A RECURSIVE RELATION IN THE COMPLEMENT OF THE $(2p+1,2)$ TORUS KNOT

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Abstract. It is known that the colored Jones polynomials of a knot in the 3-dimensional sphere satisfy recursive relations, it is also known that these recursive relations come from recurrence polynomials which have been related, by the AJ conjecture, to the geometry of the knot complement. In this paper we propose a new line of thought, by extending the concept of colored Jones polynomials to knots in the 3-dimensional manifold such as a knot complement, and then examining the case of one particular knot in the complement of the $(2p+1,2)$ torus knot for which an analogous recursive relation exists, and moreover, this relation has an associated recurrence polynomial. Part of our study consists of the writing in the standard basis of the genus two handlebody of two families of skeins in this handlebody.

1. Introduction

In 1984 V.F.R Jones has discovered a polynomial invariant for knots in the 3-dimensional sphere [13]. E. Witten has explained the Jones polynomial using a quantum field theory whose action functional is the Chern-Simons functional [20], showing that the Jones polynomial evaluated at a root of unity is the expected value of the trace of the holonomy along the knot of an $su(2)$-connection, which holonomy is computed in the fundamental representation of $SU(2)$. Witten has brought to attention the same expected value computed for other possible representations of $SU(2)$, among which a special role is played by the irreducible representations. And as there is one irreducible representation of $SU(2)$ of dimension $n+1$ for each $n \geq 0$, there is a corresponding knot invariant, called the nth colored Jones polynomial of the knot. The colored Jones polynomials of knots have been constructed rigorously in [16] and [19] using quantum groups. There exists a slightly modified version of Witten's theory, based on the Kauffman bracket [14], which can be found in [15], with its own version of colored Jones polynomials, what we prefer to call the colored Kauffman brackets.

The combinatorial nature of Witten's Chern-Simons theory is expressed in skein relations, and these skein relations have led J. Przytycki to introduce the concept of a skein module in an attempt to capture the combinatorial aspects of Chern-Simons theory [18]. Our focus is on the Kauffman bracket

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skein modules. The Kauffman bracket skein module of a 3-dimensional oriented manifold M is defined as follows. Let $\mathcal{L}$ be the set of isotopy classes of framed links in the manifold M, including the empty link. Consider the free $\mathbb{C}[t,t^{-1}]$-module with basis $\mathcal{L}$, and factor it by the smallest subspace containing all expressions of the form $\;-t\;\;-t^{-1}\,)($ and $\bigcirc+t^2+t^{-2}$, where the links in each expression are identical except in a ball in which they look like depicted. The resulting quotient is the Kauffman bracket skein module of M, denoted by $K_t(M)$.

It is in the context of Kauffman bracket skein modules that Ch. Frohman has discovered that the colored Jones polynomials (or rather the colored Kauffman brackets) of a knot in the 3-dimensional sphere are related to one another [6]. The relation was expressed as an "orthogonality" between the vector with entries the colored Jones polynomials and a vector computed from a deformed version of the A-polynomial. The orthogonality relation was further interpreted as a linear recursive relation for colored Jones polynomials by the second author in [8]. This research has been further refined by S. Garoufalidis and T.T.Q. Le in [7] using the concept of q-holonomicity, to show that such recursive relations are build in the very definition of the colored Jones polynomials. In this context they introduced the concepts of recurrence polynomials and recurrence ideals for the colored Jones polynomials of knots. It is important to point out that the Garoufalidis-Le theory is for the actual colored Jones polynomials in the way they arise in the Reshetikhin-Turaev theory based on quantum groups, and not for their slightly modified Kauffman bracket versions.

In this paper we advance the problem of finding relations among colored Jones polynomials to a different setting. We do this with the example of one knot in the complement of the $(2p+1,2)$ torus knot. Let us recall that it was shown in [4] that the Kauffman bracket skein module of the complement in the three dimensional sphere of a regular neighborhood of the $(2p+1,2)$ torus knot (in short the complement of the $(2p+1,2)$ torus knot) is a free $\mathbb{C}[t,t^{-1}]$-module with a basis consisting of the skeins

$$x^m y^n, \quad m \geq 0, \quad 0 \leq n \leq p,$$

where x and y are the curves shown in Figure 1. Our convention here and throughout the paper is that we use the blackboard framing of curves, and that the monomial $x^m y^n$ means the multicurve consisting of m parallel copies of x and n parallel copies of y.

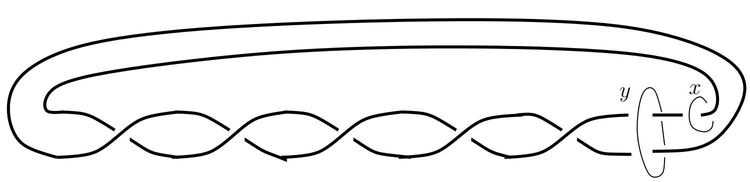

FIGURE 1

We recall the two families of (normalized) Chebyshev polynomials, those of the first kind, $T_n(\xi)$, defined by $T_n(2\cos\theta) = 2\cos n\theta$, and those of the second kind, $S_n(\xi)$, defined by $S_n(2\cos\theta) = \sin(n+1)\theta/\sin\theta$, where n ranges over all integers, positive and negative.

For the Kauffman bracket skein module of the complement of the torus knot it is then sensible to use the basis

$$S_m(x)S_n(y), \quad m \geq 0, \quad 0 \leq n \leq p,$$

motivated by the fact that the polynomial $S_n(\xi)$ is the character of the $n+1$st irreducible representation of the Lie group $SU(2)$, and so $S_m(x)S_n(y)$ would correspond, in the Kauffman bracket picture, to the link $x \cup y$ decorated by the $m+1$st and $n+1$st irreducible representations of this group (or of its quantum version).

The present paper is focussed on just one example, and this example is the knot in the complement of the $(2p+1,2)$ torus knot that is the curve y (endowed with the blackboard framing). Our main result is the following:

Theorem 1.1. *For all $n \in \mathbb{Z}$, the following identity holds in the Kauffman bracket skein module of the complement of the $(2p+1,2)$ torus knot*

$$t^{-2n-1}S_{p+n}(y) + t^{2n+1}S_{p-n-1}(y) = (-1)^n S_{2n}(x)(tS_{p-1}(y) + t^{-1}S_p(y)).$$

This formula was noticed by J. Sain and proved for $0 \leq n \leq p+1$ in [10], but unfortunately that proof does not extend to other n. In this paper we use some results of the second author and his collaborators about the Kauffman bracket skein module of the genus two handlebody to find a different proof that works in general. In the process we address in Proposition 2.3 a problem raised by R.P. Bakshi and J. Przytycki in conjuction with their work on connected sums of handlebodies from [3], the problem being about expressing a certain skein in the genus two handlebody in terms of the standard basis elements. Regarding this proposition, the authors would like to express their gratitude to T.T.Q. Le for observing and correcting two errors in the formula from the statement of the proposition.

By using Theorem 1.1 in §4 of this paper we put the colored Jones polynomials of y in the context of theory developed by Garoufalidis and Le in [7]. The question, of course, is what should the analogue of the nth colored Jones polynomial of a knot K in an arbitrary manifold be? Based on the considerations explained in [1], we define these “colored Jones polynomials” to be the skeins $S_n(K)$, $n \geq 0$, inside the skein module defined by the skein relations derived in [17] for the Reshetikhin-Turaev theory [19] (see §4 below for the definition). If we apply mutatis mutandis the method of [7], then we find the recurrence polynomial for the colored Jones polynomials of the curve y in the complement of the $(2p+1,2)$ torus knot to be

$$[L^2 - t^4(x^2-2)L + t^8]L^{2p+1} + t^{4p+8}[L^2 - (x^2-2)L + 1]M^2.$$

This simple polynomial contains all the necessary information for computing $S_n(y)$ as a linear combination with coefficients in $\mathbb{C}[t,t^{-1},x]$ of $S_0(y)$, $S_1(y)$,

..., $S_p(y)$ for $n > p$. Does there exist a geometric interpretation of this polynomial analogous to the one found in [6], [7] for the recurrence polynomials of colored Jones polynomials of knots in the 3-dimensional sphere?

2. Some sequences of skeins in the complement of the genus 2 handlebody

To enhance our ability to prove Theorem 1.1, we have to take a detour through the skein theory of the genus two handlebody. Specifically, we discuss three sequences of skeins in the Kauffman bracket skein module of the genus two handlebody, the third of which has appeared in our previous work [9]. Our discussion requires some basic knowledge about linear recursive sequences; a good reference for the necessary techniques is [2].

We interpret the genus two handlebody as the cylinder over the twice punctured disk and we represent it schematically sideways, by drawing only the two curves that trace the punctures in the cylinder. Przytycki [18] has shown that the Kauffman bracket skein module of the genus two handelbody is free with basis $x^m y^n z^k$, $m, n, k \geq 0$, where x and z are curves that are parallel to the boundaries of the two open disks that have been removed, and y is a curve parallel to the boundary of the original disk. The curves x, y, z are shown in Figure 2.

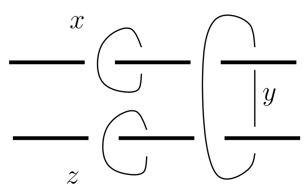

Figure 2

The first two sequences of skeins that we have in mind have been introduced in [12]. They are $X_1 * y^n$ and $Y_1 * y^n$ depicted in Figure 3. Modifying

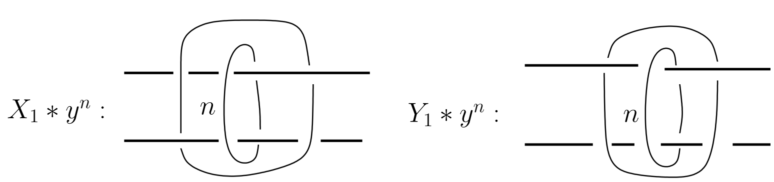

Figure 3

appropriately the argument of Lemma 2.1 in [12] we obtain

Lemma 2.1. *The sequences $X_1 * y^n$, $Y_1 * y^n$ satisfy the recursive relations*

$$X_1 * y^{n+1} = t^4 y X_1 * y^n + (t^{-2} - t^6) Y_1 * y^n + (1 - t^4)(x^2 + z^2) y^n$$
$$Y_1 * y^{n+1} = t^{-4} y Y_1 * y^n + (t^2 - t^{-6}) X_1 * y^n + 2(1 - t^{-4}) xzy^n,$$
$$X_1 * y^0 = -t^4 y - t^2 xz, \quad Y_1 * y^0 = -t^2 - t^{-2}.$$

We are interested in finding explicit formulas for $X_1 * y^n$ and $Y_1 * y^n$, but as experience has taught us, it is better to replace the "powers" of y by Chebyshev polynomials in y. So instead we will find explicit formulas for $X_1 * T_n(y)$ and $Y_1 * T_n(y)$, where T_n is the Chebyshev polynomial of first kind defined in the introduction.

Theorem 2.2. *The following formulas hold*

$$\begin{aligned} X_1 * T_n(y) &= -t^{4n+4}S_{n+1}(y) - t^{-4n}S_{n-1}(y) + t^{4n}S_{n-1}(y) \\ &+t^{-4n+4}S_{n-3}(y) - t^{4n+2}xzS_n(y) + t^{-4n+2}xzS_{n-2}(y) \\ &+(1-t^{4n})\sum_{k=0}^{n-1} t^{-4k}(x^2+z^2)S_{n-2k-1}(y) \\ &+2(1-t^{4n})\sum_{k=0}^{n-1} t^{-4k-2}xzS_{n-2k-2}(y), \quad n \geq 1, \\ Y_1 * T_n(y) &= -(t^{4n+2}+t^{-4n-2})S_n(y) + (t^{-4n}-t^{4n})xzS_{n-1}(y) \\ &+(t^{4n-2}+t^{-4n+2})S_{n-2}(y) + (1-t^{4n})(x^2+z^2)\sum_{k=0}^{n-1} t^{-4k-2}S_{n-2k-2}(y) \\ &+2(1-t^{4n})xz\sum_{k=0}^{n-1} t^{-4k-4}S_{n-2k-3}(y), \quad n \geq 1. \end{aligned}$$

Proof. The argument is based on the recursive relation for the vector

$$(X_1 * y^n, Y_1 * y^n)$$

exhibited in Lemma 2.1. We introduce the auxiliary variable w so that $y = w + w^{-1}$ (w has no geometric meaning, it is used solely for computations). The coefficient matrix of the recursive relation,

$$A = \begin{pmatrix} t^4 y & t^{-2} - t^6 \\ t^2 - t^{-6} & t^{-4} y \end{pmatrix},$$

has eigenvalues

$$\lambda_{1,2} = \frac{1}{2}[(t^4+t^{-4})y \pm (t^4-t^{-4})\sqrt{y^2-4}] = t^4 w^{\pm 1} + t^{-4} w^{\mp 1},$$

with eigenvectors $(t^2 w, 1)$ and (w^{-1}, t^{-2}), respectively. Hence this coefficient matrix is diagonalized as

$$\frac{1}{w-w^{-1}} \begin{pmatrix} t^2 w & w^{-1} \\ 1 & t^{-2} \end{pmatrix} \begin{pmatrix} t^4 w + t^{-4} w^{-1} & 0 \\ 0 & t^4 w^{-1} + t^{-4} w \end{pmatrix} \begin{pmatrix} t^{-2} & -w^{-1} \\ -1 & t^2 w \end{pmatrix}$$

Now we split the sequence $(X_1 * y^n, Y_1 * y^n)$ into $(a_n, b_n) + (c_n, d_n)$ where (a_n, b_n) satisfies the homogeneous recursive relation

$$\begin{pmatrix} a_{n+1} \\ b_{n+1} \end{pmatrix} = A \begin{pmatrix} a_n \\ b_n \end{pmatrix}, \quad a_0 = X_1 * y^0, \quad b_0 = Y_1 * y^0,$$

and (c_n, d_n) satisfies the nonhomogenous recursive relation with trivial initial condition

$$\begin{pmatrix} c_{n+1} \\ d_{n+1} \end{pmatrix} = A \begin{pmatrix} c_n \\ d_n \end{pmatrix} + \begin{pmatrix} (1-t^4)(x^2+z^2)y^n \\ 2(1-t^{-4})xzy^n \end{pmatrix}, \quad c_0 = d_0 = 0.$$

We obtain

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \frac{1}{w-w^{-1}} \begin{pmatrix} t^2 w & w^{-1} \\ 1 & t^{-2} \end{pmatrix} \begin{pmatrix} (t^4 w + t^{-4} w^{-1})^n & 0 \\ 0 & (t^4 w^{-1} + t^{-4} w)^n \end{pmatrix} \\ \times \begin{pmatrix} t^{-2} & -w^{-1} \\ -1 & t^2 w \end{pmatrix} \begin{pmatrix} -t^4(w+w^{-1}) - t^2 xz \\ -t^2 - t^{-2} \end{pmatrix}.$$

Using the equalities

$$T_n(tw + t^{-1}w^{-1}) = t^n w^n + t^{-n} w^{-n}, T_n(t^{-1}w + tw^{-1}) = t^{-n} w^n + t^n w^{-n}$$

we deduce that the "homogeneous" part of $(X_1 * T_n(y), Y_1 * T_n(y))$ is

$$\frac{1}{w-w^{-1}} \begin{pmatrix} t^2 w & w^{-1} \\ 1 & t^{-2} \end{pmatrix} \begin{pmatrix} t^{4n} w^n + t^{-4n} w^{-n} & 0 \\ 0 & t^{4n} w^{-n} + t^{-4n} w^n \end{pmatrix} \\ \times \begin{pmatrix} t^{-2} & -w^{-1} \\ -1 & t^2 w \end{pmatrix} \begin{pmatrix} -t^4(w+w^{-1}) - t^2 xz \\ -t^2 - t^{-2} \end{pmatrix}$$

Using the fact that

$$S_n(y) = \frac{w^{n+1} - w^{-n-1}}{w - w^{-1}}$$

we obtain that this is further equal to

$$\begin{pmatrix} t^{4n} S_n(y) - t^{-4n} S_{n-2}(y) & -t^2(t^{4n} - t^{-4n}) S_{n-1}(y) \\ -t^{-2}(t^{-4n} - t^{4n}) S_{n-1}(y) & t^{-4n} S_n(y) - t^{4n} S_{n-2}(y) \end{pmatrix} \begin{pmatrix} -t^4 y - t^2 xz \\ -t^2 - t^{-2} \end{pmatrix}$$

So the "homogeneous" part of $X_1 * T_n(y)$ is

$$-t^{4n+4} S_{n+1}(y) - t^{-4n} S_{n-1}(y) + t^{4n} S_{n-1}(y) + t^{-4n+4} S_{n-3}(y) \\ -t^{4n+2} xz S_n(y) + t^{-4n+2} xz S_{n-2}(y),$$

while the "homogeneous" part of $Y_1 * T_n(y)$ is

$$-(t^{4n+2} + t^{-4n-2}) S_n(y) + (t^{-4n} - t^{4n}) xz S_{n-1}(y) \\ +(t^{4n-2} + t^{-4n+2}) S_{n-2}(y).$$

On the other hand

$$\begin{pmatrix} c_n \\ d_n \end{pmatrix} = \frac{(t^2 - t^{-2})}{w - w^{-1}} \begin{pmatrix} t^2 w & w^{-1} \\ 1 & t^{-2} \end{pmatrix} \\ \times \begin{pmatrix} \dfrac{t^{4n} w^n + t^{-4n} w^{-n} - w^n - w^{-n}}{t^4 w + t^{-4} w^{-1} - w - w^{-1}} & 0 \\ 0 & \dfrac{t^{4n} w^{-n} + t^{-4n} w^n - w^n - w^{-n}}{t^4 w^{-1} + t^{-4} w - w - w^{-1}} \end{pmatrix} \\ \times \begin{pmatrix} t^{-2} & -w^{-1} \\ -1 & t^2 w \end{pmatrix} \begin{pmatrix} -t^2(x^2+z^2) \\ 2t^{-2} xz \end{pmatrix}.$$

Rewrite this expression as

$$\begin{aligned}
&\frac{t^{-2}(t^{4n}-1)}{w-w^{-1}}\begin{pmatrix} t^2w & w^{-1}\\ 1 & t^{-2}\end{pmatrix}\begin{pmatrix} \frac{w^n-t^{-4n}w^{-n}}{w-t^{-4}w^{-1}} & 0\\ 0 & \frac{w^{-n}-t^{-4n}w^{n}}{w^{-1}-t^{-4}w}\end{pmatrix}\\
&\times\begin{pmatrix} t^{-2} & -w^{-1}\\ -1 & t^2w\end{pmatrix}\begin{pmatrix} -t^2(x^2+z^2)\\ 2t^{-2}xz\end{pmatrix}\\
&=\frac{t^{-2}(t^{4n}-1)}{w-w^{-1}}\begin{pmatrix} A & B\\ C & D\end{pmatrix}\begin{pmatrix} -t^2(x^2+z^2)\\ 2t^{-2}xz\end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
A&=\frac{w^{n+1}-t^{-4n}w^{-n+1}}{w-t^{-4}w^{-1}}-\frac{w^{-n-1}-t^{-4n}w^{n-1}}{w^{-1}-t^{-4}w}\\
B&=-t^2\left(\frac{w^n-t^{-4n}w^{-n}}{w-t^{-4}w^{-1}}-\frac{w^{-n}-t^{-4n}w^{n}}{w^{-1}-t^{-4}w}\right),\\
C&=t^{-2}\left(\frac{w^n-t^{-4n}w^{-n}}{w-t^{-4}w^{-1}}-\frac{w^{-n}-t^{-4n}w^{n}}{w^{-1}-t^{-4}w}\right),\\
D&=-\frac{w^{n-1}-t^{-4n}w^{-n-1}}{w-t^{-4}w^{-1}}+\frac{w^{-n+1}-t^{-4n}w^{n+1}}{w^{-1}-t^{-4}w}.
\end{aligned}$$

And we have

$$\frac{A}{w-w^{-1}}=\sum_{k=0}^{n-1}t^{-4k}S_{n-2k-1}(y),\quad \frac{B}{w-w^{-1}}=-\sum_{k=0}^{n-1}t^{-4k+2}S_{n-2k-2}(y),$$

$$\frac{C}{w-w^{-1}}=\sum_{k=0}^{n-1}t^{-4k-2}S_{n-2k-2}(y),\quad \frac{D}{w-w^{-1}}=-\sum_{k=0}^{n-1}t^{-4k}S_{n-2k-3}(y).$$

After a final multiplication we obtain that the nonhomogeneos parts of $X_1*T_n(y)$ and $Y_1*T_n(y)$ are, respectively,

$$(1-t^{4n})\sum_{k=0}^{n-1}t^{-4k}(x^2+z^2)S_{n-2k-1}(y)+2(1-t^{4n})\sum_{k=0}^{n-1}t^{-4k-2}xzS_{n-2k-2}(y),$$

$$(1-t^{4n})(x^2+y^2)\sum_{k=0}^{n-1}t^{-4k-2}S_{n-2k-2}(y)+2(1-t^{4n})xz\sum_{k=0}^{n-1}t^{-4k-4}S_{n-2k-3}(y).$$

The conclusion follows. □

As hinted in the introduction, in the work of Bakshi and Przytycki has appeared the question of expressing the skein from Figure 4 in terms of the basis $x^my^nz^k$ of the Kauffman bracket skein module of the genus two handlebody. Again, it is advantageous to work with curves decorated by Chebyshev polynomials instead of "powers", we will therefore compute instead the skein in which y^n is replaced by $T_n(y)$, and let us call this skein σ_n. And we use the basis $S_m(x)S_n(y)S_k(z)$, $m,n,k\geq 0$.

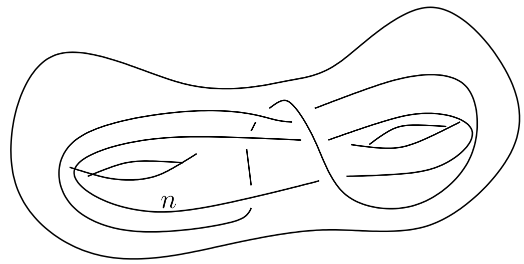

FIGURE 4

Proposition 2.3. *We have*

$$\begin{aligned}\sigma_n = {} & t^{4n+2}S_{n+1}(y) + t^{-4n-2}S_{n-1}(y) - t^{4n-2}S_{n-1}(y) - t^{-4n+2}S_{n-3}(y) \\ & + t^{4n}S_1(x)S_n(y)S_1(z) + t^{-4n}S_1(x)S_{n-2}(y)S_1(z) \\ & - t^{-4}S_1(x)S_n(y)S_1(z) + t^{-4}S_1(x)S_{n-2}(y)S_1(z) \\ & + t^{-2}(t^{4n}-1)\sum_{k=0}^{n-1} t^{-4k}(S_2(x)+S_2(z)+2)S_{n-2k-1}(y) \\ & + 2t^{-4}(t^{4n}-1)\sum_{k=0}^{n-1} t^{-4k}S_1(x)S_{n-2k-2}(y)S_1(z).\end{aligned}$$

Proof. This follows from the fact that

$$\sigma_n = tX_1 * T_n(y) + t^{-1}xzT_n(y) = tX_1 * T_n(y) + t^{-1}xz(S_n(y) - S_{n-2}(y))$$

and then applying Theorem 2.2. □

Finally, let us recall the skeins X_i from the Kauffman bracket skein module of the genus two handlebody, which were defined in [9] and are depicted in Figure 5.

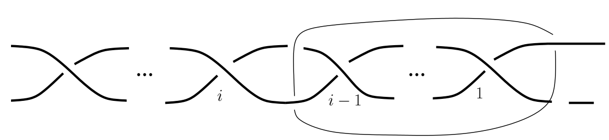

FIGURE 5

Adapting the proof of Lemma 1 from [9] to the case where $x \neq z$, we see that the X_i satisfy the recursive relation

$$X_{i+2} = t^2yX_{i+1} - t^4X_i - 2t^2xz, \quad X_0 = -t^2 - t^{-2}, X_1 = -t^4y - t^2xz.$$

and consequently

$$\begin{aligned}X_i = {} & -t^{-2i-2}S_i(y) - t^{-2i}xzS_{i-1}(y) + t^{-2i+2}S_{i-2}(y) \\ & -2t^{-2i+2}xz\sum_{k=0}^{i-2} t^{2k}S_{i-k-2}(y).\end{aligned}$$

This formula has been proved in [9]; it can be easily verified by induction, and it can be determined using the fact that $t^{-2i}S_i(y)$ and $t^{-2i-2}S_{i-1}$ form a basis for the space of solutions to the homogeneous recursion $X_{i+2} = t^2 y X_{i+1} - t^4 X_i$.

3. Proof of the main result

In this section we prove Theorem 1.1. Because $S_{-n}(y) = -S_{n-2}(y)$, we only need to check the identity

$$t^{-2n-1}S_{p+n}(y) + t^{2n+1}S_{p-n-1}(y) = (-1)^n S_{2n}(x)(tS_{p-1}(y) + t^{-1}S_p(y))$$

for $n \geq 0$. This identity has been checked already in [10] for $n = 0, 1, 2, \ldots, p+1$, but, as mentioned before, that proof cannot be extended for larger n.

We will use instead the equality derived in Figure 6. This equality trans-

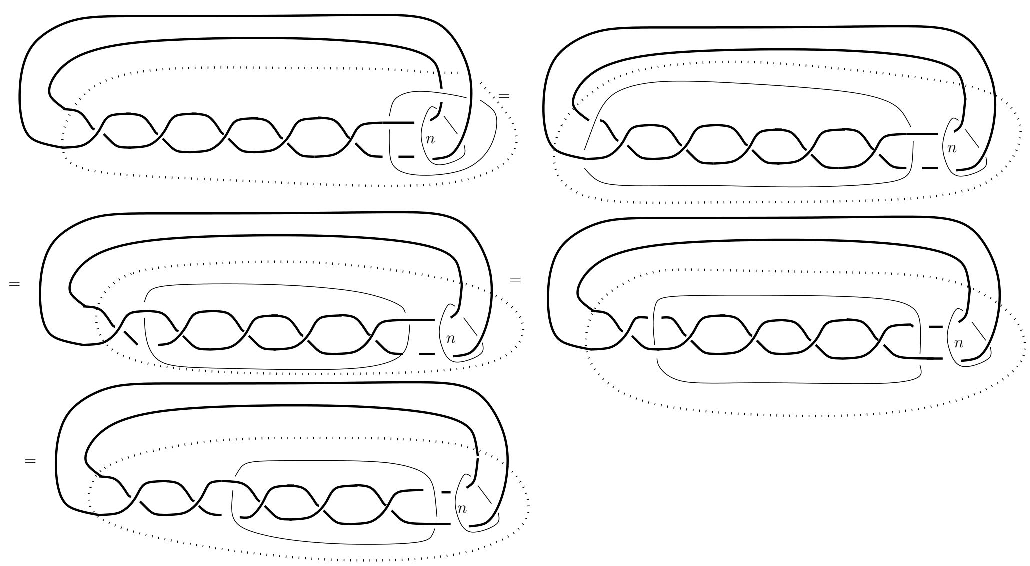

Figure 6

lates to

$$\overline{X_1 * y^n} = y^n \overline{X_{2p}},$$

where $X_1 * T_n(y)$ and X_{2p} are the skeins discussed in the previous section (viewed as lying inside the handlebody marked in the figure with dots), $\overline{\Sigma}$ denotes the mirror image of the skein Σ (with respect to a natural reflection of the handlebody onto itself) and on the right-side we have the product of y^n and $\overline{X_{2p}}$ defined by the cylinder structure of the genus two handlebody. Note that when passing to the mirror image t should be replaced by t^{-1}.

As a consequence of this identity we obtain

$$\overline{X_1 * T_n(y)} = T_n(y)\overline{X_{2p}}, \tag{3.1}$$

which is the key ingredient in the proof of Theorem 1.1.

Using the (trigonometric) identity $S_k(x)T_n(x) = S_{k+n}(x) + S_{k-n}(x)$ and Theorem 2.2 we can write (3.1) explicitly as

$$\begin{aligned}
&-t^{-4n-4}S_{n+1}(y) - t^{4n}S_{n-1}(y) + t^{-4n}S_{n-1}(y) + t^{4n-4}S_{n-3}(y)\\
&+t^{-4n-2}x^2S_n(y) - 2t^{-2}x^2S_n(y) - t^{4n-2}x^2S_{n-2}(y) + 2t^{-2}x^2S_{n-2}(y)\\
&+2t^{-2}(1-t^{-4n})x^2\sum_{k=0}^{2n-1}t^{2k}S_{n-k}(y) + t^{-4p-2}S_{2p+n}(y) + t^{-4p-2}S_{2p-n}(y)\\
&+t^{-4p}x^2S_{2p-1+n}(y) + t^{-4p}x^2S_{2p-1-n}(y) - t^{-4p+2}S_{2p-2+n}(y)\\
&-t^{-4p+2}S_{2p-2-n}(y) + 2t^{-4p+2}x^2\sum_{k=0}^{2p-2}t^{2k}[S_{2p-k-2+n}(y) + S_{2p-k-2-n}(y)]\\
&= 0.
\end{aligned}$$

Both this recursive relation and the one from the main theorem (Theorem 1.1) completely determine the values of $S_n(y)$ for $n > p$ from $S_0(y)$, $S_1(y)$, …, $S_p(y)$. To complete the proof of Theorem1.1 it suffices to show that the sequence $S_n(y)$ defined by the recursive relation from the statement of this theorem also satisfies the above identity. So from this moment on we assume that $S_n(y)$ is the sequence defined by the recursive relation from the statement of Theorem 1.1 and we prove that it satisfies this identity.

For the proof, we further transform this desired identity into

$$\begin{aligned}
&t^{-2p+2n-1}(t^{-2p-2n-1}S_{2p+n}(y) - t^{2p+2n+1}S_{n-1}(y))\\
&+t^{-2p-2n-1}(t^{-2p+2n-1}S_{2p-n}(y) + t^{2p-2n+1}S_{n-1}(y))\\
&+t^{-2p+2n-1}x^2(t^{-2p-2n+1}S_{2p-1+n}(y) - t^{2p+2n-1}S_{n-2}(y))\\
&+t^{-2p-2n-1}x^2(t^{-2p+2n+1}S_{2p-n-1}(y) + t^{2p-2n-1}S_n(y))\\
&-t^{-2p+2n-1}(t^{-2p-2n+3}S_{2p-2+n}(y) - t^{2p+2n-3}S_{n-3}(y))\\
&-t^{-2p-2n-1}(t^{-2p+2n+3}S_{2p-2-n}(y) + t^{2p-2n-3}S_{n+1}(y))\\
&= 2t^{-2}x^2\left[S_n(y) + S_{n-2}(y) + (t^{-4n}-1)\sum_{k=0}^{2n-1}t^{2k}S_{n-k}(y)\right.\\
&\left.-\sum_{k=0}^{2p-2}t^{-4p+2k+4}(S_{2p-k-2+n}(y) + S_{2p-k-2-n}(y))\right],
\end{aligned}$$

and then rewrite it as

$$\begin{aligned}
&t^{-2p+2n-1}(t^{-2p-2n-1}S_{2p+n}(y) - t^{2p+2n+1}S_{n-1}(y))\\
&+t^{-2p-2n-1}(t^{-2p+2n-1}S_{2p-n}(y) + t^{2p-2n+1}S_{n-1}(y))\\
&+t^{-2p+2n-1}x^2(t^{-2p-2n+1}S_{2p-1+n}(y) - t^{2p+2n-1}S_{n-2}(y))
\end{aligned}$$

$$
\begin{aligned}
&+t^{-2p-2n-1}x^2(t^{-2p+2n+1}S_{2p-n-1}(y)+t^{2p-2n-1}S_n(y))\\
&-t^{-2p+2n-1}(t^{-2p-2n+3}S_{2p-2+n}(y)-t^{2p+2n-3}S_{n-3}(y))\\
&-t^{2p-2n-1}(t^{-2p+2n+3}S_{2p-2-n}(y)+t^{2p-2n-3}S_{n+1}(y))\\
&=2t^{-2}x^2\left[S_n(y)-S_{n-2}(y)+(t^{-4n}-1)\sum_{k=0}^{2n-1}t^{2k}S_{n-k}(y)\right.\\
&\left.-\sum_{k=0}^{2p-2}t^{-2k}(S_{k+n}(y)+S_{k-n}(y))\right].
\end{aligned}
$$

Finally, we bring it into the form

$$
\begin{aligned}
&t^{-2p+2n-1}(t^{-2p-2n-1}S_{2p+n}(y)-t^{2p+2n+1}S_{n-1}(y))\\
&+t^{-2p-2n-1}(t^{-2p+2n-1}S_{2p-n}(y)+t^{2p-2n+1}S_{n-1}(y))\\
&+t^{-2p+2n-1}x^2(t^{-2p-2n+1}S_{2p-1+n}(y)-t^{2p+2n-1}S_{n-2}(y))\\
&+t^{-2p-2n-1}x^2(t^{-2p+2n+1}S_{2p-n-1}(y)+t^{2p-2n-1}S_n(y))\\
&-t^{-2p+2n-1}(t^{-2p-2n+3}S_{2p-2+n}(y)-t^{2p+2n-3}S_{n-3}(y))\\
&-t^{-2p-2n-1}(t^{-2p+2n+3}S_{2p-2-n}(y)+t^{2p-2n-3}S_{n+1}(y))\\
&=2t^{-2}x^2\left[(t^{-4n}-1)\sum_{k=0}^{2n-1}t^{2k}S_{n-k}(y)-\sum_{k=1}^{2p-2}t^{-2k}(S_{k+n}(y)+S_{k-n}(y))\right].
\end{aligned}
$$

We will therefore prove that the sequence $S_n(y)$ defined by the recursive relation from the statement of Theorem 1.1 satisfies this identity. Using the formula from the statement of the theorem we deduce that the left-hand side of the identity to be checked is equal to

$$
\begin{aligned}
&t^{-2p+2n-1}(-1)^{p+n}S_{2p+2n}(x)Y+t^{-2p-2n-1}(-1)^{p-n}S_{2p-2n}(x)Y\\
&+t^{-2p+2n-1}x^2(-1)^{p+n-1}S_{2p+2n-2}(x)Y\\
&+t^{-2p-2n-1}x^2(-1)^{p-n-1}S_{2p-2n-2}(x)Y\\
&-t^{-2p+2n-1}(-1)^{p+n-2}S_{2p+2n-4}(x)Y-t^{-2p-2n-1}(-1)^{p-n-2}S_{2p-2n-4}(x)Y,
\end{aligned}
$$

where we use the short hand notation $Y=t^{-1}S_p(y)+tS_{p-1}(y)$. This expression is

$$
\begin{aligned}
(-1)^{p+n+1}t^{-1}&\left[t^{-2p+2n}(S_{2p+2n-2}(x)+S_{2p+2n-4}(x))Y\right.\\
&\left.+t^{-2p-2n}(S_{2p-2n-2}(x)+S_{2p-2n-4}(x))Y\right].
\end{aligned}
$$

Now let us work on the right side of the identity. We have

$$\begin{aligned}
&\sum_{k=1}^{2p-2} t^{-2k}(S_{k+n}(y)+S_{k-n}(y)) = \sum_{k=1}^{2p-2} t^{-2k} S_{k+n}(y)\\
&+\sum_{k=1}^{2p-2} t^{-2(2p-1-k)} S_{2p-2-k-n+1}(y) = (1-t^{-4n})\sum_{k=1}^{2p-2} t^{-2k} S_{k+n}(y)\\
&+\sum_{k=1}^{2p-2} t^{-2p+1+2n}[t^{-2p+2k-2n+1} S_{2p-2-k-n+1}(y) + t^{2p-2k+2n-1} S_{k+n}(y)].
\end{aligned}$$

By using the formula from the statement of the theorem, this becomes

$$\sum_{k=1}^{2p-2} (-1)^{p-n-k-1} t^{-2p+1-2n} S_{2(p-n-k-1)}(x)Y + (1-t^{-4n})\sum_{k=1}^{2p-2} t^{-2k} S_{k+n}(y).$$

We are left to checking that the sequence $S_n(y)$ satisfies the simpler identity

$$\begin{aligned}
&[t^{4n}(S_{2p+2n-2}(x)+S_{2p+2n-4}(x)) + S_{2p-2n-2}(x) + S_{2p-2n-4}(x)\\
&+\sum_{k=1}^{2p-2} (-1)^k x^2 S_{2p-2n-2k-2}(x)]Y\\
&= (-1)^{p+n} t^{2p-1} x^2 (t^{2n}-t^{-2n})\left[\sum_{k=0}^{2n-1} t^{2k} S_{n-k}(y) + \sum_{k=1}^{2p-2} t^{-2k} S_{n+k}(y)\right],
\end{aligned}$$

Transform this into

$$\begin{aligned}
&[t^{4n}(S_{2p+2n-2}(x)+S_{2p+2n-4}(x)) + S_{2p-2n-2}(x) + S_{2p-2n-4}(x)\\
&-(-1)^p \sum_{j=1-p}^{p-2} (-1)^j x^2 S_{2j-2n}(x)]Y\\
&= (-1)^{p+n} t^{2p-1} x^2 (t^{2n}-t^{-2n})\left[\sum_{k=0}^{2n-1} t^{2k} S_{n-k}(y) + \sum_{k=1}^{2p-2} t^{-2k} S_{n+k}(y)\right],
\end{aligned}$$

then into

$$\begin{aligned}
&[t^{4n}(S_{2p+2n-2}(x)+S_{2p+2n-4}(x)) + S_{2p-2n-2}(x) + S_{2p-2n-4}(x)\\
&-S_{2p+2n-2}(x) - S_{2p+2n-4}(x) - S_{2p-2n-2}(x) - S_{2p-2n-4}(x)]Y\\
&= (-1)^{p+n} t^{2p-1} x^2 (t^{2n}-t^{-2n})\left[\sum_{k=0}^{2n-1} t^{2k} S_{n-k}(y) + \sum_{k=1}^{2p-2} t^{-2k} S_{n+k}(y)\right].
\end{aligned}$$

This can be simplified as

$$\begin{aligned}
&(S_{2p+2n-2}(x)+S_{2p+2n-4}(x))Y\\
&= (-1)^{p+n} t^{2p-2n-1} x^2 \left[\sum_{k=0}^{2n-1} t^{2k} S_{n-k}(y) + \sum_{k=1}^{2p-2} t^{-2k} S_{n+k}(y)\right].
\end{aligned}$$

And now we check that the sequence $S_n(y)$ defined by the recursive relation from the statement of Theorem 1.1 satisfies this identity by induction on n.

Set

$$A_n = \sum_{k=1}^{2n-1} t^{2k} S_{n-k}(y) + \sum_{k=1}^{2p-2} t^{-2k} S_{n+k}(y).$$

Then

$$\begin{aligned}
&A_{n+1} - t^2 A_n = t^{-4p+4} S_{n+2p-1}(y) + t^{4p+2} S_{-n}(y)\\
&= t^{-4p+4} S_{n+2p-1}(y) - t^{4p+2} S_{n-2}(y)\\
&= t^{-2p+2n+3}(t^{-2p-2n+1} S_{n+2p-1}(y) - t^{2p+2n-1} S_{n-2}(y))\\
&= (-1)^{p+n-1} t^{-2p+2n+2} S_{2n+2p-2}(x) Y,
\end{aligned}$$

where for the last equality we used the identity from the statement of the theorem. We are attempting to prove that

$$[S_{2n+2p-2}(x) + S_{2n+2p-4}(x)]Y = (-1)^{p+n} t^{2p-2n} x^2 A_n$$

holds for all n and as induction step we assume that this equality holds for n and prove it for $n+1$. But if we add the equality for n to the one for $n+1$ we obtain something obvious, as the following computation shows:

$$\begin{aligned}
&[S_{2n+2p}(x) + 2S_{2n+2p-2}(x) + S_{2n+2p-4}(x)]Y = x^2 S_{2n+2p-2}(x) Y\\
&= (-1)^{p+n+1} t^{2p-2n-2} x^2 (-1)^{p+n-1} t^{-2p+2n+2} S_{2n+2p-2}(x) Y\\
&= (-1)^{p+n} t^{2p-2n-2} x^2 [-A_{n+1} + t^2 A_n].
\end{aligned}$$

The induction is now complete because the base case was checked in [10].

4. A recurrence polynomial for the colored Jones polynomials of y

In this section we will explain how the recurrence polynomial for the colored Jones polynomials of the curve y is constructed. The nth colored Jones polynomial of a knot K in S^3 has been introduced in [20] as the quantized Wilson line associated to the knot K and the $n+1$-dimensional irreducible representation of $SU(2)$ and has been constructed rigorously in [16] and [19] using the $n+1$st irreducible representation of the quantum group of $SU(2)$. It is known that the nth colored Jones polynomial of the knot K is

$$J(K,n) = (-1)^n \langle S_n(K) \rangle \tag{4.1}$$

where $\langle \cdot \rangle$ is the Kauffman bracket. The equality (4.1) can be placed in the setting of skein modules as follows.

Alongside with the Kauffman bracket skein module $K_t(M)$ of a manifold M one can introduce the skein module of the Reshetikhin-Turaev theory, which was denoted by $RT_t(M)$ and was defined in [11] as follows. As in the case of the Kauffman bracket skein module we consider the free $\mathbb{C}[t,t^{-1}]$-module with bases the isotopy classes of framed links in M. To obtain

$RT_t(M)$ we factor this module by the submodule generated by three families of elements.

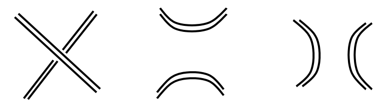

FIGURE 7

The first family consists of elements of the form

$$L \cup O - (t^2 + t^{-2})L,$$

where $L \cup O$ consits of the link L to which a trivial knot component is added. The second family consists of elements of the form

$$L - tH - t^{-1}V,$$

where L, H, V are links that are identical except inside an embedded ball in which they look as depicted in Figure 7, and additionally, the crossing in this figure comes from *different* link components of L. The third family consists of elements of the form

$$L - \epsilon(tH - t^{-1}V),$$

where again L, H, V are links that are identical except inside an embedded ball in which they look as depicted in Figure 7, but this time the crossing in the figure is the *self-crossing* of a link component of L and ϵ is the sign of that crossing. These skein relations have been derived in [17] for the Reshetikhin-Turaev theory.

Let S^3 be the 3-dimensional sphere. In $K_t(S^3)$ one has

$$S_n(K) - \langle S_n(K) \rangle \, \emptyset = 0,$$

and in $RT_t(S^3)$ one has

$$S_n(K) - J(K,n)\emptyset = 0;$$

moreover, as explained in [1], in $RT_t(S^3)$ one has $S_n(K) - (-1)^n \langle S_n(K) \rangle \, \emptyset = 0$. This gives the skein theoretical explanation for the equality (4.1). In fact it is the cabling principle stated in [17] that implies that the nth colored Jones polynomial of a knot K in S^3 is the polynomial that is the coefficient of the empty link when we write the skein $S_n(K) \in RT_t(M)$ in the basis consisting of the empty link.

We extend the definition of the colored Jones polynomial to a knot K in some arbitrary 3-dimensional oriented manifold M by stating that this polynomial is the skein $S_n(K)$ in the skein module $RT_t(M)$. Of course if $RT_t(M)$ is a free module and we know a basis, then we can express $S_n(K)$ in that basis and we obtain a family of actual polynomials, but this is less relevant for us at this moment, because we are actually concerned with how $S_n(K)$, $n \in \mathbb{Z}$, relate to one another.

As a consequence of Theorem 2.1 in [1] (see also §3.2 in the same paper) the colored Jones polynomials of the curve y satisfy

$$t^{-2n-1}S_{p+n}(y)-t^{2n+1}S_{p-n-1}(y)=S_{2n}(x)(t^{-1}S_p(y)-tS_{p-1}(y)),$$

(in the new context the factor $(-1)^n$ dissapears). If we add the corresponding relations for $n+1$ and $n-1$ and subtract the one for n multiplied by $T_2(x)=x^2-2$ we obtain the *homogeneous* recursive relation

$$\begin{aligned}&t^{-2n-3}S_{p+n+1}(y)-t^{2n+3}S_{p-n-2}(y)-t^{-2n-1}(x^2-2)S_{p+n}(y)\\&+t^{2n+1}(x^2-2)S_{p-n-1}(y)+t^{-2n+1}S_{p+n-1}(y)-t^{2n-1}S_{p-n}(y)=0.\end{aligned}$$

We rewrite this as

$$\begin{aligned}&t^{-2n-3}S_{n+p+1}(y)+t^{2n+3}S_{n-p}(y)-t^{-2n-1}(x^2-2)S_{n+p}(y)\\&-t^{2n+1}(x^2-2)S_{n-p-1}(y)+t^{-2n+1}S_{n+p-1}(y)+t^{2n-1}S_{n-p-2}(y)=0.\end{aligned}$$

We want to associate to this relation a polynomial following the guidelines of [7]. Let us first construct a coefficient ring for these polynomials.

If K is a knot and $N(K)$ is an open regular neighborhood of K in S^3, then the module $RT_t(S^3\backslash N(K))$ has also a module structure over the skein algebra of the boundary, $RT_t(\partial N(K))$, defined by gluing the cylinder over the boundary to the knot complement. But $\partial N(K)$ is the 2-dimensional torus $\mathbb{T}^2$, and so $RT_t(S^3\backslash N(K)) = RT_t(\mathbb{T}^2)$. It has been shown in [11] that the latter is canonically isomorphic to $K_t(\mathbb{T}^2)$ and the multiplicative structure of this algebra has been exhibited in [5]. When identifying the boundary of the knot complement with the standard torus, we require that the curve $(1,0)$ in $\mathbb{T}^2$ is identified with the longitude of the knot K while the curve $(0,1)$ is identified with the meridian of the knot. Note that $(0,1)\cdot\emptyset = x$, where x is the curve from Figure 1. If we consider the subalgebra of $RT_t(\mathbb{T}^2)$ generated by $(0,1)$, then this is the same as the polynomial ring $R=\mathbb{C}[t,t^{-1}][x]$, and we think of x as both the variable of the polynomial ring and as the curve x. Now we view $RT_t(S^3\backslash N(K))$ as an R-module.

For a function

$$f:\mathbb{Z}\to RT_t(S^3\backslash N(K)),$$

following [7] we define the operators

$$(Mf)(n)=t^{2n}f(n),\quad (Lf)(n)=f(n+1).$$

The operators L and M and their inverses generate the ring $\mathcal{T}_x$ which is the quotient

$$R[L,L^{-1},M,M^{-1}]/(LM=t^2ML, LL^{-1}=L^{-1}L=1, MM^{-1}=M^{-1}M=1)$$

A *recurrence polynomial* of the function f is an element $P\in\mathcal{T}_x$ such that $Pf=0$.

Turning to our particular example we notice that a recurrence polynomial in $R[L,L^{-1},M,M^{-1}]$ for the function

$$f:\mathbb{Z}\to S_n(y),$$

whose values are the colored Jones polynomials of the curve y in the complement of the $(2p+1,2)$ torus knot, is

$$t^{-3}L^{p+1}M^{-1}+t^{3}L^{-p}M-t^{-1}(x^2-2)L^{p}M^{-1}-t(x^2-2)L^{-p-1}M \\ +tL^{p-1}M^{-1}+t^{-1}L^{-p-2}M.$$

We can then find a recurrence polynomial in the variables L and M only, namely

$$\begin{aligned}
&t^{2p+5}L^{p+2}M(t^{-3}L^{p+1}M^{-1}+t^{3}L^{-p}M-t^{-1}(x^2-2)L^{p}M^{-1}\\
&-t(x^2-2)L^{-p-1}M+tL^{p-1}M^{-1}+t^{-1}L^{-p-2}M)\\
&=L^{2p+3}+t^{4p+8}L^2M^2-t^4(x^2-2)L^{2p+2}-t^{4p+8}(x^2-2)LM^2\\
&+t^8L^{2p+1}+t^{4p+8}M^2\\
&=L^{2p+3}-t^4(x^2-2)L^{2p+2}+t^8L^{2p+1}+t^{4p+8}[L^2-(x^2-2)L+1]M^2\\
&=[L^2-t^4(x^2-2)L+t^8]L^{2p+1}+t^{4p+8}[L^2-(x^2-2)L+1]M^2.
\end{aligned}$$

Remark 4.1. When $t=1$ (the "classical" setting of the Reshetikhin-Turaev theory as opposed to $t=-1$ for the Kauffman bracket) this polynomial factors as

$$(L^2-(x^2-2)L+1)(L^{2p+1}+M^2).$$

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