FROM CLASSICAL THETA FUNCTIONS TO TOPOLOGICAL QUANTUM FIELD THEORY

RĂZVAN GELCA AND ALEJANDRO URIBE

Abstract. Abelian Chern-Simons theory relates classical theta functions to the topological quantum field theory of the linking number of knots. In this paper we explain how to derive the constructs of abelian Chern-Simons theory directly from the theory of classical theta functions. It turns out that the theory of classical theta functions, from the representation theoretic point of view of A. Weil, is just an instance of Chern-Simons theory. The group algebra of the finite Heisenberg group is described as an algebra of curves on a surface, and its Schrödinger representation is obtained as an action on curves in a handlebody. A careful analysis of the discrete Fourier transform yields the Reshetikhin-Turaev formula for invariants of 3-dimensional manifolds. In this context, we give an explanation of why the composition of discrete Fourier transforms and the non-additivity of the signature of 4-dimensional manifolds under gluings obey the same formula.

1. Introduction

In this paper we construct the abelian Chern-Simons topological quantum field theory directly from the theory of classical theta functions.

It has been known for years, in the context of abelian Chern-Simons theory, that classical theta functions are related to low dimensional topology [2], [33]. Abelian Chern-Simons theory is considerably simpler than its non-abelian counterparts, and has been studied thoroughly (see for example [16] and [17]). Here we do not start with abelian Chern-Simons theory, but instead give a direct construction of the associated topological quantum field theory based on the theory of classical theta functions and using skein modules.

We consider classical theta functions in the representation theoretic point of view introduced by André Weil [31]. From this point of view, the space of theta functions is endowed with an action of a finite Heisenberg group (the Schrödinger representation), which induces, via a Stone-von Neumann theorem, the Hermite-Jacobi action of the modular group. All this structure is what we shall mean by the theory of theta functions.

1991 *Mathematics Subject Classification.* 14K25, 57R56, 57M25, 81S10, 81T45.
Key words and phrases. classical theta functions, abelian Chern-Simons theory.
Research of the first author supported by the NSF, award No. DMS 0604694.
Research of the second author supported by the NSF, award No. DMS 0805878.

We show how the finite Heisenberg group, or rather its group algebra and its Schrödinger representation on the space of theta functions, lead to algebras of curves on surfaces and their actions on spaces of curves in handlebodies. These notions are formalized using skein modules.

The Hermite-Jacobi representation of the modular group on theta functions is a discrete analogue of the metaplectic representation. The modular group acts by automorphisms that are a particular case of the Fourier-Mukai transform; in this paper we will refer to them as discrete Fourier transforms. We will show that discrete Fourier transforms can be expressed as linear combinations of curves. A careful analysis of their structure and of their relationship to the Schrödinger representation yields the Reshetikhin-Turaev formula [24] of invariants of 3-dimensional manifolds, for $U(1)$ Chern-Simons theory.

As a corollary of our point of view we obtain an explanation of why the composition of discrete Fourier transforms and the non-additivity of the signature of 4-dimensional manifolds obey the same formula.

The paper uses results and terminology from the theory of theta functions, quantum mechanics, and low dimensional topology. To make it accessible to different audiences we include a fair amount of detail.

In Section 2 we review the theory of classical theta functions on the Jacobian variety of a surface. The action of the finite Heisenberg group on theta functions can be defined via Weyl quantization of the Jacobian variety in the holomorphic polarization. In fact it has been discovered in recent years that Chern-Simons theory is related to Weyl quantization [8], [1], and this was the starting point of our paper. The next section exhibits the algebraic model of theta functions, the action of the finite Heisenberg group, and the Hermite-Jacobi action.

Up to here, played no role, but in Section 4 we show that the algebraic model of theta functions is topological in nature. We reformulate it using algebras of curves on surfaces, together with their action on skeins of curves in handlebodies which are associated to the linking number.

In Section 5 we derive a formula for the discrete Fourier transform as a skein. This formula is interpreted in terms of surgery in the cylinder over the surface. Section 6 analizes the exact Egorov identity which relates the Hermite-Jacobi action to the Schrödinger representation. This analysis shows that the topological operation of handle slides is allowed over the skeins that represent discrete Fourier transforms, and this yields in the next section the abelian Chern-Simons invariants of 3-dimensional manifolds defined by Murakami, Ohtsuki, and Okada. We point out that the abovementioned formula was introduced in an ad-hoc manner by its authors [19], our paper shows how to derive it naturally.

Section 8 shows how to associate to the discrete Fourier transform a 4-dimensional manifold. This will explain why the cocycle of the Hermite-Jacobi action is related to that governing the non-additivity of the signature of 4-dimensional manifolds [30]. Section 9 should be taken as a conclusion; it brings everything to the context of Chern-Simons theory.

2. Theta functions

We start with a closed genus g Riemann surface Σ_g, and consider a canonical basis $a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g$ for $H_1(\Sigma_g, \mathbb{R})$, like the one shown in Figure 1. To it we associate a basis in the space of holomorphic differential 1-forms $\zeta_1, \zeta_2, \ldots, \zeta_g$, defined by the conditions $\int_{a_k} \zeta_j = \delta_{jk}$, $j, k = 1, 2, \ldots, g$. The matrix Π with entries

$$\pi_{jk} = \int_{b_k} \zeta_j, \quad j, k = 1, \ldots, g,$$

is symmetric with positive definite imaginary part. This means that if $\Pi = X + iY$, then $X = X^T$, $Y = Y^T$ and $Y > 0$. The $g \times 2g$ matrix (I_g, Π) is called the period matrix of Σ_g, its columns $\lambda_1, \lambda_2, \ldots, \lambda_{2g}$, called periods, generate a lattice $L(\Sigma_g)$ in $\mathbb{C}^g = \mathbb{R}^{2g}$. The complex torus

$$\mathcal{J}(\Sigma_g) = \mathbb{C}^g / L(\Sigma_g)$$

is the *Jacobian variety* of Σ_g. The map

$$\sum_j \alpha_j a_j + \sum_j \beta_j b_j \mapsto (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$$

induces a homeomorphism $H_1(\Sigma_g, \mathbb{R})/H_1(\Sigma_g, \mathbb{Z}) \to \mathcal{J}(\Sigma_g)$.

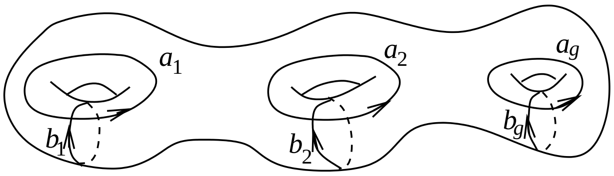

Figure 1

The complex coordinates $z = (z_1, z_2, \ldots, z_g)$ on $\mathcal{J}(\Sigma_g)$ are those inherited from $\mathbb{C}^g$. We introduce real coordinates $(x, y) = (x_1, x_2, \ldots, x_g, y_1, y_2, \ldots, y_g)$ by imposing $z = x + \Pi y$. A fundamental domain for the period lattice, in terms of the (x, y) coordinates, is simply $\{(x, y) \in [0, 1]^{2g}\}$. Moreover, $\mathcal{J}(\Sigma_g)$ has a canonical symplectic form, which in the (x, y)-coordinates is given by

$$\omega = 2\pi \sum_{j=1}^{g} dx_j \wedge dy_j.$$

The Jacobian variety with the complex structure and symplectic form ω is a Kähler manifold. The symplectic form induces a Poisson bracket for smooth functions on the Jacobian, given by $\{f, g\} = \omega(X_f, X_g)$, where X_f denotes the Hamiltonian vector field defined by $df(\cdot) = \omega(X_f, \cdot)$.

The *classical theta functions* show up when quantizing $\mathcal{J}(\Sigma_g)$ in the complex polarization, in the direction of this Poisson bracket. For the purpose of this paper, we perform this quantization in the case where Planck's constant is the reciprocal of an *even* positive integer: $h = \frac{1}{N}$ where $N = 2r$, $r \in \mathbb{N}$. The Hilbert space of the quantization consists of the holomorphic sections of a line bundle obtained as the tensor product of a line bundle with curvature $N\omega$ and the square root of the canonical line bundle. The latter is trivial for the complex torus and we ignore it. The line bundle with curvature $N\omega$ is the tensor product of a flat line bundle and the line bundle defined by the cocycle $\Lambda : \mathbb{C}^g \times L(\Sigma_g) \to \mathbb{C}^*$,

$$\begin{aligned}\Lambda(z, \lambda_j) &= 1\\ \Lambda(z, \lambda_{g+j}) &= e^{-2\pi i N z_j - \pi i N \pi_{jj}},\end{aligned}$$

$j = 1, 2, \ldots, g$. (See e.g. §4.1.2 of [5] for a discussion of how this cocycle gives rise to a line bundle with curvature $N\omega$.) We choose the trivial flat bundle to tensor with. As such, the Hilbert space can be identified with the space of entire functions on $\mathbb{C}^g$ satisfying the periodicity conditions

$$\begin{aligned}f(z + \lambda_j) &= f(z)\\ f(z + \lambda_{g+j}) &= e^{-2\pi i N z_j - \pi i N \pi_{jj}} f(z).\end{aligned}$$

We denote this space by $\boldsymbol{\Theta}_N^{\Pi}(\Sigma_g)$; its elements are called classical theta functions.[1] A basis of $\boldsymbol{\Theta}_N^{\Pi}(\Sigma_g)$ consists of the theta series

$$\theta_\mu^{\Pi}(z) = \sum_{n \in \mathbb{Z}^g} e^{2\pi i N[\frac{1}{2}(\frac{\mu}{N}+n)^T \Pi(\frac{\mu}{N}+n) + (\frac{\mu}{N}+n)^T z]}, \quad \mu \in \{0, 1 \ldots, N-1\}^g.$$

The definition of theta series will be extended for convenience to all $\mu \in \mathbb{Z}^g$, by $\theta_{\mu + N\mu'} = \theta_\mu$ for any $\mu' \in \mathbb{Z}^g$. Hence the index μ can be taken in $\mathbb{Z}_N^g$.

The inner product that makes the theta series into an orthonormal basis is

$$\langle f, g \rangle = (2N)^{g/2} \det(Y)^{1/2} \int_{[0,1]^{2g}} f(x,y)\overline{g(x,y)} e^{-2\pi N y^T Y y} dx dy. \qquad (2.1)$$

That the theta series form an orthonormal basis is a corollary of the proof of Proposition 2.1 below.

To define the operators of the quantization, we use the Weyl quantization method. This quantization method can be defined only on complex vector spaces, the Jacobian variety is the quotient of such a space by a discrete group, and the quantization method goes through. As such, the operator $\mathrm{Op}(f)$ associated to a function f on $\mathcal{J}(\Sigma_g)$ is the Toeplitz operator with symbol $e^{\frac{-h\Delta_\Pi}{4}} f$ ([7] Proposition 2.97)[2], where Δ_Π is the Laplacian on functions,

$$\Delta_\Pi = -d^* \circ d, \quad d : C^\infty(\mathcal{J}(\Sigma_g)) \to \Omega^1(\mathcal{J}(\Sigma_g)).$$

[1] In fact the precise terminology is canonical theta functions, classical theta functions being defined by a slight alteration of the periodicity condition. We use the name classical theta functions in this paper to emphasize the distinction with the non-abelian theta functions.

[2] The variable of f is not conjugated because we work in the momentum representation.

On a general Riemannian manifold this operator is given in local coordinates by the formula

$$\Delta_\Pi f = \frac{1}{\sqrt{\det(\mathbf{g})}} \frac{\partial}{\partial x^j} \left(\mathbf{g}^{jk} \sqrt{\det(\mathbf{g})} \frac{\partial f}{\partial x^k} \right),$$

where $\mathbf{g} = (\mathbf{g}_{jk})$ is the metric and $\mathbf{g}^{-1} = (\mathbf{g}^{jk})$. In the Kähler case, if the Kähler form is given in holomorphic coordinates by

$$\omega = \frac{i}{2} \sum_{j,k} \mathbf{h}_{jk}\, dz_j \wedge d\bar{z}_k,$$

then

$$\Delta_\Pi = 4 \sum_{j,k} \mathbf{h}^{jk}\, \frac{\partial^2}{\partial z_j \partial \bar{z}_k},$$

where $(\mathbf{h}^{jk}) = (\mathbf{h}_{jk})^{-1}$. In our situation, in the coordinates $z_j, \bar{z}_j$, $j = 1, 2, \ldots, g$, one computes that $(\mathbf{h}_{jk})^{-1} = Y^{-1}$ and therefore $(\mathbf{h}^{jk}) = Y$ (recall that Y is the imaginary part of the matrix Π). For Weyl quantization one introduces a factor of $\frac{1}{2\pi}$ in front of the operator. As such, the Laplace (or rather Laplace-Beltrami) operator Δ_Π is equal to

$$\frac{1}{2\pi} \sum_{j,\,k=1}^{g} Y_{jk} \left[(I_g + iY^{-1}X)\nabla_x - iY^{-1}\nabla_y \right]_j \left[(I_g - iY^{-1}X)\nabla_x + iY^{-1}\nabla_y \right]_k .$$

(A word about the notation being used: ∇ represents the usual (column) vector of partial derivatives in the indicated variables, so that each object in the square brackets is a column vector of partial derivatives. The subindices j, k are the corresponding components of those vectors.) A tedious calculation that we omit results in the following formula for the Laplacian in the (x, y) coordinates:

$$\begin{aligned} 2\pi\Delta_\Pi = \sum (Y + XY^{-1}X)_{jk}\, & \frac{\partial^2}{\partial x_j \partial x_k} \\ & -2(XY^{-1})_{jk} \frac{\partial^2}{\partial x_j \partial y_k} + Y^{jk} \frac{\partial^2}{\partial y_j \partial y_k}. \end{aligned}$$

We will only need to apply Δ_Π explicitly to exponentials, as part of the proof of the following basic proposition. Note that the exponential function

$$e^{2\pi i (p^T x + q^T y)}$$

defines a function on the Jacobian provided $p, q \in \mathbb{Z}^g$.

Proposition 2.1. *The Weyl quantization of the exponentials is given by*

$$Op\left(e^{2\pi i(p^T x + q^T y)}\right) \theta_\mu^\Pi(z) = e^{-\frac{\pi i}{N} p^T q - \frac{2\pi i}{N} \mu^T q} \theta_{\mu+p}^\Pi(z).$$

Proof. Let us introduce some useful notation local to the proof. Note that N and Π are fixed throughout.

(1) $e(t) := \exp(2\pi i N t)$,

(2) For $n \in \mathbb{Z}^g$ and $\mu \in \{0, 1, \dots N-1\}^g$, $n_\mu := n + \frac{\mu}{N}$.
(3) $Q(n_\mu) := \frac{1}{2}(n_\mu^T \Pi n_\mu)$
(4) $E_{p,q}(x,y) = e^{2\pi i (p^T x + q^T y)} = e(\frac{1}{N}(p^T x + q^T y))$.

With these notations, in the (x, y) coordinates

$$\theta_\mu(x,y) = \sum_{n \in \mathbb{Z}^g} e(Q(n_\mu))\, e(n_\mu^T (x + \Pi y)).$$

We first compute the matrix coefficients of the Toeplitz operator with symbol $E_{p,q}$, that is

$$\langle E_{p,q}\theta_\mu\, ,\, \theta_\nu \rangle$$
$$= (2N)^{g/2} \det(Y)^{1/2} \int_{[0,1]^{2g}} E_{p,q}(x,y)\theta_\mu(x,y)\overline{\theta_\nu(x,y)}\, e^{-2\pi N y^T Y y}\, dxdy.$$

Then a calculation shows that

$$E_{p,q}(x,y)\theta_\mu(x,y)\overline{\theta_\nu(x,y)}$$
$$= \sum_{m,n \in \mathbb{Z}^g} e\Big[Q(n_\mu) - \overline{Q(m_\nu)} + (n_{\mu+p} - m_\nu)^T x + \big(\frac{q^T}{N} + n_\mu^T \Pi - m_\nu^T \overline{\Pi}\big) y\Big].$$

The integral over $x \in [0,1]^g$ of the (m, n) term will be non-zero iff

$$N\Big(n_{\mu+p} - m_\nu\Big) = \mu + p - \nu + N(n - m) = 0,$$

in which case the integral will be equal to one. Therefore $\langle E_{p,q}\theta_\mu\, ,\, \theta_\nu\rangle = 0$ unless

$$[\nu] = [\mu + p],$$

where the brackets represent equivalence classes in $\mathbb{Z}_N^g$. This shows that the Toeplitz operator with multiplier $E_{p,q}$ maps θ_μ to a scalar times $\theta_{\mu+p}$. We now compute the scalar.

Taking μ in the fundamental domain $\{0, 1, \cdots, N-1\}^g$ for $\mathbb{Z}_N^g$, there is a unique representative, ν, of $[\mu + p]$ in the same domain. This ν is of the form

$$\nu = \mu + p + N\kappa$$

for a unique $\kappa \in \mathbb{Z}^g$. With respect to the previous notation, $\kappa = n - m$. It follows that

$$\langle E_{p,q}\theta_\mu\, ,\, \theta_\nu\rangle = (2N)^{g/2} \det(Y)^{1/2} \sum_{n \in \mathbb{Z}^g} \int_{[0,1]^g} e\Big[Q(n_\mu) - \overline{Q(m_\nu)}$$
$$+ \big(\frac{q^T}{N} + n_\mu^T \Pi - m_\nu^T \overline{\Pi}\big) y + i y^T Y y\Big]\, dy,$$

where $m = n - \kappa$ in the nth term.

Using that $m_\nu = n_\mu + \frac{1}{N} p$, one gets:

$$Q(n_\mu) - \overline{Q(m_\nu)} = i n_\mu^T Y n_\mu - \frac{1}{N} p^T \overline{\Pi} n_\mu - \frac{1}{N^2} \overline{Q(p)}$$

and

$$n_\mu^T \Pi - m_\nu^T \overline{\Pi} = 2i n_\mu^T Y - \frac{1}{N} p^T \overline{\Pi},$$

and so we can write

$$\langle E_{p,q}\theta_\mu\,,\,\theta_\nu\rangle = (2N)^{g/2}\det(Y)^{1/2}\, e\Big[-\frac{1}{N^2}\overline{Q(p)}\Big] \sum_{n\in\mathbb{Z}^g}\int_{[0,1]^g} dy$$

$$e\Big[i n_\mu^T Y n_\mu - \frac{1}{N} p^T \overline{\Pi} n_\mu + \big(\frac{1}{N} q^T + 2i n_\mu^T Y - \frac{1}{N} p^T \overline{\Pi}\big) y + i y^T Y y\Big].$$

Making the change of variables $w := y + n_\mu$ in the summand n, the argument of the function e can be seen to be equal to

$$i w^T Y w + \frac{1}{N}\big(q^T - p^T\overline{\Pi}\big) w - \frac{1}{N} q^T n_\mu.$$

Since q and n are integer vectors,

$$e\Big(\frac{1}{N} q^T n_\mu\Big) = e^{-2\pi i q^T \mu/N}.$$

The dependence on n of the integrand is a common factor that comes out of the summation sign. The series now is of integral over the translates of $[0,1]^n$ that tile the whole space. Therefore

$$\langle E_{p,q}\theta_\mu\,,\,\theta_\nu\rangle =$$

$$(2N)^{g/2}\det(Y)^{1/2}\, e\Big[-\frac{1}{N^2}\overline{Q(p)}\Big] e^{-2\pi i q^T \mu/N} \int_{\mathbb{R}^g} e^{-2\pi N w^T Y w + 2\pi i \left(q^T - p^T\overline{\Pi}\right) w}\, dw.$$

A calculation of the integral[3] yields that it is equal to

$$\Big(\frac{1}{2N}\Big)^{g/2} \det(Y)^{-1/2}\, e^{-\frac{\pi}{2N}(q^T - p^T\overline{\Pi})Y^{-1}(q-\overline{\Pi}p)}.$$

and so

$$\langle E_{p,q}\theta_\mu\,,\,\theta_\nu\rangle = e^{-\frac{\pi i}{N} p^T \overline{\Pi} p}\, e^{-2\pi i q^T \mu/N}\, e^{-\frac{\pi}{2N}(q^T - p^T\overline{\Pi})Y^{-1}(q-\overline{\Pi}p)}.$$

The exponent on the right-hand side is $(-\pi/N)$ times

$$\begin{aligned}
&2iq^T\mu + ip^T(X-iY)p + \frac{1}{2}\Big([q^T - p^T(X-iY)]Y^{-1}[q-(X-iY)p]\Big)\\
&= 2iq^T\mu + ip^T(X-iY)p + \frac{1}{2}\Big([q^TY^{-1} - p^TXY^{-1} + ip^T][q - Xp + iYp]\Big)\\
&= 2iq^T\mu + ip^T(X-iY)p + \frac{1}{2}\Big(q^TY^{-1}q - 2q^TY^{-1}Xp + 2iq^Tp\\
&\quad + p^TXY^{-1}Xp - 2ip^TXp - p^TYp\Big)\\
&= 2iq^T\mu + iq^Tp + \frac{1}{2}\mathcal{R}
\end{aligned}$$

where

$$\mathcal{R} := q^TY^{-1}q - 2q^TY^{-1}Xp + p^T(XY^{-1}X + Y)p.$$

[3] $\int_{\mathbb{R}^g} e^{-x^TAx + b^Tx}\,dx = \left(\frac{\pi^g}{\det A}\right)^{1/2} e^{\frac{1}{4}b^TA^{-1}b}$

That is,

$$\langle E_{p,q}\theta_\mu \,,\, \theta_\nu\rangle = e^{-\frac{2\pi i}{N}q^t\mu-\frac{\pi i}{N}q^Tp-\frac{\pi}{2N}\mathcal{R}}. \tag{2.2}$$

On the other hand, it is easy to check that $\Delta_\Pi(E_{p,q}) = -2\pi\mathcal{R}E_{p,q}$, and therefore

$$e^{-\frac{\Delta_\Pi}{4N}}(E_{p,q}) = e^{\frac{\pi}{2N}\mathcal{R}}\, E_{p,q},$$

so that, by (2.2)

$$\langle e^{-\frac{\Delta_\Pi}{4N}}(E_{p,q})\theta_\mu \,,\, \theta_\nu\rangle = e^{-\frac{2\pi i}{N}q^t\mu-\frac{\pi i}{N}q^Tp},$$

as desired. □

Let us focus on the group of quantized exponentials. First note that the symplectic form ω induces a nondegenerate bilinear form on $\mathbb{R}^{2g}$, which we denote also by ω, given by

$$\omega((p,q),(p',q')) = \sum_{j=1}^{g}\begin{vmatrix} p_j & q_j \\ p'_j & q'_j \end{vmatrix}. \tag{2.3}$$

As a corollary of Proposition 2.1 we obtain the following result.

Proposition 2.2. *Quantized exponentials satisfy the multiplication rule*

$$Op\left(e^{2\pi i(p^Tx+q^Ty)}\right)\, Op\left(e^{2\pi i(p'^Tx+q'^Ty)}\right)$$
$$= e^{\frac{\pi i}{N}\omega((p,q),(p',q'))}\, Op(e^{2\pi i((p+p')^Tx+(q+q')^Ty)}).$$

This prompts us to define the Heisenberg group

$$\mathbf{H}(\mathbb{Z}^g) = \{(p,q,k),\ p,q\in\mathbb{Z}^g, k\in\mathbb{Z}\}$$

with multiplication

$$(p,q,k)(p',q',k') = (p+p',q+q',k+k'+\omega((p,q),(p',q'))).$$

This group is a $\mathbb{Z}$-extension of $H_1(\Sigma_g,\mathbb{Z})$, with the standard inclusion of $H_1(\Sigma_g,\mathbb{Z})$ into it given by

$$\sum p_ja_j + \sum q_kb_k \mapsto (p_1,\ldots,p_g,q_1,\ldots,q_g,0).$$

The map

$$(p,q,k)\mapsto \mathrm{Op}\left(e^{\frac{\pi i}{N}k}e^{2\pi i(p^Tx+q^Ty)}\right)$$

defines a representation of $\mathbf{H}(\mathbb{Z}^g)$ on theta functions. To make this representation faithful, we factor it by its kernel.

Proposition 2.3. *The set of elements in $\mathbf{H}(\mathbb{Z}^g)$ that act on theta functions as identity operators is the normal subgroup consisting of the Nth powers of elements of the form (p,q,k) with k even. The quotient group is isomorphic to a finite Heisenberg group.*

Recall (cf. [20]) that a finite Heisenberg group H is a central extension

$$0 \to \mathbb{Z}_m \to H \to K \to 0$$

where K is a finite abelian group such that the commutator pairing $K \times K \to \mathbb{Z}_m$, $(k, k') \mapsto [\tilde{k}, \tilde{k}']$ ($\tilde{k}$, and $\tilde{k}'$ being arbitrary lifts of k and k' to H) identifies K with the group of homomorphisms from K to $\mathbb{Z}_m$.

Proof. By Proposition 2.1,

$$(p, q, k)\theta_\mu^\Pi(z) = e^{-\frac{\pi i}{N}p^Tq - \frac{2\pi i}{N}\mu^T q + \frac{\pi}{N}k}\theta_{\mu+p}^\Pi(z).$$

For (p, q, k) to act as the identity operator, we should have

$$e^{-\frac{\pi i}{N}p^Tq - \frac{2\pi i}{N}\mu^T q}\theta_{\mu+p}^\Pi(z) = \theta_\mu^\Pi(z)$$

for all $\mu \in \{0, 1, \ldots, N-1\}^g$. Consequently, p should be in $N\mathbb{Z}^g$. Then p^Tq is a multiple of N, so the coefficient $e^{-\frac{\pi i}{N}p^Tq - \frac{2\pi i}{N}\mu^T q + \frac{\pi i}{N}k}$ equals $\pm e^{-\frac{2\pi i}{N}\mu^T q + \frac{\pi i}{N}k}$. This coefficient should be equal to 1. For $\mu = (0, 0, \ldots, 0)$ this implies that $-p^Tq + k$ should be an even multiple of N. But then by varying μ we conclude that q is a multiple of N. Because N is even, it follows that p^Tq is an even multiple of N, and consequently k is an even multiple of N. Thus any element in the kernel of the representation must belong to $N\mathbb{Z}^{2g} \times (2N)\mathbb{Z}$. It is easy to see that any element of this form is in the kernel. These are precisely the elements of the form $(p, q, k)^N$ with k even.

The quotient of $\mathbf{H}(\mathbb{Z}^g)$ by the kernel of the representation is a $\mathbb{Z}_{2N}$-extension of the finite abelian group $\mathbb{Z}_N^{2g}$, thus is a finite Heisenberg group. This group is isomorphic to

$$\{(p, q, k) \mid p, q \in \mathbb{Z}_N^g, k \in \mathbb{Z}_{2N}\}$$

with the multiplication rule

$$(p, q, k)(p', q', k') = (p + p', q + q', k + k' + 2pq').$$

The isomorphism is induced by the map $F : \mathbf{H}(\mathbb{Z}^g) \to \mathbb{Z}_N^{2g} \times \mathbb{Z}_{2N}$,

$$F(p, q, k) = (p \bmod N, q \bmod N, k + pq \bmod 2N). \quad \square$$

We denote by $\mathbf{H}(\mathbb{Z}_N^g)$ this finite Heisenberg group and by $\exp(p^TP + q^TQ + kE)$ the image of (p, q, k) in it. The representation of $\mathbf{H}(\mathbb{Z}_N^g)$ on the space of theta functions is called the *Schrödinger representation*. It is an analogue, for the case of the $2g$-dimensional torus, of the standard Schrödinger representation of the Heisenberg group with real entries on $L^2(\mathbb{R})$. In particular we have

$$\begin{aligned} &\exp(p^TP)\theta_\mu^\Pi(z) = \theta_{\mu+p}^\Pi(z) \\ &\exp(q^TQ)\theta_\mu^\Pi(z) = e^{-\frac{2\pi i}{N}q^T\mu}\theta_\mu^\Pi(z) \\ &\exp(kE)\theta_\mu^\Pi(z) = e^{\frac{\pi i}{N}k}\theta_\mu^\Pi(z). \end{aligned} \tag{2.4}$$

Theorem 2.4. *(Stone-von Neumann) The Schrödinger representation of* $\mathbf{H}(\mathbb{Z}_N^g)$ *is the* unique *(up to an isomorphism) irreducible unitary representation of this group with the property that* $\exp(kE)$ *acts as* $e^{\frac{\pi i}{N}k}Id$ *for all* $k \in \mathbb{Z}$.

Proof. Let $X_j = \exp(P_j)$, $Y_j = \exp(Q_j)$, $j = 1, 2, \ldots, g$, $Z = \exp(E)$. Then $X_jY_j = Z^2Y_jX_j$, $X_jY_k = Y_kX_j$ if $j \neq k$, $X_jX_k = X_kX_j$, $Y_jY_k = Y_kY_j$, $ZX_j = X_jZ$, $ZY_j = Y_jZ$, for all i, j, and $X_j^N = Y_j^N = Z^{2N} = Id$ for all j. Because $Y_1, Y_2, \ldots, Y_g$ commute pairwise, they have a common eigenvector v. And because $Y_j^N = Id$ for all j, the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_g$ of v with respect to the $Y_1, Y_2, \ldots, Y_g$ are roots of unity. The equalities

$$\begin{aligned} Y_jX_jv &= e^{-\frac{2\pi i}{N}}X_jY_j = e^{-\frac{2\pi i}{N}}\lambda_jX_jv, \\ Y_jX_kv &= X_kY_jv = \lambda_jX_kv, \quad \text{if } j \neq k \end{aligned}$$

show that by applying X_j's repeatedly we can produce an eigenvector v_0 of the commuting system $Y_1, Y_2, \ldots, Y_g$ whose eigenvalues are all equal to 1. The irreducible representation is spanned by the vectors $X_1^{n_1}X_2^{n_2}\cdots X_g^{n_g}v_0$, $n_i \in \{0, 1, \ldots, N-1\}$. Any such vector is an eigenvector of the system $Y_1, Y_2, \ldots, Y_g$, with eigenvalues respectively $e^{\frac{2\pi i}{N}n_1}$, $e^{\frac{2\pi i}{N}n_2}, \ldots,$ $e^{\frac{2\pi i}{N}n_g}$. So these vectors are linearly independent and form a basis of the irreducible representation. It is not hard to see that the action of $\mathbf{H}(\mathbb{Z}_N^g)$ on the vector space spanned by these vectors is the Schrödinger representation. □

Proposition 2.5. *The operators* $Op\left(e^{2\pi i(p^Tx+q^Ty)}\right)$, $p, q \in \{0, 1, \ldots, N-1\}^g$ *form a basis of the space of linear operators on* $\mathbf{\Theta}_N^{\Pi}(\Sigma_g)$.

Proof. For simplicity, we will show that the operators

$$e^{\frac{\pi i}{N}p^Tq}\mathrm{Op}\left(e^{2\pi i(p^Tx+q^Ty)}\right), \quad p, q \in \{0, 1, \ldots, N-1\}^g,$$

form a basis. Denote by $M_{p,q}$ the respective matrices of these operators in the basis $(\theta_\mu^\Pi)_\mu$. For a fixed p, the nonzero entries of the matrices $M_{p,q}$, $q \in \{0, 1, \ldots, N-1\}^g$ are precisely those in the slots $(m, m+p)$, with $m \in \{0, 1, \ldots, N-1\}^g$ (here $m+p$ is taken modulo N). If we vary m and q and arrange these nonzero entries in a matrix, we obtain the gth power of a Vandermonde matrix, which is nonsingular. We conclude that for fixed p, the matrices $M_{p,q}$, $q \in \{0, 1, \ldots, N-1\}^g$ form a basis for the vector space of matrices with nonzero entries in the slots of the form $(m, m+p)$. Varying p, we obtain the desired conclusion. □

Corollary 2.6. *The algebra* $L(\mathbf{\Theta}_N^\Pi(\Sigma_g))$ *of linear operators on the space of theta functions is isomorphic to the algebra obtained by factoring* $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ *by the relation* $(0, 0, 1) = e^{\frac{i\pi}{N}}$.

Let us now recall the action of the *modular group* on theta functions. The modular group, known also as the *mapping class group*, of a simple closed surface Σ_g is the quotient of the group of homemorphisms of Σ_g by the subgroup of homeomorphisms that are isotopic to the identity map. It is at this point where it is essential that N is *even*.

The mapping class group acts on the Jacobian in the following way. An element h of this group induces a linear automorphism h_* of $H_1(\Sigma_g, \mathbb{R})$. The matrix of h_* has integer entries, determinant 1, and satisfies $h_* J_0 {h_*}^T = J_0$, where $J_0 = \begin{pmatrix} 0 & I_g \\ I_g & 0 \end{pmatrix}$ is the intersection form in $H_1(\Sigma_g, \mathbb{R})$. As such, h_* is a *symplectic linear automorphism* of $H_1(\Sigma_g, \mathbb{R})$, where the symplectic form is the intersection form. Identifying $\mathcal{J}(\Sigma_g)$ with $H_1(\Sigma_g, \mathbb{R})/H_1(\Sigma_g, \mathbb{Z})$, we see that h_* induces a symplectomorphism $\tilde{h}$ of $\mathcal{J}(\Sigma_g)$. The map $h \to \tilde{h}$ induces an action of the mapping class group of Σ_g on the Jacobian variety. This action can be described explicitly as follows. Decompose h_* into $g \times g$ blocks as

$$h_* = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then $\tilde{h}$ maps the complex torus defined by the lattice (I_g, Π) and complex variable z to the complex torus defined by the lattice (I_g, Π') and complex variable z', where $\Pi' = (\Pi C + D)^{-1}(\Pi A + B)$ and $z' = (\Pi C + D)^{-1} z$.

This action of the mapping class group of the surface on the Jacobian induces an action of the mapping class group on the finite Heisenberg group by

$$h \cdot \exp(p^T P + q^T Q + kE) = \exp[(Ap + Bq)^T P + (Cp + Dq)^T Q + kE].$$

The nature of this action is as follows: Since h induces a diffeomorphism on the Jacobian, we can compose h with an exponential and then quantize; the resulting operator is as above. We point out that if N were not even, this action would be defined only for h_* in the subgroup $Sp_\theta(2n, \mathbb{Z})$ of the symplectic group (this is because only for N even is the kernel of the map F defined in Proposition 2.3 preserved under the action of h_*).

As a corollary of Theorem 2.4, the representation of the finite Heisenberg group on theta functions given by $u \cdot \theta_\mu^\Pi = (h \cdot u)\theta_\mu^\Pi$ is equivalent to the Schrödinger representation, hence there is an automorphism $\rho(h)$ of $\boldsymbol{\Theta}_N^\Pi(\Sigma_g)$ that satisfies the *exact Egorov identity*:

$$h \cdot \exp(p^T P + q^T Q + kE) = \rho(h) \exp(p^T P + q^T Q + kE) \rho(h)^{-1}. \tag{2.5}$$

(Compare with [7], Theorem 2.15, which is the analogous statement in quantum mechanics in Euclidean space.) Moreover, by Schur's lemma, $\rho(h)$ is unique up to multiplication by a constant. We thus have a projective representation of the mapping class group of the surface on the space of classical theta functions that statisfies with the action of the finite Heisenberg group the exact Egorov identity from (2.5). This is the finite dimensional counterpart of the metaplectic representation, called the *Hermite-Jacobi action*.

Remark 2.7. We emphasize that the action of the mapping class group of Σ_g on theta functions factors through an action of the symplectic group $Sp(2n,\mathbb{Z})$.

Up to multiplication by a constant,

$$\rho(h)\theta_\mu^\Pi(z) = \exp[-\pi i z^T C(\Pi C + D)^{-1} z]\theta_\mu^{\Pi'}(z') \tag{2.6}$$

(cf. (5.6.3) in [20]). When the Riemann surface is the complex torus obtained as the quotient of the complex plane by the integer lattice, and $h = S$ is the map induced by a 90° rotation around the origin, then $\rho(S)$ is the discrete Fourier transform. In general, $\rho(h)$ is an instance of the general Fourier-Mukai transform, and like the metaplectic representation (see [15]), can be written as a composition of partial discrete Fourier transforms. For this reason, we will refer, throughout the paper, to $\rho(h)$ as a *discrete Fourier transform*.

3. Theta functions in the abstract setting

In this section we apply to the finite Heisenberg group the standard construction that gives the Schrödinger representation as left translations on the space of equivariant functions on the Heisenberg group (see for example [15]). This construction is standard in the representation theory of finite groups, it is the representation induced by an irreducible representation (i.e. a character) of a maximal abelian subgroup.

We start with a Lagrangian subspace of $H_1(\Sigma_g,\mathbb{R})$ with respect to the intersection form, which for our purpose will be the space spanned by the elements $b_1, b_2, \ldots, b_g$ of the canonical basis. Let $\mathbf{L}$ be the intersection of this space with $H_1(\Sigma_g,\mathbb{Z})$. Under the standard inclusion $H_1(\Sigma_g,\mathbb{Z}) \subset \mathbf{H}(\mathbb{Z}^g)$, $\mathbf{L}$ becomes an abelian subgroup of the Heisenberg group with integer entries. This factors to an abelian subgroup $\exp(\mathbf{L})$ of $\mathbf{H}(\mathbb{Z}_N^g)$. Let $\exp(\mathbf{L}+\mathbb{Z}E)$ be the subgroup of $\mathbf{H}(\mathbb{Z}_N^g)$ containing both $\exp(\mathbf{L})$ and the scalars $\exp(\mathbb{Z}E)$. Then $\exp(\mathbf{L}+\mathbb{Z}E)$ is a maximal abelian subgroup. Being abelian this group has only 1-dimensional irreducible representations, and these representations are characters.

In view of the Stone-von Neumann Theorem, we consider the induced representation defined by the the character $\chi_{\mathbf{L}} : \exp(\mathbf{L}+\mathbb{Z}E) \to \mathbb{C}$, $\chi_{\mathbf{L}}(l + kE) = e^{\frac{\pi i}{N}k}$. This representation is

$$\mathrm{Ind}_{\exp(\mathbf{L}+\mathbb{Z}E)}^{\mathbf{H}(\mathbb{Z}_N^g)} = C[\mathbf{H}(\mathbb{Z}_N^g)] \bigotimes_{\mathbb{C}[\exp(\mathbf{L}+\mathbb{Z}E)]} \mathbb{C}$$

with $\mathbf{H}(\mathbb{Z}_N^g)$ acting on the left in the first factor of the tensor product. Explicitly, the vector space of the representation is

the quotient of the group algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ by the vector subspace spanned by all elements of the form

$$u - \chi_{\mathbf{L}}(u')^{-1} u u'$$

with $u \in \mathbf{H}(\mathbb{Z}_N^g)$ and $u' \in \exp(\mathbf{L}+\mathbb{Z}E)$. We denote the vector space which is the quotient by $\mathcal{H}_{N,g}(\mathbf{L})$, and let $\pi_{\mathbf{L}} : \mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)] \to \mathcal{H}_{N,g}(\mathbf{L})$ be the quotient map.

The left regular action of the Heisenberg group $\mathbf{H}(\mathbb{Z}_N^g)$ on its group algebra descends to an action on $\mathcal{H}_{N,g}(\mathbf{L})$.

Proposition 3.1. *The map $\theta_\mu^\Pi(z) \mapsto \pi_{\mathbf{L}}(\exp(\mu^T P))$, $\mu \in \mathbb{Z}_N^g$ defines a unitary map between the space of theta functions $\mathbf{\Theta}_N^\Pi(\Sigma_g)$ and $\mathcal{H}_{N,g}(\mathbf{L})$, which intertwines the Schrödinger representation and the left action of the finite Heisenberg group.*

Proof. It is not hard to see that $\mathbf{\Theta}_N^\Pi(\Sigma_g)$ and $\mathcal{H}_{N,g}(\mathbf{L})$ have the same dimension. Also, for $\mu \neq \mu' \in \mathbb{Z}_N^g$, $\exp(\mu^T P)$ and $\exp(\mu'^T P)$ are not equivalent modulo $\exp(\mathbf{L}+\mathbb{Z}E)$, hence the map from the statement is an isomorphism of finite dimensional spaces. The norm of $\pi_{\mathbf{L}}(\exp(\mu^T P))$ is one, hence this map is unitary. We have

$$\exp(p^T P)\exp(\mu^T P) = \exp((p+\mu)^T)P)$$

and

$$\exp(q^T Q)\exp(\mu^T P) = e^{-\frac{\pi i}{N}q^T\mu}\exp(\mu^T P)\exp(q^T Q).$$

It follows that

$$\begin{aligned}\exp(p^T P)\pi_{\mathbf{L}}(\exp(\mu^T P)) &= \pi_{\mathbf{L}}((p+\mu)^T P)\\ \exp(q^T Q)\pi_{\mathbf{L}}(\exp(\mu^T P)) &= e^{-\frac{\pi i}{N}q^T\mu}\pi_{\mathbf{L}}(\exp(\mu^T P))\end{aligned}$$

in agreement with the Schrödinger representation (2.4). □

We rephrase the Hermite-Jacobi action in this setting. To this end, fix an element h of the mapping class group of the Riemann surface Σ_g. Let $\mathbf{L}$ be the subgroup of $H_1(\Sigma_g,\mathbb{Z})$ associated to a canonical basis as explained in the beginning of this section, which determines the maximal abelian subgroup $\exp(\mathbf{L}+\mathbb{Z}E)$.

The automorphism of $H_1(\Sigma_g, mathbbZ)$ defined by $a_j \mapsto h_*(a_j)$, $b_j \mapsto h_*(b_j)$, $j = 1,2,\ldots,g$, maps isomorphically $\mathbf{L}$ to $\mathbf{L}' = h_*(\mathbf{L})$, and thus allows us to identify in a canonical fashion $\mathcal{H}_{N,g}(\mathbf{L})$ and $\mathcal{H}_{N,g}(\mathbf{L}')$. Given this identification, we can view the discrete Fourier tranform as a map $\rho(h)$: $\mathcal{H}_{N,g}(\mathbf{L})$ and $\mathcal{H}_{N,g}(\mathbf{L}')$.

The discrete Fourier transform should map an element $\mathbf{u}$ mod $\ker(\pi_{\mathbf{L}})$ in $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]/\ker(\pi_{\mathbf{L}})$ to $\mathbf{u}$ mod $\ker(\pi_{h_*(\mathbf{L})})$ in $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]/\ker(\pi_{h_*(\mathbf{L})})$. In this form the map is not well defined, since different representatives for the class of $\mathbf{u}$ might yield different images. The idea is to consider all possible liftings of $\mathbf{u}$ and average them. For lifting the element $\mathbf{u}$ mod $\ker(\pi_{\mathbf{L}})$ we use the section of $\pi_{\mathbf{L}}$ defined as

$$s_{\mathbf{L}}(\mathbf{u} \bmod \ker(\pi_{\mathbf{L}})) = \frac{1}{2N^{g+1}}\sum_{u_1\in\exp(\mathbf{L}+\mathbb{Z}E)}\chi_{\mathbf{L}}(u_1)^{-1}\mathbf{u}u_1. \tag{3.1}$$

Then, up to multiplication by a constant

$$\rho(h)\,(\mathbf{u} \bmod \ker(\pi_{\mathbf{L}})) = \frac{1}{2N^{g+1}} \sum_{u_1\in \exp(\mathbf{L}+\mathbb{Z}E)} \chi_{\mathbf{L}}(u_1)^{-1}\mathbf{u}u_1 \bmod \ker(\pi_{h_*(\mathbf{L})}). \tag{3.2}$$

This formula identifies $\rho(h)$ as a Fourier transform. That this map agrees with the one defined by (2.6) up to multiplication by a constant follows from Schur's lemma, since both maps satisfy the exact Egorov identity (2.5).

4. A topological model for theta functions

The finite Heisenberg group, the equivalence relation defined by the kernel of $\pi_{\mathbf{L}}$, and the Schrödinger representation can be given topological interpretations, which we will explicate below. First, a heuristical discussion that explains how to arrive at the topological model.

The Heisenberg group. The group $\mathbf{H}(\mathbb{Z}^g)$ is a $\mathbb{Z}$-extension of the abelian group $H_1(\Sigma_g,\mathbb{Z})$. The bilinear form ω from (2.3), which defines the cocycle of this extension, is the intersection form in $H_1(\Sigma_g,\mathbb{Z})$. Cycles in $H_1(\Sigma_g,\mathbb{Z})$ can be represented by families of non-intersecting simple closed curves on the surface. As vector spaces, we can identify $\mathbb{C}[\mathbf{H}(\mathbb{Z}^g)]$ with $\mathbb{C}[t,t^{-1}]H_1(\Sigma_g,\mathbb{Z})$, where t is an abstract variable whose exponent equals the last coordinate in the Heisenberg group.

We start with an example on the torus. Here and throughout the paper we agree that (p,q) denotes the curve of slope q/p on the torus, oriented from the origin to the point (p,q) when viewing the torus as a quotient of the plane by integer translations. Consider the multiplication

$$(1,0)(0,1) = t(1,1),$$

shown graphically in Figure 2. The product curve $(1,1)$ can be obtained by cutting open the curves $(1,0)$ and $(0,1)$ at the crossing and joining the ends in such a way that the orientations agree. This operation is called *smoothing of the crossing*. It is easy to check that this works in general, for arbitrary surfaces, and so whenever we multiply two families of curves we introduce a coefficient of t raised to the algebraic intersection number of the two families and we smoothen all crossings. Such algebras of curves, with multiplication related to polynomial invariants of knots, were first considered in [27].

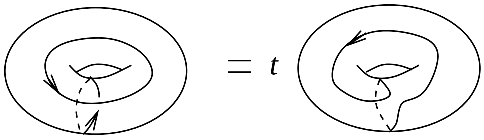

FIGURE 2

The group $\mathbf{H}(\mathbb{Z}_N^g)$ is a quotient of $\mathbf{H}(\mathbb{Z}^g)$, but can also be viewed as an extension of $H_1(\Sigma_g, \mathbb{Z}_N)$. As such, the elements of $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ can be represented by families of non-intersecting simple closed curves on the surface with the convention that any N parallel curves can be deleted. The above observation applies to this case as well, provided that we set $t = e^{\frac{i\pi}{N}}$.

It follows that the space of linear operators $L(\mathbf{\Theta}_N^\Pi(\Sigma_g))$ can be represented as an algebra of simple closed curves on the surface with the convention that any N parallel curves can be deleted. The multiplication of two families of simple closed curves is defined by introducing a coefficient of $e^{\frac{i\pi}{N}}$ raised to the algebraic intersection number of the two families and smoothing the crossings.

Theta functions. Next, we examine the space of theta functions, in its abstract framework from Section 3. To better understand the factorization modulo the kernel of $\pi_{\mathbf{L}}$, we look again at the torus. If the canonical basis is $(1,0)$ and $(0,1)$ with $\mathbf{L} = \mathbb{Z}(0,1)$, then an equivalence modulo $\ker(\pi_{\mathbf{L}})$ is shown in Figure 3. If we map the torus to the boundary of a solid torus in such a way that $\mathbf{L}$ becomes null-homologous, then the first and last curves from Figure 3 are homologous in the solid torus. To keep track of t we apply a standard method in topology which consists of framing the curves. A framed curve in a manifold is an embedding of an annulus. One can think of the curve as being one of the boundary components of the annulus, and then the annulus itself keeps track of the number of ways that the curve twists around itself. Changing the framing by a full twist amounts to multiplying by t or t^{-1} depending whether the twist is positive or negative. Then the equality from Figure 3 holds in the solid torus. It is not hard to check for a general surface Σ_g the equivalence relation modulo $\ker(\pi_{\mathbf{L}})$ is of this form in the handlebody bounded by Σ_g in such a way that $\mathbf{L}$ is null-homologous.

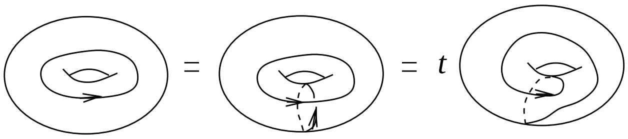

FIGURE 3

The Schrödinger representation. One can frame the curves on Σ_g by using the blackboard framing, namely by embedding the annulus in the surface. As such, the Schrödinger representation is the left action of an algebra of framed curves on a surface on the vector space of framed curves in the handlebody induced by the inclusion of the surface in the handlebody. We will make this precise using the language of skein modules [21].

Let M be an orientable 3-dimensional manifold, with a choice of orientation. A framed link in M is a smooth embedding of a disjoint union of finitely many annuli. The embedded annuli are called link components. We consider oriented framed links. The orientation of a link component is an orientation of one of the circles that bound the annulus. When M is the cylinder over a surface, we represent framed links as oriented curves with the blackboard framing, meaning that the annulus giving the framing is always parallel to the surface.

Let t be a free variable, and consider the free $\mathbb{C}[t,t^{-1}]$-module with basis the isotopy classes of framed oriented links in M including the empty link $\emptyset$. Let $\mathcal{S}$ be the the submodule spanned by all elements of the form depicted in Figure 4, where the two terms in each skein relation depict framed links that are identical except in an embedded ball, in which they look as shown. The ball containing the crossing can be embedded in any possible way. To normalize, we add to $\mathcal{S}$ the element consisting of the difference between the unknot in M and the empty link $\emptyset$. Recall that the unknot is an embedded circle that bounds an embedded disk in M and whose framing annulus lies inside the disk.

Definition 4.1. The result of the factorization of the free $\mathbb{C}[t,t^{-1}]$-module with basis the isotopy classes of framed oriented links by the submodule $\mathcal{S}$ is called the *linking number skein module* of M, and is denoted by $\mathcal{L}_t(M)$. The elements of $\mathcal{L}_t(M)$ are called *skeins*.

In other words, we are allowed to smoothen each crossing, to change the framing provided that we multiply by the appropriate power of t, and to identify the unknot with the empty link.

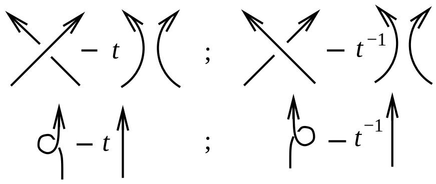

FIGURE 4

The "linking number" in the name is motivated by the fact that the skein relations from Figure 4 are used for computing the linking number. These skein modules were first introduced by Przytycki in [22] as one-parameter deformations of the group algebra of $H_1(M,\mathbb{Z})$. Przytycki computed them for all 3-dimensional manifolds.

Lemma 4.2. *Any trivial link component, namely any link component that bounds a disk disjoint from the rest of the link in such a way that the framing is an annulus inside the disk, can be deleted.*

Proof. The proof of the lemma is given in Figure 5. □

If $M=\Sigma_g\times[0,1]$, the cylinder over a surface, then the identification

$$\Sigma_g\times[0,1]\cup\Sigma_g\times[0,1]\approx\Sigma\times[0,1]$$

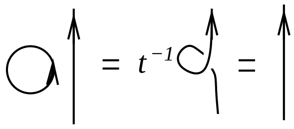

FIGURE 5

obtained by gluing the boundary component $\Sigma_g \times \{0\}$ in the first cylinder to the boundary component $\Sigma_g \times \{1\}$ in the second cylinder by the identity map induces a multiplication on $\mathcal{L}_t(\Sigma_g \times [0,1])$. This turns $\mathcal{L}_t(\Sigma_g \times [0,1])$ into an algebra, called the *linking number skein algebra.* As such, the product of two skeins is obtained by placing the first skein on top of the second. The nth power of an oriented, framed, simple closed curve consists then of n parallel copies of that curve. We adopt the same terminology even if the manifold is not a cylinder, so γ^n stands for n parallel copies of γ. Additionally, γ^{-1} is obtained from γ by reversing orientation, and $\gamma^{-n} = (\gamma^{-1})^n$.

Definition 4.3. For a fixed positive integer N, we define the *reduced linking number skein module* of the manifold M, denoted by $\widetilde{\mathcal{L}}_t(M)$, to be the quotient of $\mathcal{L}_t(M)$ obtained by imposing that $\gamma^N = \emptyset$ for every oriented, framed, simple closed curve γ, and by setting $t = e^{\frac{\pi i}{N}}$. As such, $L = L'$ whenever L' is obtained from L by removing N parallel link components.

Remark 4.4. As a rule followed throughout the paper, whenever we talk about skein modules, t is a free variable, and when we talk about reduced skein modules, t is a root of unity. The isomorphisms $\mathcal{L}_t(S^3) \cong \mathbb{C}[t, t^{-1}]$ and $\widetilde{\mathcal{L}}_t(S^3) \cong \mathbb{C}$ allow us to identify the linking number skein module of S^3 with the set of Laurent polynomials in t and the reduced skein module with $\mathbb{C}$.

For a closed, oriented, genus g surface Σ_g, consider a canonical basis of its first homology $a_1, a_2, \ldots, a_g, b_1, b_2, \ldots, b_g$ (see Section 1). The basis elements are oriented simple closed curves on the surface, which we endow with the blackboard framing. Let H_g be a genus g handlebody and $h_0 : \Sigma_g \to \partial H_g$ be a homeomorphism that maps $b_1, b_2, \ldots, b_g$ to null homologous curves. Then $a_1, a_2, \ldots a_g$ is a basis of the first homology of the handlebody. Endow these curves in the handlebody with the framing they had on the surface.

The linking number skein module of a 3-manifold M with boundary is a module over the skein algebra of a boundary component Σ_g. The module structure is induced by the identification

$$\Sigma_g \times [0,1] \cup M \approx M$$

where $\Sigma_g \times [0,1]$ is glued to M along $\Sigma_g \times \{0\}$ by the identity map. This means that the module structure is induced by identifying $\Sigma_g \times [0,1]$ with a regular neighborhood of the boundary of M. The product of a skein in a regular neighborhood of the boundary and a skein in the interior is the union of the two skeins. This module structure descends to relative skein modules.

In particular $\mathcal{L}_t(\Sigma_g \times [0,1])$ acts on the left on $\mathcal{L}_t(H_g)$, with the action induced by the homeomorpism $h_0 : \Sigma_g \to \partial H_g$, and this action descends to relative skein modules.

Theorem 4.5. *(a) The linking number skein module $\mathcal{L}_t(\Sigma_g \times [0,1])$ is a free $\mathbb{C}[t,t^{-1}]$-module with basis*

$$a_1^{m_1}a_2^{m_2}\cdots a_g^{m_g}b_1^{n_1}b_2^{n_2}\cdots b_g^{n_g}, \quad m_1, m_2, \ldots, m_g, n_1, n_2, \ldots, n_g \in \mathbb{Z}.$$

(b) The linking number skein module $\mathcal{L}_t(H_g)$ is a free $\mathbb{C}[t,t^{-1}]$-module with basis

$$a_1^{m_1}a_2^{m_2}\cdots a_g^{m_g}, \quad m_1, m_2, \ldots, m_g \in \mathbb{Z}.$$

(c) The algebras $\mathcal{L}_t(\Sigma_g \times [0,1])$ and $\mathbb{C}[\mathbf{H}(\mathbb{Z}^g)]$ are isomorphic, with the isomorphism defined by the map

$$t^k\gamma \mapsto ([\gamma], k).$$

where γ ranges over all skeins represented by oriented simple closed curves on Σ_g (with the blackboard framing) and $[\gamma]$ is the homology class of this curve in $H_1(\Sigma_g, \mathbb{Z}) = \mathbb{Z}^{2g}$.

Proof. Parts (a) and (b) are consequences of a general result in [22]; we include their proof for sake of completeness.
(a) We bring all skeins in the blackboard framing (of the surface). A skein t^kL, where L is an oriented framed link in the cylinder over the surface is equivalent modulo the skein relations to a skein $t^{k+m}L'$ where L' is an oriented framed link such that the projection of L' onto the surface has no crossings, and m is the sum of the positive crossings of the projection of L minus the sum of negative crossings. Moreover, because any embedded ball can be isotoped to be a cylinder over a disk, any skein t^nL'' that is equivalent to t^kL and in which L'' is a framed link with no crossings has the property that $n = k + m$.

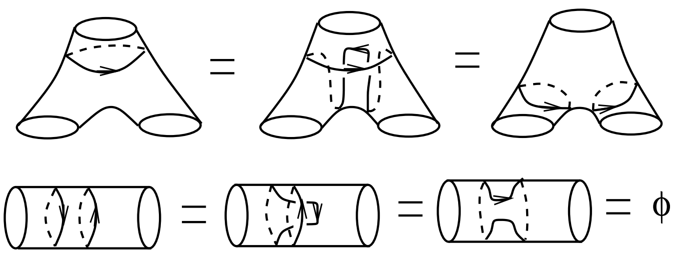

Figure 6

If L is an oriented link with blackboard framing whose projection onto the surface has no crossings, and if it is null-homologous in $H_1(\Sigma_g \times [0,1], \mathbb{Z})$, then L is equivalent modulo skein relations to the empty skein. This follows from the computations in Figure 6 given the fact that the closed orientable surface Σ_g can be decomposed into pairs of pants and annuli.

View Σ_g as a sphere with g punctured tori attached to it. Then L is equivalent to a link L' that consists of simple closed curves on these tori, which therefore is of the form

$$(p_1,q_1)^{k_1}(p_2,q_2)^{k_2}\cdots(p_g,q_g)^{k_g},$$

where (p_j,q_j) denotes the curve of slope p_j/q_j on the jth torus. This last link is equivalent, modulo skein relations, to

$$t^{\sum_j k_jp_jq_j}a_1^{k_1p_1}a_2^{k_2p_2}\cdots a_g^{k_gp_g}b_1^{k_1q_1}b_2^{k_2q_2}\cdots b_g^{k_gq_g}. \tag{4.1}$$

It is not hard to check that if we change the link by a Reidemeister move and then resolve all crossings, we obtain the same expression (4.1). Hence the result does only depend on the link and not on how it projects to the surface. This completes the proof of (a).

(b) This case is analogous to (a), given that a genus g handlebody is the cylinder over a disk with g punctures.

(c) Recall Corollary 2.6. The fact that the specified map is a linear isomorphism follows from (a). It is straightforward to check that the multiplication rule is the same. □

Remark 4.6. Explicitly, the map

$$t^ka_1^{m_1}a_2^{m_2}\cdots a_g^{m_g}b_1^{n_1}b_2^{n_2}\cdots b_g^{n_g}\mapsto(m_1,m_2,\ldots,m_g,n_1,n_2,\ldots,n_g,k),$$

for all $m_j,n_j,k\in\mathbb{Z}$, defines an isomorphism of the algebras $\mathcal{L}_t(\Sigma_g\times[0,1])$ and $\mathbb{C}[\mathbf{H}(\mathbb{Z}^g)]$.

Theorem 4.7. *(a) The reduced linking number skein module $\widetilde{\mathcal{L}}_t(\Sigma_g\times[0,1])$ is a finite dimensional vector space with basis*

$$a_1^{m_1}a_2^{m_2}\cdots a_g^{m_g}b_1^{n_1}b_2^{n_2}\cdots b_g^{n_g},\quad m_1,m_2,\ldots,m_g,n_1,n_2,\ldots,n_g\in\mathbb{Z}_N.$$

(b) The reduced linking number skein module $\widetilde{\mathcal{L}}_t(H_g)$ is a finite dimensional vector space with basis

$$a_1^{m_1}a_2^{m_2}\cdots a_g^{m_g},\quad m_1,m_2,\ldots,m_g\in\mathbb{Z}_N.$$

Moreover, there is a linear isomorphism of the spaces $\widetilde{\mathcal{L}}_t(H_g)$ and $\mathbf{\Theta}_N^\Pi(\Sigma_g)$ given by

$$\gamma\to\theta_{[\gamma]},$$

where γ ranges among all oriented simple closed curves in B_g^2 with the blackboard framing and $[\gamma]$ is the homology class of γ in $H_1(H_g,\mathbb{Z}_N^g)=\mathbb{Z}_N^g$.
(c) The algebra isomorphism defined in Theorem 4.5 factors to an algebra isomorphism of $\widetilde{\mathcal{L}}_t(\Sigma_g\times[0,1])$ and $L(\mathbf{\Theta}_N^\Pi(\Sigma_g))$, the algebra of linear operators on the space of theta functions. The isomorphism defined in (b) intertwines the left action of $\widetilde{\mathcal{L}}_t(\Sigma_g\times[0,1])$ on $\widetilde{\mathcal{L}}_t(H_g)$ and the Schrödinger representation.

Proof. (a) Using Theorem 4.5 we can identify $\mathcal{L}_t(\Sigma_g \times [0,1])$ with $\mathbb{C}[\mathbf{H}(\mathbb{Z}^g)]$. Setting t to a $2N$th root of unity and deleting any N parallel copies of a link component are precisely the relations by which we factor the Heisenberg group in Proposition 2.3. The only question is whether factoring by this additional relation before applying the other skein relations factors any further the skein module. However, we see that when a curve is crossed by N parallel copies of another curve, there is no distinction between overcrossings and undercrossings. Hence if a link contains N parallel copies of a curve, we can move this curve so that it is inside a cylinder $\Sigma_g \times [0, \epsilon]$ that does not contain other link components and we can also resolve all self-crossings of this curve without introducing factors of t. Then we can delete the curve without introducing new factoring relations. This proves (a).

For (b), notice that we factor $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$ to obtain $\widetilde{\mathcal{L}}_t(H_g)$ by the same relations by which we factor $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ to obtain $\mathcal{H}_{N,g}(\mathbf{L})$ in Section 3.

(c) An easy check shows that that the left action of the skein algebra of the cylinder over the surface on the skein module of the handlebody is the same as the one from Propositions 2.1 and 3.1. $\square$

Remark 4.8. The isomorphism between the reduced skein module of the handlebody and the space of theta functions is given explicitly by

$$a_1^{n_1} a_2^{n_2} \cdots a_g^{n_g} \mapsto \theta^{\Pi}_{n_1, n_2, \ldots, n_g}, \quad \text{for all } n_1, n_2, \ldots, n_g \in \mathbb{Z}_N.$$

In view of Theorem 4.7 we endow $\widetilde{\mathcal{L}}_t(H_g)$ with the Hilbert space structure of the space of theta functions.

Now we turn our attention to the discrete Fourier transform, and translate in topological language formula (3.2). Let h be an element of the mapping class group of the surface Σ_g. The action of the mapping class group on the finite Heisenberg group described in Section 2 becomes the action on skeins in the cylinder over the surface given by

$$\sigma \mapsto h(\sigma),$$

where $h(\sigma)$ is obtained by replacing each framed curve of the skein σ by its image through the homeomorphism h.

Consider h_1 and h_2 two homeomorphisms of Σ_g onto the boundary of the handlebody H_g such that $h_2 = h \circ h_1$. These homeomorphisms extend to embeddings of $\Sigma_g \times [0,1]$ into H_g which we denote by h_1 and h_2 as well. The homeomorphisms h_1 and h_2 define the action of the reduced skein module of the cylinder over the surface on the reduced skein module of the handlebody in two different ways, i.e. they give two different constructions of the Schrödinger representations. By the Stone-von Neumann theorem, these are unitary equivalent; they are related by the isomorphism $\rho(h)$. We want to give $\rho(h)$ a topological definition. To this end, let us take a closer look at the lifting map $s_{\mathbf{L}}$ defined in (3.1). First, it is standard to remark that one should only average over $\exp(\mathbf{L}+\mathbb{Z}E)/\exp(\mathbb{Z}E)=\exp(\mathbf{L})$, hence

$$s_{\mathbf{L}}(\mathbf{u} \bmod \ker(\pi_{\mathbf{L}})) = \frac{1}{N^g}\sum_{u_1\in\exp(\mathbf{L})}\mathbf{u}u_1.$$

If $\mathbf{u}=u\in\mathbf{H}(\mathbb{Z}_N^g)$, then, as a skein, u is of the form γ^k where γ is a framed oriented curve on $\Sigma_g=\partial H_g$ and k is an integer. The element $\hat{u}=u \bmod \ker(\pi_{\mathbf{L}}(u))$ is just this skein viewed as lying inside the handlebody; it consists of k parallel framed oriented curves in H_g.

On the other hand, as a skein, u_1 is of the form $b_1^{n_1}b_2^{n_2}\ldots b_g^{n_g}$, and as such, the product uu_1 becomes after smoothing all crossings another lift of the skein $\hat{u}$ to the boundary obtained by lifting γ to the boundary and then taking k parallel copies. Such a lift is obtained by pushing $\hat{u}$ inside a regular neighborhood of the boundary and then viewing it as an element in $\widetilde{\mathcal{L}}_t(\Sigma_g\times[0,1])$. When u_1 ranges over all $\exp(\mathbf{L})$ we obtain all possible lifts of $\hat{u}$ to the boundary obtained by pushing γ to the boundary and then taking k parallel copies.

Theorem 4.9. *For a skein of the form γ^k in $\widetilde{\mathcal{L}}_t(H_g)$, where γ is a curve in H_g and k a positive integer, consider all possible liftings to $\widetilde{\mathcal{L}}(\Sigma_g\times[0,1])$ using h_1, obtained by pushing the curve γ to the boundary and then taking k parallel copies. Take the average of these liftings and map the average by h_2 to $\widetilde{\mathcal{L}}(H_g)$. This defines a linear endomorphism of $\widetilde{\mathcal{L}}_t(H_g)$ which is, up to multiplication by a constant, the discrete Fourier transform $\rho(h)$.*

Proof. The map defined this way intertwines the Schrödinger representations defined by h_1 and h_2, so the theorem is a consequence of the Stone-von Neumann theorem. □

Example: We will exemplify this by showing how the S-map on the torus acts on the theta series

$$\theta_1^{\Pi}(z)=\sum_{n\in\mathbb{Z}} e^{2\pi i N\left[\frac{\Pi}{2}\left(\frac{1}{N}+n\right)^2+z\left(\frac{1}{N}+n\right)\right]}$$

(in this case Π is a just a complex number with positive imaginary part). This theta series is represented in the solid torus by the curve shown in Figure 7. The N linearly independent liftings of this curve to the boundary

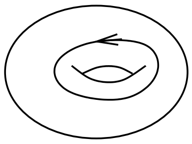

FIGURE 7

are shown in Figure 8.

The S-map sends these to those in Figure 9, which, after being pushed inside the solid torus, become the skeins from Figure 10.

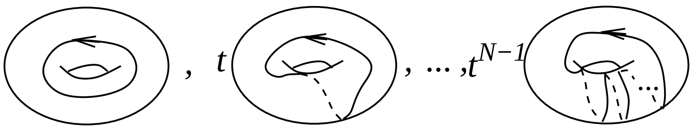

Figure 8

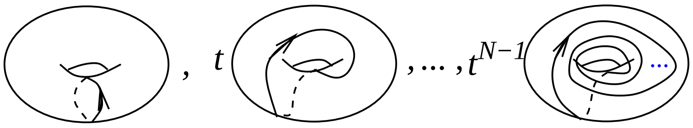

Figure 9

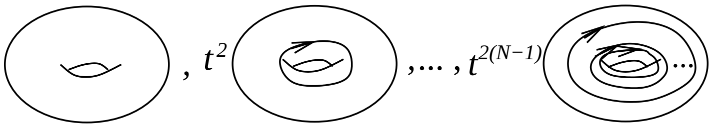

Figure 10

Note that in each skein the arrow points the opposite way as for $\theta_1(z)$. Using the identity $\gamma^N = \emptyset$, we can replace j parallel strands by $N - j$ parallel strands with opposite orientation. Hence these skeins are $t^{2j}\theta_{N-j}$, $j = 1, \ldots, N$ (note also that $\theta_0(z) = \theta_N(z)$). Taking the average we obtain

$$\rho(S)\theta_1(z) = \frac{1}{N}\sum_{j=0}^{N-1} e^{\frac{2\pi i j}{N}}\theta_{N-j}(z) = \frac{1}{N}\sum_{j=0}^{N-1} e^{-\frac{2\pi i j}{N}}\theta_j(z),$$

which is, up to a multiplication by a constant, the standard discrete Fourier transform of $\theta_1(z)$.

5. The discrete Fourier transform as a skein

As a consequence of Proposition 2.5, $\rho(h)$ can be represented as an element in $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$. Furthermore, Theorem 4.7 implies that $\rho(h)$ can be represented as left multiplication by a skein $\mathcal{F}(h)$ in $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$. The skein $\mathcal{F}(h)$ is unique up to a multiplication by a constant. We wish to find an explicit formula for it.

Theorem 4.7 implies that the action of the group algebra of the finite Heisenberg group can be represented as left multiplication by skeins. Using this fact, the exact Egorov identity (2.5) translates to

$$h(\sigma)\mathcal{F}(h) = \mathcal{F}(h)\sigma \tag{5.1}$$

for all skeins $\sigma \in \widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$.

By the Lickorish twist theorem (Chapter 9 in [23]), every homeomorphism of the surface Σ_g is isotopic to a product of Dehn twists along the $3g - 1$

curves depicted in Figure 11. We recall that a Dehn twist is the homemorphism obtained by cutting the surface along the curve, applying a full rotation on one side, then gluing back.

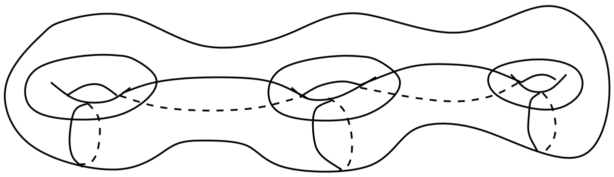

FIGURE 11

The curves from Figure 11 are nonseparating, and any two can be mapped into one another by a homeomorphism of the surface. Thus in order to understand $\mathcal{F}(h)$ in general, it suffices to consider the case $h = T$, the positive Dehn twist along the curve b_1 from Figure 1. The word positive means that after we cut the surface along b_1 we perform a full rotation of the part on the left in the direction of the arrow. Because $T(\sigma) = \sigma$ for all skeins that do not contain curves that intersect b_1, it follows that $\rho(T)$ commutes with all such skeins. It also commutes with the multiples of b_1 (viewed as a skein with the blackboard framing). Hence $\rho(T)$ commutes with all operators of the form $\exp(pP + qQ + kE)$ with p_1, the first entry of p, equal to 0. This implies that

$$\rho(T) = \sum_{j=0}^{N-1} c_j \exp(jQ_1).$$

To determine the coefficients c_j, we write the exact Egorov identity (2.5) for $\exp(P_1)$. Since $T \cdot \exp(P_1) = \exp(P_1 + Q_1)$ this identity reads

$$\exp(P_1 + Q_1) \sum_{j=0}^{N-1} c_j \exp(jQ_1) = \sum_{j=0}^{N-1} c_j \exp(jQ_1) \exp(P_1).$$

We transform this further into

$$\sum_{j=0}^{N-1} c_j e^{\frac{\pi i}{N} j} \exp[P_1 + (j+1)Q_1] = \sum_{j=0}^{N-1} c_j e^{-\frac{\pi i}{N} j} \exp(P_1 + jQ_1),$$

or, taking into account that $\exp(P_1) = \exp(P_1 + NQ_1)$,

$$\sum_{j=0}^{N-1} c_{j-1} e^{\frac{\pi i}{N}(j-1)} \exp(P_1 + jQ_1) = \sum_{j=0}^{N-1} c_j e^{-\frac{\pi i}{N} j} \exp(P_1 + jQ_1),$$

where $c_{-1} = c_{N-1}$. It follows that $c_j = e^{\frac{\pi i}{N}(2j-1)} c_{j-1}$ for all j. Normalizing so that $\rho(T)$ is a unitary map and $c_0 > 0$ we obtain $c_j = N^{-1/2} e^{\frac{\pi i}{N} j^2}$, and hence

$$\mathcal{F}(T) = N^{-1/2}\sum_{j=0}^{N-1} e^{\frac{\pi i}{N}j^2}\exp(jQ_1).$$

Turning to the language of skein modules, and taking into account that any Dehn twist is conjugate to the above twist by an element of the mapping class group, we conclude that if T is a positive Dehn twist along the simple closed curve γ on Σ_g, then

$$\mathcal{F}(T) = N^{-1/2}\sum_{j=0}^{N-1} t^{j^2}\gamma^j.$$

This is the same as the skein

$$\mathcal{F}(T) = N^{-1/2}\sum_{j=0}^{N-1}(\gamma^+)^j$$

where γ^+ is obtained by adding one full positive twist to the framing of γ (the twist is positive in the sense that, as skeins, $\gamma^+ = t\gamma$).

This skein has an interpretation in terms of surgery. Consider the curve $\gamma^+ \times \{1/2\} \subset \Sigma_g \times [0,1]$ with framing defined by the blackboard framing of γ^+ on Σ_g. Take a solid torus which is a regular neighborhood of the curve on whose boundary the framing determines two simple closed curves. Remove it from $\Sigma_g \times [0,1]$, then glue it back in by a homeomorphism that identifies its meridian (the curve that is null-homologous) to one of the curves determined by the framing. This operation, called surgery, yields a manifold that is homeomorphic to $\Sigma_g \times [0,1]$, such that the restriction of the homeomorphism to $\Sigma_g \times \{0\}$ is the identity map, and the restriction to $\Sigma_g \times \{1\}$ is the Dehn twist T.

The reduced linking number skein module of the solid torus H_1 is, by Theorem 4.7, an N-dimensional vector space with basis $\emptyset, a_1, \ldots, a_1^{N-1}$. Alternately, it is the vector space of 1-dimensional theta functions with basis $\theta_0^\Pi(z), \theta_1^\Pi(z), \ldots, \theta_{N-1}^\Pi(z)$, where Π in this case is a complex number with positive imaginary part. We introduce the element

$$\Omega = N^{-1/2}\sum_{j=0}^{N-1} a_1^j = N^{-1/2}\sum_{j=0}^{N-1}\theta_j^\tau(z) \tag{5.2}$$

in $\widetilde{\mathcal{L}}_t(H_1) = \mathbf{\Theta}_N^\Pi(\Sigma_1)$. As a diagram, Ω is the skein depicted in Figure 12 multiplied by $N^{-1/2}$. If S is the homemorphism on the torus induced by the 90° rotation of the plane when viewing the torus as the quotient of the plane by the integer lattice, then $\Omega = \rho(S)\emptyset$. In other words, Ω is the (standard) discrete Fourier transform of $\theta_0^\Pi(z)$.

For an arbitrary framed link L, we denote by $\Omega(L)$, the skein obtained by replacing each link component by Ω. In other words, $\Omega(L)$ is the sum of framed links obtained from L by replacing its components, in all possible

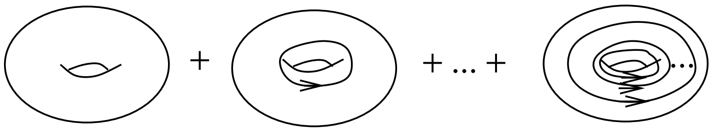

FIGURE 12

ways, by $0, 1, \ldots, N-1$ parallel copies. The skein Ω is called the coloring of L by Ω.

Here are two properties of Ω that will be used in the sequel.

Proposition 5.1. *a) The skein $\Omega(L)$ is independent of the orientations of the components of L.*
b) The skein relation from Figure 13 holds, where the n parallel strands point in the same direction.

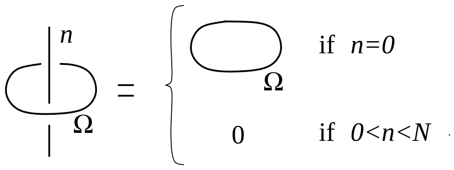

FIGURE 13

Proof. a) The computation in Figure 6 implies that if we switch the orientation on the j parallel curves that represent $\theta_j^{\Pi}(z)$ we obtain $\theta_{N-j}^{\Pi}(z)$. Hence by changing the orientation on all curves that make up Ω we obtain the skein

$$N^{-1/2}[\theta_0^{\Pi}(z) + \theta_{N-1}^{\Pi}(z) + \theta_{N-2}^{\Pi}(z) + \cdots + \theta_1^{\Pi}(z)],$$

which is, again, Ω.

b) When $n = 0$ there is nothing to prove. If $n \neq 0$, then by resolving all crossings in the diagram we obtain n vertical parallel strands with the coefficient

$$N^{-1/2} \sum_{j=0}^{N-1} t^{\pm 2nj} = N^{-1/2} \cdot \frac{t^{2Nj} - 1}{t^2 n - 1}.$$

where the signs in the exponents are either all positive, or all negative. Since t^2 is a primitive Nth root of unity, this is equal to zero. Hence the conclusion. $\square$

Up to this point we have proved the following result:

Lemma 5.2. *For a Dehn twist T, $\mathcal{F}(T)$ is the skein obtained by coloring the surgery framed curve γ^+ of T by Ω.*

Since by the Lickorish twist theorem every element h of the mapping class group is a product of twists, we obtain the following skein theoretic description of the discrete Fourier transform induced by the map h.

Proposition 5.3. *Let h be an element of the mapping class group of Σ_g obtained as a composition of Dehn twists $h = T_1 T_2 \cdots T_n$. Express each Dehn twist T_j by surgery on a curve γ_j as above, and consider the link $L_h = \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ which expresses h as surgery on the framed link L_h in $\Sigma_g \times [0,1]$. Then the discrete Fourier transform $\rho(h) : \widetilde{\mathcal{L}}_t(H_g) \to \widetilde{\mathcal{L}}_t(H_g)$ is given by*

$$\rho(h)\beta = \Omega(L_h)\beta.$$

6. The Egorov identity and handle slides

Next, we give the Egorov identity a topological interpretation in terms of handle slides. For this we look at its skein theoretical version (5.1). We start again with an example on the torus.

Example: For the positive twist T and the operator represented by the curve $(1,0)$ the exact Egorov identity reads

$$\rho(T)(1,0) = (1,1)\rho(T),$$

which is described graphically in Figure 14.

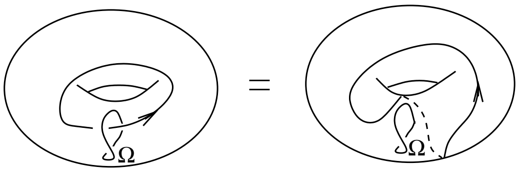

Figure 14

The diagram on the right is the same as the one from Figure 15. As such,

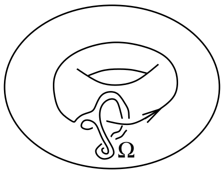

Figure 15

the curve $(1,1)$ is obtained by sliding the curve $(1,0)$ along the surgery curve of the positive twist. Here is the detailed description of the operation of sliding a framed knot along another using a Kirby band-sum move.

The *slide* of a framed knot K_0 along the framed knot K, denoted by $K_0 \# K$, is obtained as follows. Let K_1 be a copy of K obtained by pushing K in the direction of its framing. Take an embedded $[0,1]^3$ that is disjoint from K, K_0, and K_1 except for the opposite faces $F_i = [0,1]^2 \times \{i\}$, $i = 0, 1$ and which are embedded in ∂K_0 respectively ∂K_1. F_i is embedded in the annulus K_i such that $[0,1] \times \{j\} \times \{i\}$ is embedded in ∂K_i. Delete from $K_0 \cup K_1$ the faces F_i and add the faces $\{j\} \times [0,1] \times [0,1]$. The framed knot obtained this way is $K_0 \# K$. Saying it less rigorously but more intuitively, we cut the knots K_0 and K_1 and join together the two open strands by pulling them along the sides of an embedded rectangle (band) which does not intersect the knots. Figure 16 shows the slide of a trefoil knot over a figure-eight knot, both with the blackboard framing. When the knots are oriented, we perform the slide so that the orientations match. One should point out that there are many ways in which one can slide one knot along the other, since the band that connects the two knots is not unique.

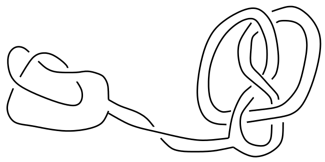

FIGURE 16

For a closed curve α in $\Sigma_g = \Sigma_g \times \{0\}$, the curve $h(\alpha)$ is obtained from α by slides over the components of the surgery link of h. Indeed, if h is the twist along the curve γ, with surgery curve γ^+, and if α and γ intersect on Σ_g at only one point, then $h(\alpha) = \alpha \# \gamma^+$. If the algebraic intersecton number of α and γ is $\pm k$, then $h(\alpha)$ is obtained from α by performing k consecutive slides along γ^+. The general case follows from the fact that h is a product of twists.

It follows that the exact Egorov identity is a particular case of slides of framed knots along components of the surgery link. In fact, the exact Egorov identity covers all cases of slides of one knot along another knot colored by Ω, and we have

Theorem 6.1. *Let M be a 3-dimensional manifold, σ an arbitrary skein in $\widetilde{\mathcal{L}}_t(M)$ and K_0 and K two oriented framed knots in M disjoint from σ. Then, in $\widetilde{\mathcal{L}}_t(M)$, one has the equality*

$$\sigma \cup K_0 \cup \Omega(K) = \sigma \cup (K_0 \# K) \cup \Omega(K),$$

however one does the band sum $K_0 \# K$.

Remark 6.2. The knots from the statement of the theorem should be understood as representing elements in $\widetilde{\mathcal{L}}_t(M)$.

Proof. Isotope K_0 along the embedded $[0,1]^3$ that defines $K_0 \# K$ to a knot K_0' that intersects K. There is an embedded punctured torus $\Sigma_{1,1}$ in M, disjoint from σ, which contains $K_0' \cup K$ on its boundary, as shown in Figure 17 a). In fact, by looking at a neighborhood of this torus, we can find an embedded $\Sigma_{1,1} \times [0,1]$ such that $K_0' \cup K \subset \Sigma_{1,1} \times \{0\}$. The boundary of this cylinder is a genus 2 surface Σ_2, and K_0' and K_1 lie in a punctured torus of this surface and intersect at exactly one point. By pushing off K_0' to a knot isotopic to K_0 (which we identify with K_0), we see that we can place K_0 and K in an embedded $\Sigma_2 \times [0,1]$ such that $K_0 \in \Sigma_2 \times \{0\}$ and $K \in \Sigma_2 \times \{1/2\}$.

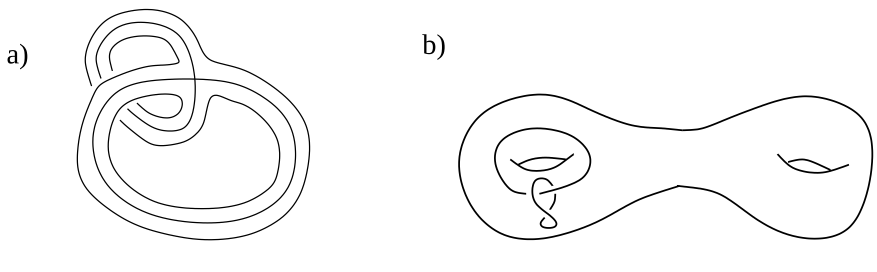

FIGURE 17

By performing a twist in $\Sigma_{1,1} \times [0,1]$ we can change the framing of K in such a way that K_0 and K look inside $\Sigma_2 \times [0,1]$ like in Figure 17 b). Then K_0 is mapped to $K_0 \# K$ in $\Sigma_2 \times \{1\}$ by the Dehn twist of Σ_2 with surgery diagram K. Hence the equality

$$K_0 \cup \Omega(K) = (K_0 \# K) \cup \Omega(K)$$

in $\Sigma_2 \times [0,1]$ is just the exact Egorov identity, which we know is true . By embedding $\Sigma_2 \times [0,1]$ in $\Sigma_{1,1} \times [0,1]$ we conclude that this equality holds in $\Sigma_{1,1} \times [0,1]$. By applying the inverse of the twist, embedding $\Sigma_{1,1} \times [0,1]$ in M, and adding σ, we conclude that the identity from the statement holds as well. $\square$

The operation of sliding one knot along another is related to the surgery description of 3-dimensional manifolds (for more details see [23]). Let us recall the basic facts.

We use the standard notation B^n for an n-dimensional (unit) ball and S^n for the n-dimensional sphere. Every 3-dimensional closed manifold is the boundary of a 4-dimensional manifold obtained by adding 2-handles $B^2 \times B^2$ to B^4 along the solid tori $B^2 \times S^1$ [14]. On the boundary S^3 of B^4, when adding a handle we remove a solid torus from S^3 (the one which we identify with $B^2 \times S^1$) and glue back the solid torus $S^1 \times B^2$. The curve $\{1\} \times S^1$ in the solid torus $B^2 \times S^1$ that was removed becomes the meridian (i.e. null-homologous curve) on the boundary of $S^1 \times B^2$.

This procedure of constructing 3-dimensional manifold is called Dehn surgery with integer coefficients. The curve $\{1\} \times S^1$ together with the core of the solid torus $B^2 \times S^1$ bound an embedded annulus which defines a framed link component in S^3. As such, the information for Dehn surgery with integer coefficients can be encoded in a framed link in S^3.

If K_0 is a knot inside a 3-dimensional manifold M obtained by surgery on S^3 and if the framed knot K is a component of the surgery link, then

$K_0 \# K$ is the slide of K_0 over the 2-handle corresponding to K. Indeed, when we slide K_0 along the handle we push one arc close to K, then move it to the other side of the handle by pushing it through the meridinal disk of the surgery solid torus. The meridian of this solid torus is parallel to the knot K (when viewed in S^3), so by sliding K_0 across the handle we obtain $K_0 \# K$. In particular, the operation of sliding one 2-handle over another corresponds to sliding one link component of the surgery link along another.

In conclusion, we can say that the Egorov identity allows handle-slides along surgery link components colored by Ω. We will make use of this fact in the next section.

7. The topological quantum field theory associated to theta functions

Theorem 6.1 has two direct consequences:

- the definition of a topological invariant for closed 3-dimensional manifolds,
- the existence of an isomorphism between the reduced linking number skein modules of 3-dimensional manifolds with homeomorphic boundaries.

We have seen in the previous section that handle-slides correspond to changing the presentation of a 3-dimensional manifold as surgery on a framed link. Kirby's theorem [12] states that two framed link diagrams represent the same 3-dimensional manifold if they can be transformed into one another by a sequence of isotopies, handle slides and additions/deletions of the trivial link components U_+ and U_- described in Figure 18. A trivial link component corresponds to adding a 2-handle to B^4 in a trivial way, and on the boundary, to taking the connected sum of the original 3-dimensional manifold and S^3.

Figure 18

Theorem 6.1 implies that, given a framed link L in S^3, the element $\Omega(L) \in \mathcal{L}_t(S^3) = \mathbb{C}$ is an invariant of the 3-dimensional manifold obtained by performing surgery on L, modulo addition and subtraction of trivial 2-handles. This ambiguity can be removed by using the linking matrix of L as follows.

Recall that the linking matrix of an oriented framed link L has the (i, j) entry equal to the linking number of the ith and jth components for $i \neq j$ and the (i, i) entry equal to the writhe of the ith component, namely to the linking number of the ith component with a push-out of this component in the direction of the framing. The signature $\mathrm{sign}(L)$ of the linking matrix does not depend on the orientations of the components of L, and is equal to the signature sign(W) of the 4-dimensional manifold W obtained by adding 2-handles to B^4 as specified by L. We remind the reader that the signature of sign(W) is the signature of the intersection form in $H_2(W, \mathbb{R})$.

When adding a trivial handle via U_+ respectively U_-, the signature of the linking matrix, and hence of the 4-dimensional manifold, changes by $+1$ respectively -1.

Proposition 7.1. *In any* 3*-dimensional manifold, the following equalities hold*

$$\Omega(U_+) = e^{\frac{\pi i}{4}}\emptyset, \quad \Omega(U_-) = e^{-\frac{\pi i}{4}}\emptyset.$$

Consequently $\Omega(U_+) \cup \Omega(U_-) = \emptyset$.

Proof. We have

$$\Omega(U_+) = N^{-1/2}\sum_{j=0}^{N-1} t^{j^2}\emptyset.$$

Because N is even,

$$\sum_{j=0}^{N-1} t^{j^2} = \sum_{j=0}^{N-1} e^{\frac{\pi i}{N}j^2} = \sum_{j=0}^{N-1} e^{\frac{\pi i}{N}(N+j)^2} = \sum_{j=N}^{2N-1} t^{j^2}.$$

Hence

$$\sum_{j=0}^{N-1} t^{j^2} = \frac{1}{2}\sum_{j=0}^{2N-1} e^{\frac{2\pi i}{2N}j^2}.$$

The last expression is a Gauss sum, which is equal to $e^{\frac{\pi i}{4}}N^{1/2}$ (see [13] page 87). This proves the first formula.

On the other hand,

$$\Omega(U_-) = N^{-1/2}\sum_{j=0}^{N-1} e^{-\frac{\pi i}{N}j^2}\emptyset$$

which is the complex conjugate of $\Omega(U_+)$. Hence, the second formula. □

Theorem 7.2. *Given a closed, oriented,* 3*-dimensional manifold* M *obtained as surgery on the framed link* L *in* S^3*, the number*

$$Z(M) = e^{-\frac{\pi i}{4}sign(L)}\Omega(L)$$

is a topological invariant of the manifold M*.*

Proof. Using Proposition 7.1 we can rewrite

$$Z(M) = \Omega(U_+)^{-b_+}\Omega(U_-)^{-b_-}\Omega(L)$$

where b_+ and b_- are the number of positive, respectively negative eigenvalues of the linking matrix. This quantity is invariant under the addition of trivial handles, and also under handleslide because of Theorem 6.1, so it is a topological invariant of M. □

Remark 7.3. This is the Murakami-Ohtsuki-Okada invariant [19]! Here we were able to derive its existence from quantum mechanical considerations.

The second application of the exact Egorov identity is the construction of a Sikora isomorphism, which identifies the reduced linking number skein modules of two manifolds with homeomorphic boundaries. Let us point out that such an isomorphism was constructed for reduced Kauffman bracket skein modules in [25].

Theorem 7.4. *Let M_1 and M_2 be two 3-dimensional manifold with homeomorphic boundaries. Then*

$$\widetilde{\mathcal{L}}_t(M_1) \cong \widetilde{\mathcal{L}}_t(M_2).$$

Proof. Because the manifolds M_1 and M_2 have homeomorphic boundaries, there is a framed link $L_1 \subset M_1$ such that M_2 is obtained by performing surgery on L_1 in M_1. Let N_1 be a regular neighborhood of L_1 in M, which is the union of several solid tori, and let N_2 be the union of the surgery tori in M_2. The cores of these tori form a framed link $L_2 \subset M_2$, and M_1 is obtained by performing surgery on L_2 in M_2. Every skein in M_1, respectively M_2 can be isotoped to one that misses N_1, respectively N_2. The homeomorphism $M_1 \backslash N_1 \cong M_2 \backslash N_2$ yields an isomorphism

$$\phi : \widetilde{\mathcal{L}}_t(M_1 \backslash N_1) \to \widetilde{\mathcal{L}}_t(M_1 \backslash N_1).$$

However, this does not induce a well defined map between $\widetilde{\mathcal{L}}_t(M_1)$ and $\widetilde{\mathcal{L}}_t(M_2)$ because a skein can be pushed through the N_i's. To make this map well defined, the skein should not change when pushed through these regular neighborhoods. To this end we use we use Theorem 6.1 and define $F_1 : \widetilde{\mathcal{L}}_t(M_1) \to \widetilde{\mathcal{L}}_t(M_1)$ by

$$F_1(\sigma) = \phi(\sigma) \cup \Omega(L_1)$$

and $F_1 : \widetilde{\mathcal{L}}_t(M_2) \to \widetilde{\mathcal{L}}_t(M_1)$ by

$$F_2(\sigma) = \phi^{-1}(\sigma) \cup \Omega(L_2).$$

By Proposition 5.1 b) we have

$$\Omega(L_1) \cup \Omega(\phi^{-1}(L_2)) = \emptyset \in \widetilde{\mathcal{L}}_t(M_1),$$

since each of the components of $\phi^{-1}(L_2)$ is a meridian in the surgery torus, hence it surrounds exactly once the corresponding component in L_1. This implies that $F_2 \circ F_1 = Id$. A similar argument shows that $F_1 \circ F_2 = Id$. $\square$

One should note that the Sikora isomorphism depends on the surgery diagram. Now it is easy to describe the reduced linking number skein module of any manifold.

Proposition 7.5. *For every oriented* 3*-dimensional manifold* M *having the boundary components* Σ_{g_i}, $i = 1, 2, \dots, n$, *one has*

$$\widetilde{\mathcal{L}}_t(M) \cong \bigotimes_{i=1}^{n} \mathbb{C}^{Ng_i}.$$

Proof. If M has no boundary component then $\widetilde{\mathcal{L}}_t(M) = \widetilde{\mathcal{L}}_t(S^3) = \mathbb{C}$, and if M is bounded by a sphere, then $\widetilde{\mathcal{L}}_t(M) = \widetilde{\mathcal{L}}_t(B^3) = \mathbb{C}$, where B^3 denotes the 3-dimensional ball. If M has one genus g boundary component with $g \geq 1$, then $\widetilde{\mathcal{L}}_t(M) = \widetilde{\mathcal{L}}_t(H_g) = \mathbb{C}^{Ng}$ by Theorem 4.7.

To tackle the case of more boundary components we need the following result:

Lemma 7.6. *Given two oriented* 3*-dimensional manifolds* M_1 *and* M_2*, and let* $M_1 \# M_2$ *be their connected sum. The map*

$$\widetilde{\mathcal{L}}_t(M_1) \otimes \widetilde{\mathcal{L}}_t(M_2) \to \widetilde{\mathcal{L}}_t(M_1 \# M_2)$$

defined by $(\sigma, \sigma') \mapsto \sigma \cup \sigma'$ *is an isomorphism.*

Proof. In $M_1 \# M_2$, the manifolds M_1 and M_2 are separated by a 2-dimensional sphere S^2_{sep}. Every skein in $M_1 \# M_2$ can be written as $\sum_{j=0}^{N-1} \sigma_j$, where each σ_j intersects S^2_{sep} in j strands pointing in the same direction. A trivial skein colored by Ω is equal to the empty link. But when we slide it over S^2_{sep} it turns $\sum_{j=0}^{N-1} \sigma_j$ into σ_0. This shows that the map from the statement is onto.

On the other hand, the reduced linking number skein module of a regular neighborhood of S^2_{sep} is $\mathbb{C}$ since every skein can be resolved to the empty link. This means that, in $M_1 \# M_2$, if a skein that lies entirely in M_1 can be isotoped to a skein that lies entirely in M_2, then this skein is a scalar multiple of the empty skein. So if $\sigma_1 \cup \sigma_1' = \sigma_2 \cup \sigma_2'$, then $\sigma_1 = \sigma_2$ in $\widetilde{L}_t(M_1)$ and $\sigma_1' = \sigma_2'$ in $\widetilde{L}_t(M_2)$. Hence the map is one-to-one, and we are done. $\square$

Returning to the theorem, an oriented 3-dimensional manifold with n boundary components can be obtained as surgery on a connected sum of n handlebodies. The conclusion of the proposition follows by applying the lemma. $\square$

If M is a 3-dimensional manifold without boundary, then Theorem 7.4 shows that

$$\widetilde{\mathcal{L}}_t(M) \cong \widetilde{\mathcal{L}}_t(S^3) = \mathbb{C}.$$

If we describe M as surgery on a framed link L with signature zero, which is always possible by adding trivial link components with framing ± 1, then the Sikora isomorphism maps the empty link in M to the vector

$$Z(M) = \Omega(L) \in \widetilde{\mathcal{L}}_t(S^3) = \mathbb{C}.$$

More generally, M can be endowed with a framing defined by the signature of the 4-dimensional manifold W that it bounds constructed as explained before. If L is the surgery link that gives rise to M and W, then to the framed manifold $(M, \mathrm{sign}(W)) = (M, m)$ we can associate the invariant

$$Z(M, m) = \Omega(L) \in \widetilde{\mathcal{L}}_t(S^3).$$

The Sikora isomorphism associated to L identifies this invariant with the empty link in M.

All these can be generalized to manifolds with boundary. A 3-dimensional manifold M with boundary can be obtained by performing surgery on a framed link L in the complement N of n handlebodies embedded in S^3, $n \geq 1$. We can again endow M with a framing by filling in the missing handlebodies in S^3 and constructing the 4-dimensional manifold M with the surgery instructions from L. To the manifold $(M, \mathrm{sign}(W)) = (M, m)$ we can associate the skein $\emptyset \in \widetilde{\mathcal{L}}_t(M)$. A Sikora isomorphism allows us to identify this vector with $\Omega(L) \in \widetilde{\mathcal{L}}_t(N)$. Another Sikora isomorphism allows us to identify $\widetilde{\mathcal{L}}_t(N)$ with the reduced linking number skein module of the connected sum of handlebodies, namely with $\otimes_{i=1}^n \mathbb{C}^{Ng_i}$, where g_i, $i = 1, 2, \ldots, n$ are the generra of the boundary components of M. Via Proposition 7.5, $\Omega(L)$ can be identified with a vector

$$Z(M, m) \in \bigotimes_{i=1}^{n} \mathbb{C}^{Ng_i}.$$

There is another way to see this identification of the linking number skein module with $\otimes_{i=1}^n \mathbb{C}^{Ng_i}$, done in the spirit of [24]. In this alternative formalism, $\Omega(L)$ acts as a linear functional on

$$\Omega(L) : \bigotimes_{i=1}^{n} \widetilde{L}_t(H_{g_i}) \to \mathbb{C},$$

by gluing handlebodies to N as to obtain S^3. In this setting, the formalism developed in [28] Chapter IV applies to show that the vector

$$Z(M, m) \in \left(\oplus_{i=1}^{N} \mathbb{C}^{Ng_i}\right)^* = \oplus_{i=1}^{n} \mathbb{C}^{Ng_i}$$

is well defined once the framing m is fixed.

This construction fits Atiyah's formalism of a topological quantum field theory (TQFT) [3] with anomaly [28]. In this formalism

- to each surface $\Sigma = \Sigma_{g_1} \cup \Sigma_{g_2} \cup \cdots \cup \Sigma_{g_n}$ we associate the vector space

$$V(\Sigma) = \bigotimes_{i=1}^{n} \mathbb{C}^{Ng_i}$$

  which is isomorphic to the reduced linking number skein module of any 3-dimensional manifold that Σ bounds.

- to each framed 3-dimensional manifold (M, m) we associate the empty link in $\widetilde{\mathcal{L}}_t(M)$. As a vector in $V(\partial M)$, this is $Z(M, m)$.

Atiyah's axioms are easy to check. *Functoriality* is obvious. The fact that Z is *involutory* namely that $Z(\Sigma^*) = Z(\Sigma)^*$ where Σ^* denotes Σ with opposite orientation follows by gluing a manifold M bounded by Σ to a manifold M^* bounded by Σ^* and using the standard pairing

$$\widetilde{\mathcal{L}}_t(M) \times \widetilde{\mathcal{L}}_t(M^*) \to \widetilde{\mathcal{L}}_t(M \cup M^*) = \mathbb{C}.$$

Let us check the *multiplicativity* of Z for disjoint union. If Σ and Σ' are two surfaces, we can consider disjoint 3-dimensional manifolds M and M' such that $\partial M = \Sigma$ and $\partial M' = \Sigma'$. Then

$$Z(\Sigma \cup \Sigma') = \widetilde{\mathcal{L}}_t(M \cup M') = \widetilde{\mathcal{L}}_t(M) \otimes \widetilde{\mathcal{L}}_t(M') = Z(\Sigma) \otimes Z(\Sigma').$$

Also $\emptyset \in \widetilde{\mathcal{L}}_t(M \cup M')$ equals $\emptyset \otimes \emptyset \in \widetilde{\mathcal{L}}_t(M) \otimes \widetilde{\mathcal{L}}_t(M')$. If we now endow M and M' with framings m, respectively m', then

$$Z(M \cup M', m + m') = Z(M, m) \otimes Z(M', m'),$$

since the Sikora isomorphism acts separately on the skein modules of M and M'.

What about *multiplicativity* for manifolds glued along a surface? Let M_1 and M_2 be 3-dimensional manifolds with $\partial M_1 = \Sigma \cup \Sigma'$ and $M_2 = \Sigma^* \cup \Sigma''$, and assume that M_1 is glued to M_2 along Σ. Then the empty link in $M \cup M'$ is obtained as the union of the empty link in M with the empty link in M'. It follows that

$$Z(M_1 \cup M_2, m_1 + m_2) = e^{-\frac{i\pi}{4}\tau} \langle Z(M_1, m_1), Z(M_2, m_2) \rangle,$$

where $\langle , \rangle$ is the contraction

$$V(\Sigma') \otimes V(\Sigma) \otimes V(\Sigma)^* \otimes V(\Sigma'') \to V(\Sigma') \otimes V(\Sigma''),$$

and τ expresses the anomaly of the TQFT and depends on how the signature of the surgery link changes under gluing (or equivalently, on how the signatures of the 4-dimensional manifolds bounded by the given 3-dimensional manifolds change under the gluing, see Section 8).

Finally, $Z(\emptyset) = \mathbb{C}$, because the only link in the void manifold is the empty link. Also, if $M = \Sigma \times [0, 1]$, then the empty link in M is the surgery diagram of the identity homeomorphism of Σ and hence $Z(M, 0)$ can be viewed as the identity map in $\mathrm{End}(V(\Sigma))$.

This TQFT is *hermitian.* Indeed, $V(\partial M)$, being a space of theta functions, has the inner product introduced in Section 2. On the other hand, if M is a 3-dimensional manifold and M^* is the same manifold but with reversed orientation, then the surgery link L^* of M^* is the mirror image of the surgery link L of M. The invariant of M is computed by smoothing the crossings in L while the invariant of M^* is computed by smoothing the crossings in L^*, whatever was a positive crossing in L becomes a negative crossing in L^* and vice-versa. Hence $\mathrm{sign}(L^*) = -\mathrm{sign}(L)$. Also, because for $t = e^{\frac{i\pi}{N}}$ one has $t^{-1} = \bar{t}$, and hence $\Omega(L^*) = \overline{\Omega(L)}$. It follows that

$$Z(M^*, -m) = \overline{Z(M, m)}$$

as desired.

8. The Hermite-Jacobi action and the non-additivity of the signature of 4-dimensional manifolds

An interesting coincidence in mathematics is the fact that the Segal-Shale-Weil cocycle of the metaplectic representation [15] and the non-additivity of the signature of 4-dimensional manifolds under gluings [30] are both described in terms of the Maslov index. We will explain this coincidence by showing how to resolve the projectivity of the Hermite-Jacobi action using 4-dimensional manifolds. Note that the theory of theta functions explains the coincidence of the cocycle of the metaplectic representation and the cocycle of the Hermite-Jacobi action.

Each element h of the mapping class group of Σ_g can be represented by surgery on a link $L_h \in \Sigma_g \times [0,1]$. This means that mapping cylinder of h is obtained as the surgery on $\Sigma_g \times [0,1]$ along L_h; it is homeomorphic to $\Sigma_g \times [0,1]$, where the homeomorphism is the identity map on $\Sigma_g \times \{0\}$ and h on $\Sigma_g \times \{1\}$. The link L_h is not necessarily obtained from writing h as a composition of Dehn twists, as in Proposition 5.3.

Theorem 8.1. *Let h be an element of the mapping class group of Σ_g obtained by surgery on the framed link L_h in $\Sigma_g \times [0,1]$. Then the discrete Fourier transform $\rho(h) : \widetilde{\mathcal{L}}_t(H_g) \to \widetilde{\mathcal{L}}_t(H_g)$ is given by*

$$\rho(h)\beta = \Omega(L_h)\beta.$$

Proof. Write $h = T_1 T_2 \cdots T_n$, where T_j are Dehn twists obtained a surgeries along the curves γ_j, $j = 1, 2, \ldots, n$. If we glue two handlebodies so as to obtain a sphere, the gluing defines a nondegenerate pairing $[\cdot,\cdot]\widetilde{\mathcal{L}}_t(H_g) \times \widetilde{\mathcal{L}}_t(H_g) \to \mathbb{C}$. If we consider basis elements e_j, e_k in $\widetilde{\mathcal{L}}_t(H_g)$, then $[\Omega(L_h)e_j, e_k]$ completely determines the operator defined by $\Omega(L_h)$. Because of the invariance under handle slides, we can transform $\Omega(L_h)$ into $\Omega(\gamma_1)\Omega(\gamma_2)\cdots\Omega(\gamma_n)$ such that

$$[\Omega(L_h)e_j, e_k] = [\Omega(\gamma_1)\Omega(\gamma_2)\cdots\Omega(\gamma_n)e_j, e_k].$$

Hence the operators defined by $\Omega(L_h)$ and $\Omega(\gamma_1)\Omega(\gamma_2)\cdots\Omega(\gamma_n)$ are equal. The conclusion now follows by applying Proposition 5.3. $\square$

Fix a Lagrangian subspace $\mathbf{L}$ of $H_1(\Sigma_g, \mathbb{R})$ and consider the closed 3-dimensional manifold M obtained by gluing to the surgery of $\Sigma_g \times [0,1]$ along L the handlebodies H_g^0 and H_g^1 such that $\partial H_g^0 = \Sigma_g \times \{0\}$, $\partial H_g^1 = \Sigma_g \times \{1\}$, and $\mathbf{L}$ respectively $h_*(\mathbf{L})$ are the kernels of the inclusion of Σ_g into H_g^0 respectively H_g^1. M is the boundary of a 4-dimensional manifold W obtained by adding 2-handles to the 4-dimensional ball as prescribed by L.

The discrete Fourier transform $\rho(h)$ is a skein in $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$ which is uniquely determined once we fix the signature of W. Hence to the pair (h,n) where h is an element of the mapping class group and n is an integer, we can associate uniquely a skein $\mathcal{F}(h,n)$, its discrete Fourier transform. We identify the pair (h,n) with $(h, \mathrm{sign}(W))$ where W is a 4-dimensional manifold defined as above. Note that by adding trivial 2-handles we can enforce $\mathrm{sign}(W)$ to be any integer.

Let us consider the $\mathbb{Z}$-extension of the mapping class group defined by the multiplication rule

$$(h', \mathrm{sign}(W')) \circ (h, \mathrm{sign}(W)) = (h' \circ h, \mathrm{sign}(W' \cup W))$$

where W' and W are glued in such a way that $H_g^0 \in W'$ is identified with H_g^1 in W. If $\mathbf{L}'$ is the Lagrangian subspace of $H_1(\Sigma_g, \mathbb{R})$ used for defining W', then necessarily $\mathbf{L}' = h_*(\mathbf{L})$. Recall Wall's formula for the non-additivity of the signature of 4-dimensional manifolds

$$\mathrm{sign}(W' \cup W) = \mathrm{sign}(W') + \mathrm{sign}(W) - \tau(\mathbf{L}, h_*(\mathbf{L}), h'_* \circ h_*(\mathbf{L})),$$

where τ is the Maslov index. By using this formula we obtain

$$\begin{aligned}
&\mathcal{F}(h' \circ h, \mathrm{sign}(W' \cup W)) \\
&\quad = \mathcal{F}(h' \circ h, \mathrm{sign}(W') + \mathrm{sign}(W) - \tau(\mathbf{L}, h_*(\mathbf{L}), h'_* \circ h_*(\mathbf{L})) \\
&\quad = e^{-\frac{i\pi}{4}\tau(\mathbf{L}, h_*(\mathbf{L}), h'_* \circ h_*(\mathbf{L}))}\mathcal{F}(h' \circ h, \mathrm{sign}(W') + \mathrm{sign}(W)) \\
&\quad = e^{-\frac{i\pi}{4}\tau(\mathbf{L}, h_*(\mathbf{L}), h'_* \circ h_*(\mathbf{L}))}\mathcal{F}(h', \mathrm{sign}(W'))\mathcal{F}(h, \mathrm{sign}(W)),
\end{aligned}$$

where for the second step we changed the signature of the 4-dimensional manifold associated to $h \circ h'$ by adding trivial handles and used Proposition 7.1.

Or equivalently, if we let $\rho(h, \mathrm{sign}(W))$ be discrete the Fourier transform associated to h, normalized by the (signature of) the manifold W, then

$$\rho(h' \circ h, \mathrm{sign}(W' \cup W)) = e^{-\frac{i\pi}{4}\tau(\mathbf{L}, h_*(\mathbf{L}), h'_* \circ h_*(\mathbf{L}))}\rho(h', \mathrm{sign}(W'))\rho(h, \mathrm{sign}(W)).$$

This formula is standard in the theory of the Fourier-Mukai transform; in it we recognize the cocycle of the Segal-Shale-Weil representation

$$c(h, h') = e^{-\frac{i\pi}{4}\tau(\mathbf{L}, h_*(\mathbf{L}), h'_* \circ h_*(\mathbf{L}))}$$

used for resolving the projective ambiguity of the Hermite-Jacobi action, as well as for resolving the projective ambiguity of the metaplectic representation.

9. Theta functions and abelian Chern-Simons theory

By Jacobi's inversion theorem and Abel's theorem [6], the Jacobian of a surface Σ_g parametrizes the set of divisors of degree zero modulo principal divisors. This is the moduli space of stable line bundles, which is the same as the moduli space $\mathcal{M}_g(U(1))$ of flat $u(1)$-connections on the surface (in the trivial $U(1)$-bundle).

The moduli space $\mathcal{M}_g(U(1))$ has a complex structure defined as follows (see for example [11]). The tangent space to $\mathcal{M}_g(U(1))$ at an arbitrary point is $H^1(\Sigma_g, \mathbb{R})$, which, by Hodge theory, can be identified with the space of real-valued harmonic 1-forms on Σ_g. The complex structure is given by

$$J\alpha = - * \alpha,$$

where α is a harmonic form. In local coordinates, if $\alpha = udx + vdy$, then $J(udx + vdy) = vdx - udy$.

If we identify the space of real-valued harmonic 1-forms with the space of holomorphic 1-forms $H^{(1,0)}(\Sigma_g)$ by the map Φ given in local coordinates by $\Phi(udx + vdy) = (u - iv)dz$, then the complex structure becomes multiplication by i in $H^{(1,0)}(\Sigma_g)$.

The moduli space is a torus obtained by exponentiation

$$\mathcal{M}_{U(1)} = H^1(\Sigma_g, \mathbb{R})/\mathbb{Z}^{2g}.$$

If we choose a basis of the space of real-valued harmonic forms $\alpha_1, \alpha_2, \ldots, \alpha_g$, $\beta_1, \ldots, \beta_g$ such that

$$\int_{a_j} \alpha_k = \delta_{jk}, \quad \int_{b_j} \alpha_k = 0, \quad \int_{a_j} \beta_k = 0, \quad \int_{b_j} \beta_k = \delta_{jk}, \tag{9.1}$$

then the above $\mathbb{Z}^{2g}$ is the period matrix of this basis.

On the other hand, if $\zeta_1, \zeta_2, \ldots, \zeta_g$ are the holomorphic forms introduced in Section 2, and if $\alpha_j' = \Phi^{-1}(\zeta_j)$ and $\beta_j' = \Phi^{-1}(-i\zeta_j)$, $j = 1, 2, \ldots, g$ then one can compute that

$$\int_{a_j} \alpha_k' = \delta_{jk}, \int_{b_j} \alpha_k' = \mathrm{Re}\ \pi_{jk}, \int_{a_j} \beta_k' = 0, \int_{b_j} \beta_k' = \mathrm{Im}\ \pi_{jk}.$$

The basis $\alpha_1', \ldots, \alpha_g', \beta_1', \ldots, \beta_g'$ determines coordinates (X', Y') in the tangent space to $\mathcal{M}_{U(1)}$. If we consider the change of coordinates $X' + iY' = X + \Pi Y$, then the moduli space is the quotient of $\mathbb{C}^g$ by the integer lattice $\mathbb{Z}^{2g}$. This is exactly what has been done in Section 2 to obtain the jacobian variety. This shows that the complex structure on the Jacobian variety coincides with the standard complex structure on the moduli space of flat $u(1)$-connections on the surface.

The moduli space $\mathcal{M}_g(\Sigma_g)$ has a symplectic structure defined by the Atiyah-Bott form [4]. This form is given by

$$\omega(\alpha, \beta) = - \int_{\Sigma_g} \alpha \wedge \beta,$$

where α, β are real valued harmonic 1-forms, i.e. vectors in the tangent space to $\mathcal{M}_g(\Sigma_g)$. If α_j, β_j, $j = 1, 2, \ldots, g$ are as in (9.1), then $\omega(\alpha_j, \alpha_k) = \omega(\beta_j, \beta_j) = 0$ and $\omega(\alpha_j, \beta_k) = \delta_{jk}$ (which can be seen by identifying the space of real-valued harmonic 1-forms with $H^1(\Sigma_g, \mathbb{R})$ and using the topological definition of the cup product). This shows that the Atiyah-Bott form coincides with the symplectic form introduced in Section 2.

For a $u(1)$-connection A and curve γ on the surface, we denote by $\mathrm{hol}_\gamma(A)$ the holonomy of A along γ. The map $A \mapsto \mathrm{trace}(\mathrm{hol}_\gamma(A))$ induces a function on the Jacobian variety called Wilson line[4]. If $[\gamma] = (p, q) \in H_1(\Sigma_g, \mathbb{Z})$, then the Wilson line associated to γ is the function $(x, y) \mapsto \exp 2\pi i(p^T x + q^T y)$. These are the functions on the Jacobian variety of interest to us.

The goal is to quantize the moduli space of flat $u(1)$-connections on the closed Riemann surface Σ_g endowed with the Atiyah-Bott symplectic form. One procedure has been outlined in Section 2; it is Weyl quantization on the $2g$-dimensional torus in the holomorphic polarization.

Another quantization procedure has been introduced by Witten in [33] using Feynman path integrals. In his approach, states and observables are defined by path integrals of the form

$$\int_A e^{\frac{i}{\hbar}L(A)}\mathrm{trace}(\mathrm{hol}_\gamma(A))\mathcal{D}A,$$

where $L(A)$ is the Chern-Simons lagrangian

$$L(A) = \frac{1}{4\pi}\int_{\Sigma_g\times[0,1]} \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right).$$

According to Witten, states and observables should be representable as skeins in the skein modules of the linking number discussed in Section 4.

Witten's quantization model is symmetric with respect to the action of the mapping class group of the surface, a property shared by Weyl quantization in the guise of the exact Egorov identity (2.5). As we have seen, the two quantization models coincide.

It was Andersen [1] who pointed out that the quantization of the Jacobian that arises in Chern-Simons theory coincides with Weyl quantization. For non-abelian Chern-Simons theory, this phenomenon was first observed by the authors in [8].

In the sequel of this paper, [9], we will conclude that, for non-abelian Chern-Simons, the algebra of quantum group quantizations of Wilson lines on a surface and the Reshetikhin-Turaev representation of the mapping class group of the surface are analogues of the group algebra of the finite Heisenberg group and of the Hermite-Jacobi action. We will show how the element Ω corresponding to the group $SU(2)$ can be derived by studying the discrete sine transform.

References

[1] J.E. Andersen, *Deformation quantization and geometric quantization of abelian moduli spaces*, Commun. Math. Phys., **2**55(2005), 727–745.

[2] M.F. Atiyah, *The Geometry and Physics of Knots*, Accademia Nationale de Lincei, Cambridge Univ. Press, 1990.

[3] M.F. Atiyah, *Topological quantum field theory*, Publications Mathématiques de l'I.H.É.S., **68**(1988), 175–186.

[4]Since we work with the group $U(1)$, the holonomy is just a complex number of absolute value 1, and the trace is the number itself.

[4] M.F. Atiyah, R. Bott, *The Yang-Mills equations over a Riemann surface*, Phil. Trans. Royal. Soc. A, **308** (1982), 523–615.

[5] A. Bloch, F. Golse, T. Paul, A. Uribe, *Dispersionless Toda and Toeplitz operators*, Duke Math. J. **117** no. 1 (2003), 157–196.

[6] H.M Farkas, I. Kra, *Riemann Surfaces*, second ed., Springer, 1991.

[7] G. Folland, *Harmonic Analysis in Phase Space*, Princeton University Press, Princeton 1989.

[8] R. Gelca, A. Uribe, *The Weyl quantization and the quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus are the same*, Commun. Math. Phys., **233**(2003), 493–512.

[9] R. Gelca, A. Uribe, *Quantum mechanics and nonabelian theta functions for the gauge group $SU(2)$*, preprint.

[10] Ph. Griffiths, J. Harris, *Principles of Algebraic Geometry*, John Wiley and Sons, 1994.

[11] N. Hitchin, *Flat connections and geometric quantization*, Commun. Math. Phys., **131** (1990), 347–380.

[12] R. Kirby, *A calculus for framed links in S^3*, Inventiones Math. **45** (1978), 35–56.

[13] S. Lang, *Algebraic Number Theory*, second ed., Springer 1994.

[14] W.B.R. Lickorish, *A representation of orientable combinatorial* 3*-manifolds*, Ann. of Math. **76** (1962), 531-540.

[15] G. Lion, M. Vergne, *The Weil Representation, Maslov Index, and Theta Series*, Birkhäuser, 1980.

[16] M. Manoliu, *Abelian Chern-Simons theory I, A topological quantum field theory*, J. Math. Phys., **39**(1998), 170-206.

[17] M. Manoliu, *Abelian Chern-Simons theory II, A functional integral approach*, J. Math. Phys., **39**(1998), 207-217.

[18] D. Mumford, *Tata Lectures on Theta*, Birkhäuser, 1983.

[19] H. Murakami, T. Ohtsuki, M. Okada, *Invariants of three-manifolds derived from linking matrices of framed links*, Osaka J. Math., **29**(1992), 545-572.

[20] A. Polishchuk, *Abelian Varieties, Theta Functions and the Fourier Transform*, Cambridge Univ. Press, 2003.

[21] J.H. Przytycki, *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci. **39(1-2)** (1991) 91–100.

[22] J.H. Przytycki, *A q-analogue of the first homology group of a 3-manifold*, in *Contemporary Mathematics* 214, Perspectives on Quantization (Proceedings of the joint AMS-IMS-SIAM Conference on Quantization, Mount Holyoke College, 1996), Ed. L.A. Coburn, M.A. Rieffel, AMS 1998, 135–144.

[23] D. Rolfsen, *Knots and Links*, AMS Chelsea Publ., 2003.

[24] N. Reshetikhin, V. Turaev, *Invariants of* 3*-manifolds via link polynomials and quantum groups*, Invent. Math., **103** (1991), 547–597.

[25] A. Sikora, *Skein modules and TQFT* in "Knots in Hellas '98," Proc. of the Int. Conf. on Knot Theory, Eds. C. McA. Gordon et al., World Scientific, 2000.

[26] J. Śniatycki, *Geometric Quantization and Quantum Mechanics*, Springer, 1980.

[27] V.G. Turaev, *Algebras of loops on surfaces, algebras of knots, and quantization*, Adv. Ser. in Math. Phys., **9** (1989), eds. C.N. Yang, M.L. Ge, 59–95.

[28] V.G. Turaev, *Quantum Invariants of Knots and* 3*-manifolds*, de Gruyter Studies in Math., Berlin–New York, 1994.

[29] K. Walker, *On Witten's 3-manifold invariants*, preprint 1991.

[30] C.T.C. Wall, *Non-additivity of the signature*, Invent. Math. **7** (1969), 269–274.

[31] A. Weil, *Sur certains groupes d'operateurs unitaires*, Acta Math. **111** (1964), 143–211.

[32] J. Weitsman, *Quantization via real polarization of the moduli space of flat connections and Chern-Simons gauge theory in genus one*, Commun. Math. Phys., **137** (1991), 175–190.

[33] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys., **121** (1989), 351–399.

Department of Mathematics and Statistics, Texas Tech University, Lubbock, TX 79409

E-mail address: rgelca@gmail.com

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

E-mail address: uribe@math.lsa.umich.edu