\rho(h)$ could be deduced like in the case of the torus. The idea is to write $\rho(h)$ as a composition of twists along nonseparating curves using the Lickorish twist theorem, then examine each twist separately.

If $\gamma$ is such a curve, with corresponding twist $T_\gamma$, and if $\rho(T_\gamma)$ is represented by a skein $\mathcal{F}(T_\gamma)$, then $\mathcal{F}(T_\gamma)$ commutes with all skeins that do not intersect $\gamma$. One can show that on each eigenspace of $\mathrm{Op}(W_\gamma)$ these skeins span the algebra of all linear operators. Hence the skeins commuting with $\mathcal{F}(T_\gamma)$ span the algebra of all operators that commute with $\mathrm{Op}(W_\gamma)$. Consequently, $\mathcal{F}(T_\gamma)$ is a polynomial in $\gamma$. Next we can restrict ourselves to a solid torus containing $\gamma$ and follow the steps from the computation in §6.1 to deduce the formula for $\mathcal{F}(T_\gamma)$.

As explained in [40] and [43], the projective representation of the mapping class group can be made into a true representation by passing to a $\mathbb{Z}$-extension of the mapping class group. Like in the case of classical theta functions, this extension can be defined in terms of either the Maslov index or the signature of 4-manifolds. To outline the similarity with classical theta functions we recall briefly this construction. The details can be found in [43].

Fix a rigid structure on the surface $\Sigma_g$ and consider the subspace $\mathbf{L}$ of $H_1(\Sigma_g,\mathbb{R})$ spanned by the curves that dissect the surface into pairs of pants. Then $\mathbf{L}$ is a Lagrangian subspace of $H_1(\Sigma_g,\mathbb{R})$ with respect to the intersection form. The composition of extended homeomorphisms is defined by

$$(h',n')\circ(h,n)=(h'\circ h, n+n'+\boldsymbol{\tau}(\mathbf{L},h(\mathbf{L}),h'\circ h(\mathbf{L})),$$

where $\boldsymbol{\tau}$ is the Maslov index with respect to the intersection pairing. We translate this into entirely topological language.

Let $h_1,h_2,h_3$ be the homeomorphisms of $\Sigma_g$ onto the bondaries of the 3-manifolds $N_1,N_2$, respectively $N_3$ such that $\mathbf{L}$, $h(\mathbf{L})$, respectively $h'\circ h(\mathbf{L})$ are the kernels of the morphisms induced in first homology. Consider the closed 3-manifolds

$$M=N_1\cup_{h_1}(\Sigma_g\times[0,1])\cup_{h_2}N_2 \text{ and } M'=N_2\cup_{h_2}(\Sigma_g\times[0,1])\cup_{h_3}N_3.$$

Let also $W$ and $W'$ be 4-dimensional manifolds such that $\partial W=M$, $\mathrm{sign}(W)=n$, $\partial W'=M'$, $\mathrm{sign}(W')=n'$, where sign denotes the signature of the 4-dimensional manifold. The elements of the $\mathbb{Z}$-extension of the mapping class group can be put in the form $(h,\mathrm{sign}(W))$. Wall's formula for the non-addititivity of the signature implies that the multiplication is given by

$$(h',\mathrm{sign}(W'))\circ(h,\mathrm{sign}(W))=(h'\circ h,\mathrm{sign}(W'\cup_{N_2}W)).$$

For completeness, we conclude our discussion with the proof that the bilinear pairing used in the definition of the quantum group quantization from §5.2 is nondegenerate. We first give a description of the inner product on the Hilbert space $\mathcal{H}_r(\Sigma_g)=\widetilde{RT}_t(H_g)$ by diagrams, following an idea in [13].

The handlebody $H_g$ has a natural orientation reversing symmetry $s$ that leaves its core invariant. Glue two copies of $H_g$ along their boundaries by the restriction of $s$ to the boudary to obtain a connected sum of $g$ copies of $S^1\times S^2$, denoted $\#_gS^1\times S^2$. This induces a pairing

$$\langle\ ,\ \rangle_0:\widetilde{RT}_t(H_g)\times\widetilde{RT}_t(H_g)\to\widetilde{RT}_t(\#_gS^1\times S^2).$$

The manifold $\#_gS^2\times S^2$ is obtained from $S^3$ by performing surgery on the trivial link with $g$ components. Identifying $\widetilde{RT}_t(\#_gS^1\times S^2)$ with $\widetilde{RT}_t(S^3)$ via Sikora's isomorphism as in the proof of Theorem 6.7, we deduce that the pairing $\langle\ ,\ \rangle_0$ takes values in $\mathbb{C}$.

The pairing of two basis elements is given by the Reshetikhin-Turaev invariant of a graph like the one in Figure 25. We argue on this figure, but one should keep in mind that there are many different graphs that can be the cores of the same handlebody. By Proposition 6.5 c), in order for this Reshetikhin-Turaev invariant to be nonzero, in each pair of edges linked by a circle colored by $\Omega_{SU(2)}$ the colors must be equal. This is because in order for the tensor product $V^{j_i} \otimes V^{k_i}$ to contain a copy of $V^1$, the dimensions of the two irreducible representations must be equal. Note also that because we work in $S^3$, the pairs of edges like the $V^{j_3}$ and $V^{k_3}$ are also linked by a circle colored by $\Omega_{SU(2)}$, namely the circle that links $V^{j_4}$ and $V^{k_4}$. In general, the edges coresponding to decomposition circles that do not disconnect the surface fall in this category.

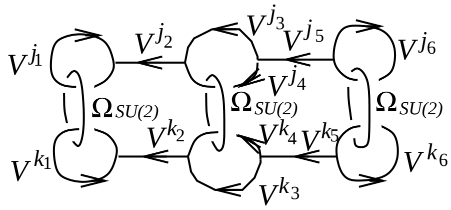


Figure 25

Let us examine next the pairs of edges that are not linked by surgery circles, such as those colored by $V^{j_2}$, $V^{k_2}$ in the figure. In general, the edges that come from decomposition circles that disconnect the surface fall in this category. Rotating the graph by 90° and evaluating the Reshetikhin-Turaev invariant by the rules we obtain a homomorphism $\mathbb{C} = V^1 \to V^{j_2} \otimes V^{k_2}$. This homomorphism is nonzero if and only if $j_2 = k_2$. We conclude that the pairing of two distinct basis elements is zero. On the other hand, computing the pairing of a basis element with itself we can trace a $V^1$ from the bottom to the top, and the value of the pairing is $\Omega_{SU(2)}(O) = \eta^{-g}$. Hence $\langle\ ,\ \rangle_0 = \eta^{-g}\langle\ ,\ \rangle$, where $\langle\ ,\ \rangle$ is the inner product.

The bilinear pairing $[\ ,\ ]$ from §5.2 is defined by gluing two copies of $H_g$ along an orientation reversing homeomorphism as to obtain $S^3$. The homeomorphism is of the form $s \circ h$, so $[e_i, e_j] = \langle e_i, \rho(h)e_j\rangle$. Because $\rho(h)$ is an automorphism of the Hilbert space of the quantization, the pairing is nondegenerate.

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