NONCOMMUTATIVE TRIGONOMETRY AND THE A-POLYNOMIAL OF THE TREFOIL KNOT

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1. Introduction

The noncommutative generalization of the A-polynomial of a knot of Cooper, Culler, Gillet, Long and Shalen [4] was introduced in [6] . This generalization consists of a finitely generated left ideal of polynomials in the quantum plane, the noncommutative A-ideal, and was defined based on Kauffman bracket skein modules, by deforming the ideal generated by the A-polynomial with respect to a parameter. The deformation was possible because of the relationship between the skein module with the variable t of the Kauffman bracket evaluated at -1 and the $SL(2,C)$-character variety of the fundamental group, which was explained in [2]. The purpose of the present paper is to compute the noncommutative A-ideal for the left- and right-handed trefoil knots. As it will be seen below, this reduces to trigonometric operations in the noncommutative torus, the main device used being the product-to-sum formula for noncommutative cosines.

The computation of the A-ideal relies on understanding how the skein algebra of the cylinder over a torus acts on the skein module of the knot complement. For the left-handed trefoil the action is described in Theorem 1. In Theorems 2 and 4 we describe the peripheral ideal of the left-handed trefoil knot, which is the ideal that anihilates the empty skein. Finally, Theorem 3 lists three generators for the noncommutative A-ideal of the left-handed trefoil, when t is not an eighth root of unity. The case $t=-1$ makes the object of Theorem 5. In the last section, we list the analogous results for the right-handed trefoil. The proofs of the results are sketchy, since we preferred to skip long routine computations, and focussed only on main ideas.

2. Preliminary facts

Througout this paper t will either denote the variable of the ring of Laurent polynomials $\mathbb{C}[t,t^{-1}]$, or a fixed complex number, with the distinction specified whenever necessary. A *framed link* in an orientable manifold M

is a disjoint union of embedded annuli. In the case where the manifold is the cylinder over the torus, framed links will be identified with curves, using the convention that the annulus is parallel to the surface. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold M, including the empty link. Consider the free module over $\mathbb{C}[t,t^{-1}]$ with basis $\mathcal{L}$, and factor it by the smallest submodule containing all expressions of the form  $-\,t$  $-\,t^{-1}$  and  $+\,t^2+t^{-2}$, where the links in each expression are identical except in a ball in which they look like depicted. This quotient is denoted by $K_t(M)$ and is called the Kauffman bracket skein module of the manifold [8]. In the case of a cylinder over a surface, the skein module has an algebra structure induced by the operation of gluing one cylinder on top of the other. The operation of gluing the cylinder over ∂M to M induces a $K_t(\partial M\times I)$-left module structure on $K_t(M)$. We denote by $*$ the multiplication in $K_t(\partial M\times I)$ and by $\cdot$ the left action of this algebra on $K_t(M)$.

Let us discuss in more detail two structure results about the Kauffman bracket skein algebra of the torus and the Kauffman bracket skein module of the complement of the trefoil knot. For this we need the Chebyshev polynomials of first and second type. The Chebyshev polynomials of first type are T_n, $n\geq 0$, defined by $T_0(x)=2$, $T_1(x)=x$, and $T_{n+1}(x)=xT_n-T_{n-1}$. Recall that they arise when expressing $2\cos n\alpha$ as a function of $2\cos\alpha$. The Chebyshev polynomials of second type S_n, $n\geq 0$ satisfy the same reccurence relation, but with $S_0(x)=1$ and $S_1(x)=x$. They arise when writing $\sin{(n+1)\alpha}/\sin\alpha$ as a function of $\cos\alpha$, and satisfy $S_n=T_n+T_{n-2}+T_{n-4}+\cdots$, where the sum ends in a 1 (not 2) if n is even. Extend both polynomials by the reccurence relation to all indices $n\in\mathbb{Z}$. Note that $T_{-n}=T_n$, while $S_{-n}=-S_{n-2}$.

For a link γ in a skein module we will denote by γ^n the link consisting of n parallel copies of γ, and extend the notation to polynomials. For a pair of integers (p,q), we denote by $(p,q)_T$ the element of the Kauffman bracket skein module of the cylinder over the torus defined in the following way. Let n be the greatest common divisor of p and q, $p' = p/n$, $q' = q/n$ and let (p', q') be the simple closed curve of slope p'/q' on the torus. Then $(p, q)_T = T_n((p', q'))$, i.e. the formal polynomial in $K_t(\mathbb{T}^2 \times I)$ obtained by replacing the variable of the Chebyshev polynomial by the (p', q')-curve on the torus. Define analogously $(p, q)_{JW} = S_n((p', q'))$. The index is motivated by the fact that $S_n((p'q'))$ is the (p', q')-curve colored by the n-th Jones-Wenzl idempotent.

As an $\mathbb{C}[t, t^{-1}]$-module, $K_t(\mathbb{T}^2 \times I)$ is spanned by the elements $(p, q)_T$, $p \geq 0$. It was proved in [5] that $K_t(\mathbb{T}^2 \times I)$ is canonically isomorphic to the subalgebra of the noncommutative torus [3] spanned by noncommutative cosines. Here the torus is deformed with respect to the parameter t. The isomorphism is a consequence of the *product-to-sum* formula for skeins in $K_t(\mathbb{T}^2 \times I)$ (which is actually the product-to-sum formula for noncommutative cosines):

$$(p, q)_T * (r, s)_T = t^{\left|{pq \atop rs}\right|}(p + r, q + s)_T + t^{-\left|{pq \atop rs}\right|}(p - r, q - s)_T.$$

This formula will be essential for all computations in the paper.

It was proved in [1] that the Kauffman bracket skein module of the complement of the trefoil (whether left or right), is the free module generated by x^n and $x^n y$, where $n \geq 0$, and x and y are the curves shown in Fig. 1. As we will see below, it is more useful to consider a different basis of

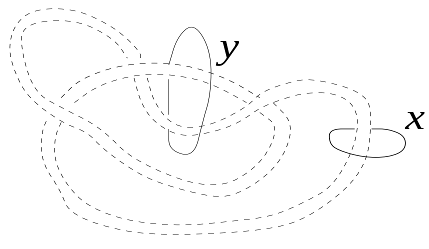

Figure 1.

this module, namely $S_n(x)$ and $S_n(x)y$, $n \geq 0$. This recalls the case of the unknot [5], where again the Chebyshev polynomials of second type helped produce concise formulas.

If M is the complement of a knot K, then one can introduce a noncommutative generalization of the A-polynomial of K defined in [4], in the following way (see [6]). Denote by π the map between skein modules induced by the inclusion $(\partial M \times I) \subset M$ and let $I_t(K)$ be the kernel of π. The noncommutative A-ideal of K is defined to be the left ideal obtained by extending $I_t(K)$ to the subalgebra of the noncommutative torus consisting of trigonometric polynomials, and then intersecting it with $\mathbb{C}_t[l, m]$ (i.e. by contracting to the quantum plane). Recall that the algebra of noncommutative trigonometric polynomials is $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$, while $\mathbb{C}_t[l, m]$ is usually called the quantum plane [7], where in both l and m satisfy $lm = t^2 ml$. The classical A-polynomial is obtained by working with $t = -1$, replacing l and m by $-l$ and $-m$ respectively, and taking the generator of the radical of the part of the ideal that has Krull dimension equal to 1. The fact that this is indeed the A-polynomial follows from Bullock's [2] characterization of the Kauffman bracket skein module at $t = -1$ as the affine $SL(2, C)$-character variety ring of the fundamental group (see also [9]), together with the fact that $K_{-1}(\mathbb{T}^2 \times I)$ has no nilpotents. It is important to stress out the fact that, as it usually happens in the theory of skein modules, the A-ideal for $t = -1$ is not obtained by simply substituting in formulas the variable t by its particular value [6]. Rather the computation has to be done separately for particular values of t.

With the above notation, the elements $e_{p,q} = t^{-pq} l^p m^q$ in $\mathbb{C}_t[l, l^{-1}, m, m^{-1}]$ are called noncommutative exponentials, the elements $\cos_t(p, q) = (e_{p,q} + e_{-p,-q})/2$ are called noncommutative cosines, and the elements $\sin_t(p.q) = (e_{p,q} - e_{-p,-q})/2$ are called noncommutative sines. The morphism from $K_t(\mathbb{T}^2 \times I)$ to the noncommutative torus maps $(p, q)_T$ to $2\cos_t(p, q)$, and under this identification the computations become trigonometric manipulations of $\cos_t(p, q)$ and $\sin_t(np, nq)/\sin_t(p, q)$, hence the title of the paper.

3. The $K_t(\mathbb{T}^2 \times I)$-module structure of the skein module of the complement of the left-handed trefoil knot

In this section K will denote the left-handed trefoil knot and M the complement of a regular neighborhood of this knot.

Lemma 1. *In the Kauffman bracket skein module of the complement of the trefoil knot, the following equality holds*

$$y^2 = -t^2 S_2(x)y - t^4 S_2(x) + S_0(x).$$

Proof. The element y^2 can be obtained by the Kauffman bracket skein relation for the unknot like in Fig. 2. To compute the skein on the right side, one can proceed like in Fig. 3. It is not hard to see that in the end one obtains $-t^4y + t^4x^2y + t^6x^2$. Since the unknot with a positive twist from Fig. 1 is equal to $(-t^3)(-t^2-t^{-2})$, we obtain $y^2 = 1+t^4+t^2y-t^2x^2y-t^4x^2$. Grouping these terms appropriately we obtain the formula from the statement. $\square$

Lemma 2. *In the Kauffman bracket skein module of the complement of the trefoil knot, the following equality holds*

$$y^3 = t^4 S_4(x)y + 2S_0(x)y + t^6 S_4(x) + t^{10} S_0(x).$$

Proof. To compute y^3 we use the relation from Fig. 4. It is not hard to see that the skein on the left is equal to $(t + t^5)y$, while the skein on the right can be transformed like in Fig. 5.

In the last line of Fig. 5, the second skein is equal to $t^4x^2y^2$ and the third to t^6x^2y. Since we already computed y^2, it only remains to compute the remaining one skein. To this end we write a Kauffman bracket skein relation as in Fig. 6. The first diagram from the right side can be easily transformed like in Fig. 7, and applying the skein relation we obtain that this is equal to $-t^{-2}x^2 - t^{-4}y$. For the diagram on the left we use the computation from Fig. 8 to conclude that is equal to $t^7 + t^3 - 2t^3x^2 - tx^2y$. Hence the missing skein is equal to $t^8 + t^4 - 2t^4x^2 + x^2 + t^{-2}y - t^4x^2y$. In the end we deduce

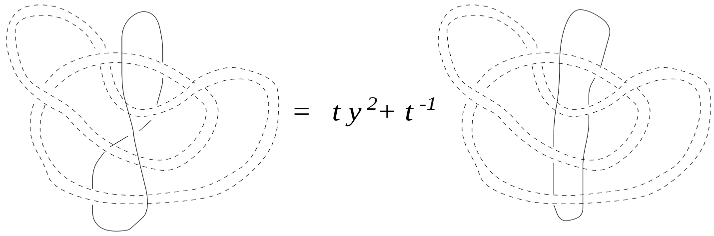

Figure 2.

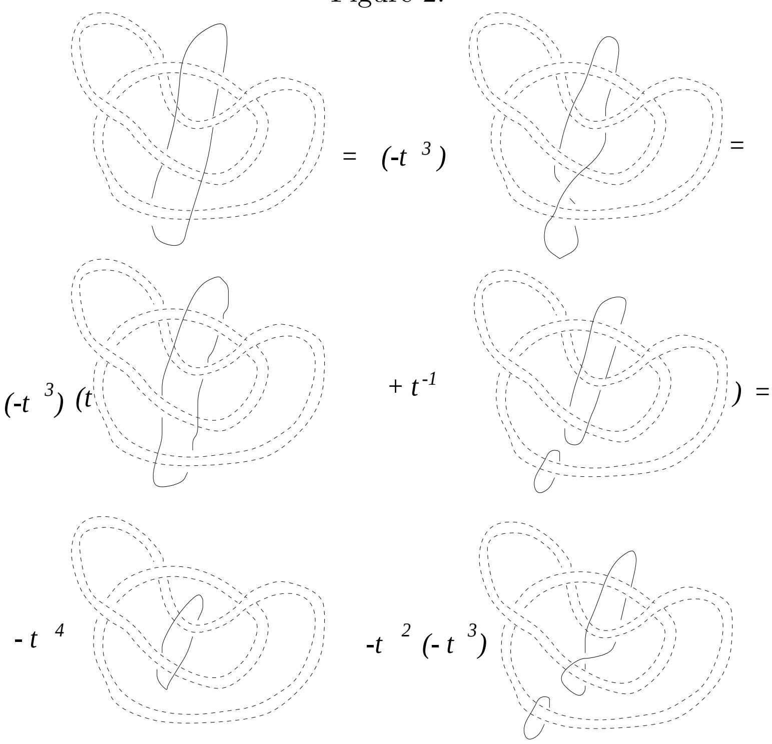

Figure 3.

that $y^3 = t^{10} + t^6 - 3t^6x^2 + t^6x^4 + 2y + t^4y - 2t^4x^2y - t^4x^2y + t^4x^4y$, and by grouping terms we obtain the formula from the statement.

□

Lemma 3. *The following equalities hold*

$$a).\ \pi((1,0)_T) = t^6S_6(x) - t^2S_0(x) + t^4S_4(x)y - S_0(x)y.$$

$$b).\ \pi((1,-1)_T) = t^5S_5(x) + t^3S_3(x)y.$$

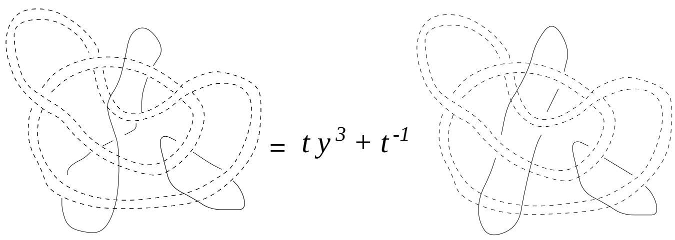

Figure 4.

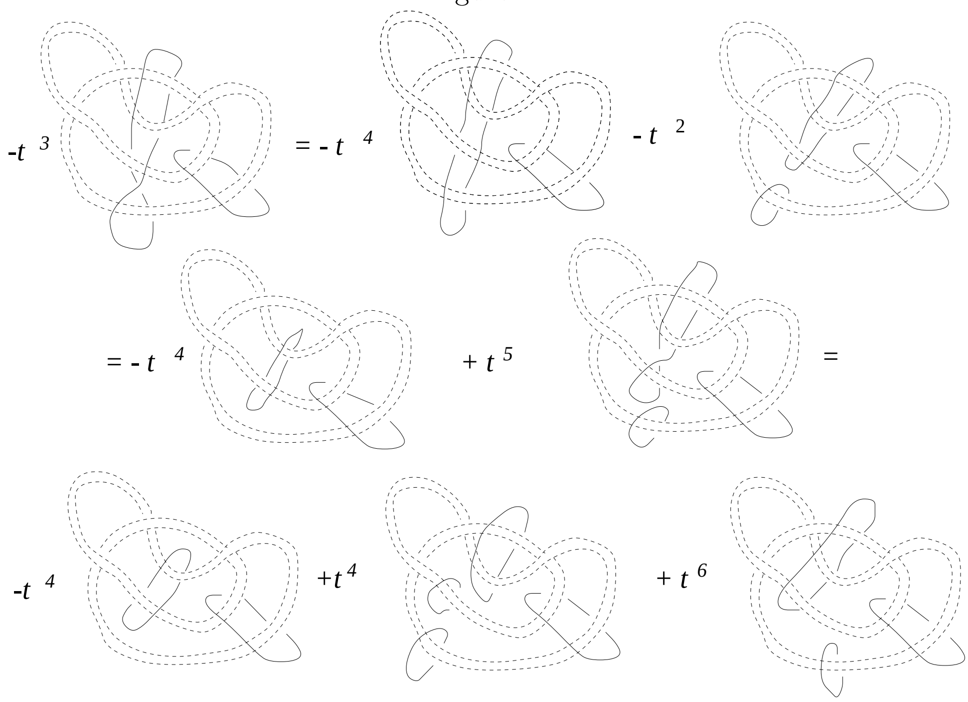

Figure 5.

Proof. Using the Kauffman bracket skein relation we obtain the expansion from Fig. 9. Using the computation from Fig. 7 we see that the first diagram is equal to $(-t^{-2}m^2 - t^{-4}y)^3$. The second and the fourth diagram are equal to $(-t^2 - t^{-2})^2$ and $(-t^2 - t^{-2})$, respectively. It remains to find the value of the third diagram. The computations from Fig. 10 and Fig. 11, combined with that of Fig. 7, show that the desired value of diagram is $1 + t^{-4} + t^{-6}x^2y + t^{-8}x^4 + t^{-10}y$. On the other hand, the first skein from Fig. 7 is almost the image in the skein module of the knot complement of the $(1, 0)$ curve on the torus. To obtain the image of the curve multiply this

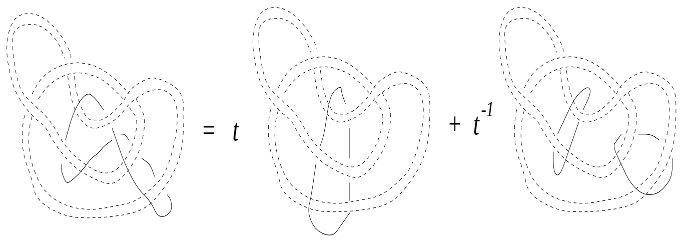

Figure 6.

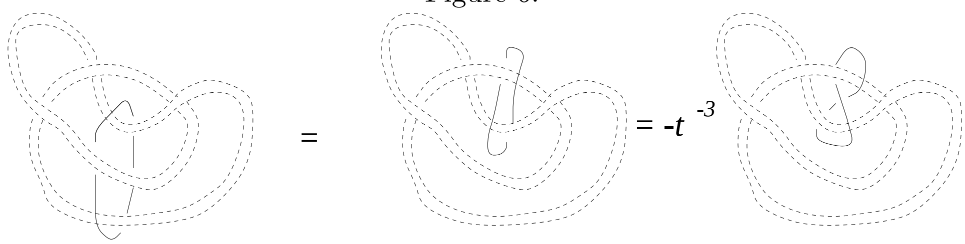

Figure 7.

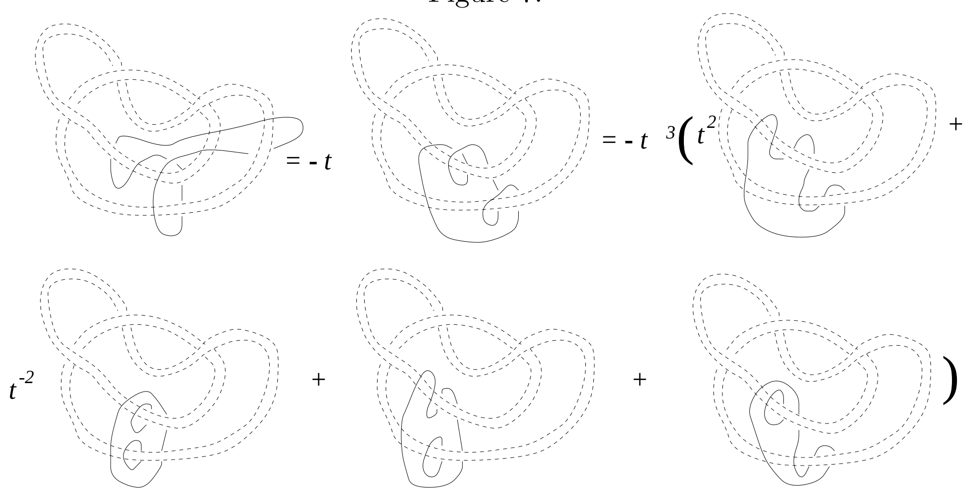

Figure 8.

skein by the framing factor of $-t^9$, induced by the twistings of the $(1,0)$ curve around the torus, and a) follows. The proof of b). is analogous, even simpler, and is left to the reader.

□

Lemma 4. *For any $q \in \mathbb{Z}$ one has*

$$\pi((1,q)_T) = t^{q+6}S_{q+6}(x) - t^{q+2}S_q(x) + t^{q+4}S_{q+4}(x)y - t^qS_q(x)y.$$

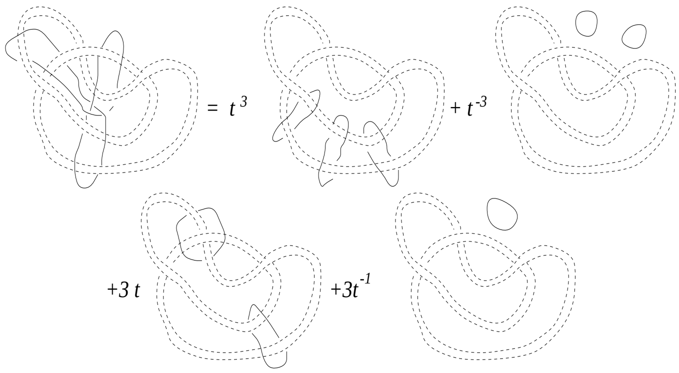

Figure 9.

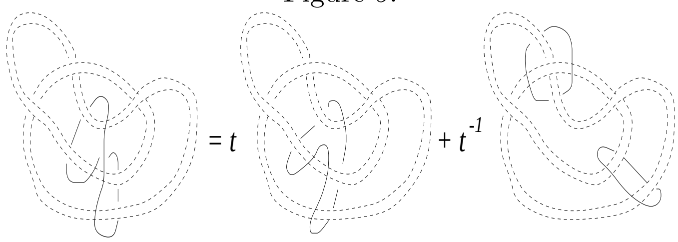

Figure 10.

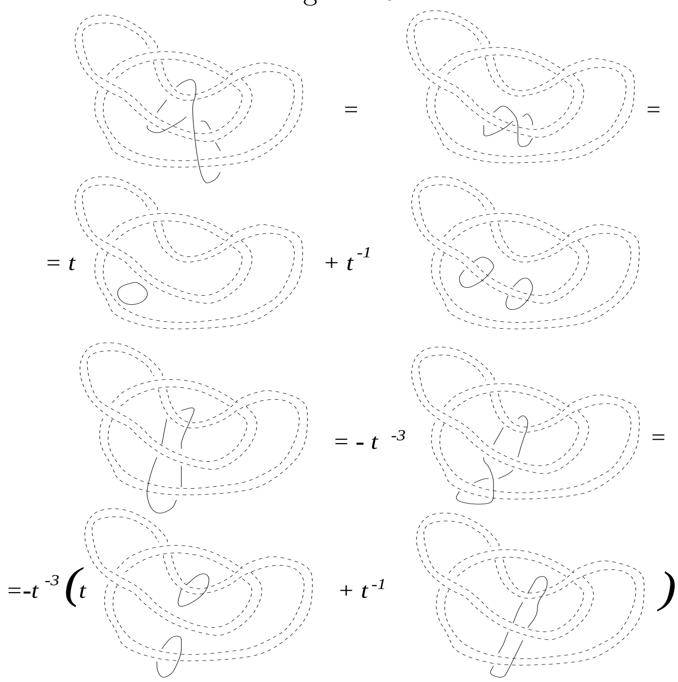

Figure 11.

Proof. The proof is by induction on q, based on Lemma 3 and the *product-to-sum* formula:

$$(1,q)_T * (0,1)_T = t(1,q+1)_T + t^{-1}(1,q-1)_T.$$

Here we use the fact that $\pi((0,1)_T) = x$. □

An element of the skein module of a manifold is called peripheral if it is the image through the inclusion map of an element of the skein algebra of the cylinder over the boundary. A skein module is peripheral if all of its elements are peripheral. As a consequence of Lemma 4 we obtain:

Lemma 5. *In $K_t(M)$, one has*

$$(t^4 - t^{-4})y = \pi\left(t^4(1,-4)_T - t^{-2}(1,-2)_T + t^2(0,4)_T - t^6(0,2)_T - t^6 + t^{-2}\right).$$

So if t is not an eighth root of unity, then y is peripheral, hence the skein module $K_t(M)$ is peripheral.

Let m, n be two integers. Denote by $x_{n,m}$ the skein in the mapping cylinder of a pair of pants given in Fig. 12. Here if m or n is negative, the kinks are in the other direction. Recall that the mapping cylinder of a pair of

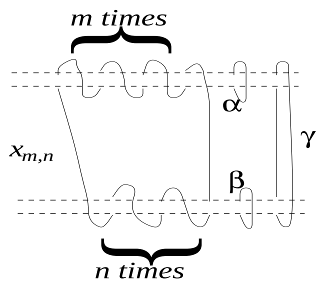

Figure 12.

pants is the free module with basis $\alpha^m\beta^n\gamma^p$, $m,n,p \geq 0$ where the curve α, β and γ are shown in Fig. 12.

Lemma 6.

$$\begin{aligned} x_{m,n} &= (-t^{-2})^{m+n-1} S_m(\alpha)S_n(\beta) - (t^{-2})^{m+n} S_{m-1}(\alpha)S_{n-1}(\beta)\gamma + \\ &\quad (-t^{-2})^{n+m+1} S_{m-2}(\alpha)S_{n-2}(\beta). \end{aligned}$$

Proof. Sliding the strand one might produce a kink which, when resolved via the Kauffman bracket skein relation, produces two configurations with less complexity. Hence an induction on both m and n works to prove the formula. □

Lemma 7. *For $q \in \mathbb{Z}$ one has*

$$(1,q)_T \cdot y = t^q S_{q-2}(x) - t^{q+8} S_{q+6}(x) + t^{q-2} S_{q-2}(x) y - t^{q+6} S_{q+4}(x) y.$$

Proof. Note that we have the equality from Fig. 13. If we denote by u_{n-3} the skein from Fig. 14 multiplied by $(-t^3)^n \cdot (-t^9)$, then the image of $(1,p)_T \cdot y$ in $K_t(M)$ is obtained from the sum in Fig. 13 by capping each term of the sum with u_{p-3}. Let us denote by $a_1, a_2, \cdots, a_{12}$ be the twelve skeins of the sum respectively, and let us denote the capping operation by $\circ$. Then one can easily see that

$$x_1 \circ u_n = \pi((1,n)_T)$$
$$x_2 \circ u_n = (-t^{-3}) x \pi((1,n+1)_T)$$
$$x_3 \circ u_n = (-t^3) x \pi((1,n-1)_T).$$

Also using Lemma 6 we obtain

$$x_4 \circ u_n = t^{n+6} x^2 S_{n+6}(x) + t^{n+4} x S_{n+5}(x) y$$
$$x_5 \circ u_n = t^{n+6} x S_2(x) S_{n+5}(x) + t^{n+4} x^2 S_{n+4}(x) + t^{n+2} x S_{n+3}(x)$$
$$x_6 \circ u_n = t^{n+10} x S_{n+5}(x) - t^{n+6} x S_{n+3}(x)$$
$$x_7 \circ u_n = t^{n+10} x^2 S_{n+4}(x) + t^{n+8} x S_{n+3}(x) y$$
$$x_8 \circ u_n = -t^{n+8} x^3 S_{n+5}(x) - t^{n+6} x^2 S_{n+4}(x) y$$
$$x_9 \circ u_n = -t^{n+8} x S_{n+5}(x) - t^{n+6} S_{n+4}(x) y$$
$$x_{10} \circ u_n = -t^{n+8} x S_{n+5}(x) - t^{n+6} S_{n+4}(x) y$$
$$x_{11} \circ u_n = -t^{n+8} S_{n+6}(x) + t^{n+4} S_{n+4}(x)$$
$$x_{12} \circ u_n = -t^{n+8} S_2(x) S_{n+4}(x) - t^{n+6} x S_{n+3}(x) y - t^{n+4} S_{n+2}(x).$$

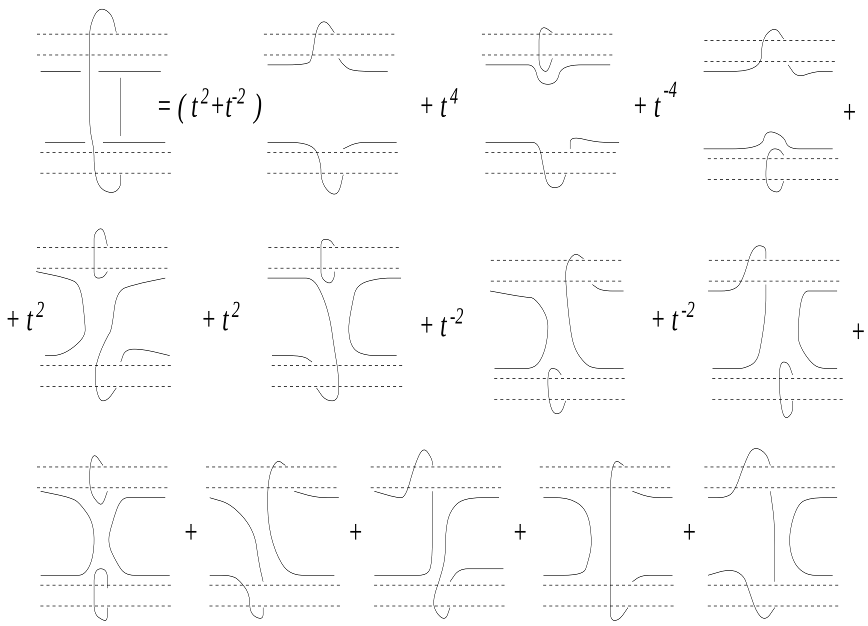

Figure 13.

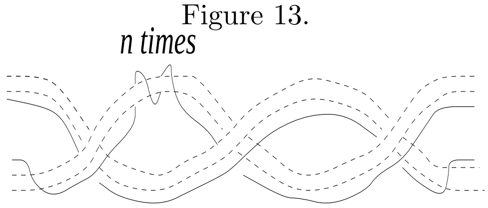

Figure 14.

On the other hand, $xS_n(x) = S_{n+1}(x) + S_{n-1}(x)$ and

$$x\pi((p,q)_T) = \pi((p,q)_T * (0,1)_T) = t^p\pi((p,q+1)_T) + t^{-p}\pi((p,q-1)_T).$$

Using these observations and Lemma 4, we see how to write all right-hand sides in the above relations in the basis of $K_t(M)$. A lengthy computation with many cancellations produces the formula from the statement. $\square$

Define ϵ_k to be equal to 1 if k is odd, and 0 if k is even.

Theorem 1. *If $p > 0$ then*

$$\pi((p,q)_T) = t^{6p^2+pq}S_{q+6p}(x) + t^{6p^2+pq-2}S_{q+6p-2}(x)y +$$
$$\sum_{k=1}^{2p-1}(-1)^{[\frac{k+1}{2}]}t^{6p^2-([\frac{k}{2}]+1)(6[\frac{k}{2}]+4\epsilon_k)+pq}S_{q+6p-3k+\epsilon_k-2}(x)(1+t^{-2}y).$$

The action of $K_t(\mathbb{T}^2\times I)$ on $K_t(M)$ is determined by

$$
\begin{aligned}
&(p,q)_T\cdot y = -t^{6p^2+pq+2}S_{q+6p}(x) - t^{6p^2+pq}S_{q+6p-2}(x)y + \\
&\sum_{k=1}^{2p-2}(-1)^{\left[\frac{k-1}{2}\right]}t^{6p^2-(\left[\frac{k}{2}\right]+1)(6\left[\frac{k}{2}\right]+4\epsilon_k)+pq}S_{q+6p-3k+\epsilon_k-2}(x)(t^2+y) + \\
&(-1)^{p-1}S_q(x)t^{-2q+pq}(t^2+y).
\end{aligned}
$$

Proof. The two equations are proved simultaneously by induction. By Lemma 4 and Lemma 7, they are true for $p=1$, and the inductive step is based on the *product-to-sum* formula

$$(1,0)_T*(p,q)_T = t^p(p+1,q)_T + t^{-p}(p-1,q)_T.$$

and the fact that, by Lemma 7,

$$(1,0)_T\cdot y = -1 - t^8S_6(x) - t^{-2}y - t^6S_4(x)y.$$

□

4. The noncommutative A-ideal of the left-handed trefoil

The considerations below hold if t is a the polynomial variable or if t is a complex number, which is not an eighth root of unity. For $m\geq 1$, denote by $I_t^m(K)$ the intersection of $I_t(K)$ with the linear span of $(p,q)_T$, $0\leq p\leq m$, $q\in\mathbb{Z}$.

Lemma 8. *Every element in $I_t^1(K)$ is of the form $p((0,1))*\tau$, where*

$$\tau = (1,-5)_T - t^{-8}(1,-1)_T + t^{-3}(0,5)_T - t(0,1)_T.$$

and p is a polynomial with complex coefficients .

Proof. There is a part of $I_t^1(K)$ that arises via Lemmas 4 and 5, namely that spanned by the elements

$$
\begin{aligned}
\phi_q \quad = \quad & (t^4-t^{-4})((1,q)_T - t^{q+6}(0,q+6)_{JW} + t^{q+2}(0,q)_{JW}) - \\
& (t^{q+4}(0,q+4)_{JW} - t^q(0,q)_{JW})*(t^4(1,-4)_T - t^{-2}(1,-2)_T + \\
& +t^2(0,4)_T - t^6(0,2)_T - t^6 + t^{-2}).
\end{aligned}
$$

For example, for $q = -5$ we obtain τ, and for $q = -6$, we get the element in the kernel

$$(1,-6)_T + t^{-2}(1,-4)_T - t^{-8}(1,-2)_T - t^{-10}(1,0)_T + t^{-4}(0,6)_T +$$
$$t^{-4}(0,4)_T - (0,2)_T - 2,$$

and it is not hard to see that it is equal to $(0,1)_T * \tau$. On the other hand, using the *product-to-sum* formula, we can write $(0,1)_T * \phi_q = t^{-1}\phi_{q+1} + t\phi_{q-1}$ so an induction in both direction shows that all elements ϕ_q are of the form $p((0,1)_T) * \tau$. Note also that if $q \geq 0$, then the terms with extreme second coordinate that appear in the formula of ϕ_q are $t^{-2}(1, q+2)_T$ and $t^{2q+2}(1, -q-8)_T$.

Among all elements in the part of $I_t^1(K)$ not spanned by ϕ_q's, choose one, ψ, such that the maximum of $|q+1|$ with $(1,q)_T$ appearing in the writing of ψ, is minimal. Let $a(1, q+2)_T + (b(1, -q-4)_T$ be the part where the maximum is attained (here $q \geq 0$). Note that in $K_t(M)$, $\pi(\psi)$ had the coordinate of $S_{q+6}(x)y$ equal to $at^{q+6} + bt^{q-8}$, hence a and b are in the same proportion as those of ϕ_q. But then by subtracting a multiple of q, we can eliminate them, contradicting the minimality of ψ. This shows that there are no other elements in $I_t^1(K)$, and the lemma is proved. □

Theorem 2. *The ideal $I_t(K)$ is generated by*

$$(1,-5)_T - t^{-8}(1,-1)_T + t^{-3}(0,5)_T - t(0,1)_T,$$
$$(2,-6)_T - (t^6 + t^{-6})(1,0)_T + (t^4 + t^{-4})(1,-6)_T + (0,6)_T - 2(t^4 + t^{-4}),$$
$$(2,-7)_T + t^{-5}(1,-7)_T + (t^{-5} - t^{-1})(1,-3)_T - t^5(1,-1)_T +$$
$$+(t^2 - t^{-2})(0,3)_T - t^{-6}(0,1)_T.$$

Proof. Let us denote by $\mathcal{J}$ the ideal generated by the three skeins from the statement. The fact that $\mathcal{J} \subset I_t(K)$ follows from from Lemmas 4, 5 and 8. Let us prove the equality. First, we check that there are no other generators in $I_t^2(K)$. By Lemma 8, we only need to check the part of $I_t^2(K)$ of elements that actually contain $(2,q)_T$'s. But using the *product-to-sum* formula

$$(2,q)_T * (0,1)_T = t^2(2,q+1)_T + t^{-2}(2,q-1)_T$$

we can prove inductively that each such element in $I_t^2(K)$ can be reduced modulo $\mathcal{J}$ to one in $I_t^1(K)$, and the latter is in $\mathcal{J}$ by Lemma 8.

We will prove that $I_t^m(K)$ is contained in $\mathcal{J}$ by induction on m. First note that $I_t(K)$ is spanned by the elements that arise by using the first formula from Theorem 1, in which we replace x by $(0,1)_T$ and y by its value given in Lemma 5. Indeed, the right-hand side of the formulas produces only elements in $(0,n)_T$, $(1,n)_T$, we can eliminate all $(p,q)_T$'s that appear in the writing of an elements in the kernel using them and reduce such an element modulo $\mathcal{J}$ to an element that only contains $(p,q)_T$ with $p \leq 2$, and then use the first part of the proof.

On the other hand, using the *product-to-sum* formula

$$(1,0)_t * (p,q)_T = t^q(p+1,q)_T + t^{-q}(p-1,q)_T$$

we see that inductively we can reduce any element in $I_t^m(K)$ to one in $I_t^2(K)$ modulo $\mathcal{J}$, and the conclusion follows. □

Theorem 3. *The ideal $\mathcal{A}_t(K)$ is generated by*

$$[m^4(l+t^{10}) - t^{-4}(l+t^2)](l-t^6m^6),$$
$$(l+t^{24})(l+t^{10})(l+t^2)(l-t^6m^6),$$
$$(m^2-t^{-22})(l+t^{10})(l+t^2)(l-t^6m^6).$$

Proof. The first generator of $I_t(K)$ given in Theorem 2 gives rise to the element

$$t^5(lm^{-5}+l^{-1}m^5) - t^{-7}(lm^{-1}+l^{-1}m) + t^{-3}(m^5+m^{-5}) - t(m+m^{-1})$$

in the subring of the noncommutative torus consisting of trigonometric polynomials. After multiplying by t^{11} this element contracts to

$$t^{-4}l^2 + t^{16}m^{10} - t^{-16}l^2m^4 - t^4m^6 + t^{-2}lm^{10} + t^{-2}l - t^2lm^6 - t^2lm^4$$

in the quantum plane. It is not hard to see that it factors as $[m^4(l+t^{10}) - t^{-4}(l+t^2)](l-t^6m^6)$. The other two generators of $I_t(K)$ give rise to two more generators of $\mathcal{A}_t(K)$. The ones from the statement are obtained from these two after some algebraic manipulations which involve also the first generator. □

Note the presence of the factor $(l - t^6m^6)$, which is the noncommutative analogue of the factor of the classical A-polynomial that stands for the irreducible $SL(2, C)$-representations of the fundamental group.

Now let $t = -1$. Note the presence of a discontinuity, due to Lemma 5. In this case, the same arguments apply *mutatis mutandis* to prove the following two results.

Theorem 4. *The ideal* $I_{-1}(K)$ *is generated by* $(1,-4)_T - (1,-2)_T + (0,4)_T - (0,2)_T - 2$ *and* $(2,-6)_T - (0,6)_T$.

Theorem 5. *The ideal* $\mathcal{A}_{-1}(K)$ *is generated by* $(l^2-1)(l+1)(l-m^6)$ *and* $(m^2-1)(l+1)(l-m^6)$.

Note that the classical A-polynomial is obtained by replacing l by $-l$ and m by $-m$, and then taking the common factors of the two generators (i.e. eliminating the embedded primes). The change of sign in the variables is due to the fact that the relationship between skein modules and character varieties is established by the negative of the trace.

5. The case of the right-handed trefoil

When taking the mirror image the (p, q) curve in the boundary of the left-handed trefoil becomes the $(p, -q)$ curve on the boundary of the right-handed trefoil. Also, in the Kauffman bracket skein relation, taking the mirror image changes t to t^{-1}. So for K the right-handed trefoil knot we obtain the following results.

Theorem 6. *If $p > 0$ then*

$$\pi((p,q)_T) = t^{-6p^2+pq}S_{-q+6p}(x) + t^{-6p^2+pq+2}S_{-q+6p-2}(x)y +$$
$$\sum_{k=1}^{2p-1}(-1)^{\left[\frac{k+1}{2}\right]}t^{-6p^2+(\left[\frac{k}{2}\right]+1)(6\left[\frac{k}{2}\right]+4\epsilon_k)+pq}S_{-q+6p-3k+\epsilon_k-2}(x)(1+t^2y).$$

The action of $K_t(\mathbb{T}^2\times I)$ on $K_t(M)$ is determined by

$$(p,q)_T\cdot y = -t^{-6p^2+pq+2}S_{-q+6p}(x) - t^{-6p^2+pq}S_{-q+6p-2}(x)y +$$
$$\sum_{k=1}^{2p-2}(-1)^{\left[\frac{k-1}{2}\right]}t^{-6p^2+(\left[\frac{k}{2}\right]+1)(6\left[\frac{k}{2}\right]+4\epsilon_k)+pq}S_{-q+6p-3k+\epsilon_k-2}(x)(t^{-2}2+y) +$$
$$(-1)^{p-1}S_{-q}(x)t^{-2q+pq}(t^{-2}+y).$$

Theorem 7. *If t is not an eighth root of unity, the ideal $I_t(K)$ is generated by*

$$(1,5)_T - t^8(1,1)_T + t^3(0,5)_T - t^{-1}(0,1)_T,$$
$$(2,6)_T - (t^6+t^{-6})(1,0)_T + (t^4+t^{-4})(1,6)_T + (0,6)_T - 2(t^4+t^{-4}),$$
$$(2,7)_T + t^5(1,7)_T + (t^5-t)(1,3)_T - t^{-5}(1,1)_T -$$
$$-(t^2-t^{-2})(0,3)_T - t^6(0,1)_T.$$

Theorem 8. *If t is not an eighth root of unity, the ideal $\mathcal{A}_t(K)$ is generated by*

$$[m^4(l+t^{10}) - t^{-4}(l+t^2)](lm^6-t^6),$$
$$(l+t^{24})(l+t^{10})(l+t^2)(lm^6-t^6),$$
$$(m^2-t^{-22})(l+t^{10})(l+t^2)(lm^6-t^6).$$

Theorem 9. *The ideal $I_{-1}(K)$ is generated by $(1,4)_T-(1,2)_T+(0,4)_T-(0,2)_T-2$ and $(2,6)_T-(0,6)_T$, and the ideal $\mathcal{A}_{-1}$ is generated by $(l^2-1)(l+1)(lm^6-1)$ and $(m^2-1)(l+1)(lm^6-1)$*

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