ON THE FORMULA OF THE QUANTUM INVARIANT FOR THREE MANIFOLDS WITH BOUNDARY

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ABSTRACT

The paper shows how the formula of the quantum invariant for three manifolds with boundary can be deduced from the axioms of the a topological quantum field theory with corners. The topological quantum field theory we use is the one related to the Kauffman bracket.

Keywords: topological quantum field theory, 3-manifolds, knot theory

1. Introduction

In [3] a topological quantum field theory (TQFT) with corners was constructed, based on the Kauffman bracket, which gives rise to the smooth TQFT from [6] and [1]. As in the case of homology, the value of a TQFT on a manifold on a 3-manifolds can be computed using cut and paste techniques and a knowledge of a small collection of information about how to glue the invariant. This information is called the basic data.

In the present paper we deduce from axioms a formula for the invariant of a 3-manifold with boundary. Note that the case of closed 3-manifolds has been discussed in [3]. Our methods can also be applied to deduce the Reshetikhin-Turaev formula in the case of the construction based on the representations of the quantized $sl(2, \mathbf{C})$ (see [4]). In addition we explain what is the canonical choice of the constant in front of the invariant.

We start by briefly recalling some facts about TQFT's with corners that will be needed in the sequel. For a more extended treatment, the reader can consult [8], [2], [3]. All surfaces and 3-manifolds are supposed to be compact, oriented and piece-wise linear.

A DAP-decomposition of a surface consists out of a family of simple closed curves in its interior which cut the surface into elementary surfaces (disks, annuli and pairs of pants), an ordering of these elementary surfaces; and for each elementary surface an ordering and parametrization by S^1 of its boundary components, such that the parametrizations agree with the orientation of the elementary surface and such that parametrizations coming from neighboring elementary surfaces are the complex conjugate of the other; and finally a collection of disjoint arcs (called seams) in each elementary surface, joining the point $e^{i\epsilon}$ on the j-th boundary component, to $e^{-i\epsilon}$ on the (j+1)-st, for some small $\epsilon > 0$.

An example of such a DAP-decomposition is given in Fig. 1. The simple closed curves, together with the boundary components of the surface will be called decomposition curves.

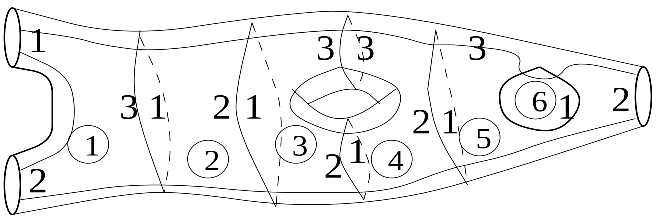

Figure 1.

A pair (Σ, D), where Σ is a surface and D is a DAP-decomposition is called an extended surface (e-surface). A transformation of one DAP-decomposition into another is called a move. An extended morphism (e-morphism) is a pair (f, n) with f a composition of a homeomorphism and a move and n an integer. Extended morphisms compose as usual functions, with the only difference that the integer gets modified by means of the negative of Wall's nonadditivity function ([8], [4]).

An extended 3-manifold (shortly e-3-manifold) is a triple (M, D, n) where M is a 3-manifold, D is a DAP-decomposition of its boundary and n is an integer called framing. Recall that when we glue e-3-manifolds the framing gets modified by the negative of Wall's nonadditivity function [8].

For a finite set of labels $\mathcal{L}$ one defines the category of labeled extended surfaces (le-surfaces) with objects e-surfaces with boundary components labeled by elements of $\mathcal{L}$

and with morphisms the e-morphisms that preserve the labeling. A TQFT with corners consists out of a modular functor V from the category of le-surfaces to the category of finite dimensional vector spaces, and a partition function Z which associates to each 3-manifold a vector in its boundary (which eventually is a number when the manifold is closed). The two are supposed to satisfy the 10 axioms from [8]. Note that the DAP-decomposition plays the role of the basis of the vector space.

The TQFT with corners we consider makes use of the skein spaces of the Kauffman bracket [5] and has been constructed in [4]. We remind the reader of the following facts from there.

For a surface Σ with $2n$ points on the boundary ($n \geq 0$) we call a link diagram in Σ an immersed 1-dimensional compact manifold L with $L \cap \partial\Sigma = \partial L$ consisting of the $2n$ points, and the only singular points of the immersion are transverse double points in the interior of Σ for which "over" and "under" information has been recorded.

Let $r \in \mathbf{N}$, $r \geq 2$ and $t = e^{\pi i/2r}$. The skein vector space of Σ is defined to be the complex vector space generated by all link diagrams, factored by

a) $L \cup$(trivial link)$= -(t^2 + t^{-2})L$,

b) $L_1 = tL_2 + t^{-1}L_3$

where L_1, L_2 and L_3 are three link diagrams that coincide except in a small disk where they are like in Fig. 2.

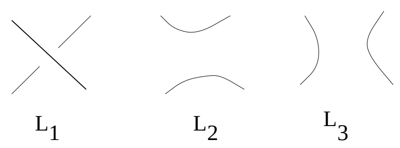

Figure 2.

We make the convention that whenever we write a number n next to a strand we actually mean that we have n parallel strands there.

Recall that the skein space of the plane is $\mathbf{C}$ and the skein space of the disk with $2n$ points on the boundary can be identified with the Temperley-Lieb algebra TL_n by viewing the disk as a rectangle with n points on one side and n on the opposite and the multiplication defined by the juxtaposition of rectangles. The algebra TL_n is generated by the elements $1, e_1, e_2, \cdots, e_{n-1}$ of the form described in Fig. 3.

For each $0 \leq n \leq r_1$, TL_n contains an idempotent element $f^{(n)}$, called the Jones-Wenzl idempotent [10], with the property that

1) $f^{(n)}e_i = 0 = e_i f^{(n)}$, $0 \leq i \leq n-1$,

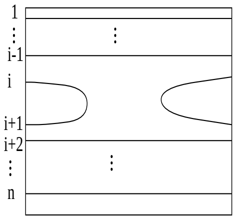

Figure 3.

2) $f^{(n)} - 1$ belongs to the algebra generated by $e_1, e_2, \cdots, e_{n-1}$,

3) $\Delta_n = (-1)^n [n+1]$

where $[n] = (t^{2n} - t^{-2n})/(t^2 - t^{-2})$ and Δ_n is the quantum trace of $f^{(n)}$ (i.e. the value of the diagram obtained by closing $f^{(n)}$ with n parallel strands). In diagrams the Jones-Wenzl idempotents will be represented like in Fig. 4.

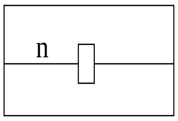

Figure 4.

Denote by $d_n = i^n \sqrt{[n+1]}$ and $X = i\sqrt{2r}/(t^2 - t^{-2})$, thus $X^2 = \sum d_n^4$.

The label set of our TQFT is $\mathcal{L} = \{0, 1, \cdots, r-2\}$. To a disk with boundary labeled by m one associates the vector space $V_m = 0$ if $m \neq 0$ and $V_m = \mathbf{C}$ if $m = 0$. Let β_0 be the basis element equal to the empty diagram.

To an annulus with boundary components labeled by m and n one associates $V_{mn} = 0$ if $m \neq n$ and V_{mm} the vector space spanned by the diagram from Fig. 5. We denote by β_{mm} the product of this diagram and $1/d_a$.

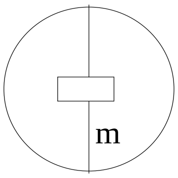

Figure 5.

To a pair of pants with boundary components labeled by m, n and p one associates the vector space V_{mnp} spanned by the Kauffman triad described in Fig. 6. It is either 0 or 1-dimensional. When it is 1-dimensional the triple (m, n, p) is called admissible. In this latter case we let β_{abc} be the product of the Kauffman triad and $(d_{x+y+z}!d_{x-1}!d_{y-1}!d_{z-1}!)^{-1}(d_{y+z-1}!$

$d_{z+x-1}!d_{x+y-1}!$) where $x+y=m$, $y+z=n$, $z+x=p$ and $d_a! = d_a d_{a-1}\cdots d_1$. In diagrams we use for β_{mnp} the notation from Fig. 7. Note that β_{mnp} has been normalized such that the value of the diagram that looks like the greek letter θ is equal to 1, and that this normalization differs from the one from [6]. In what follows we will identify the vectors β_{mnp}, β_{pmn} and β_{npm} (and subsequently their vector spaces) by the rotation of the disk.

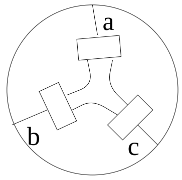

Figure 6.

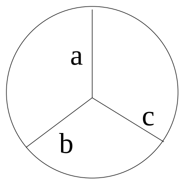

Figure 7.

To the move F from Fig. 8. we associate via the modular functor the morphism $F : \bigoplus_p V_{pmn} \otimes V_{pkl} \to \bigoplus_q V_{qlm} \otimes V_{qnk}$ whose entries are defined in Fig. 9.

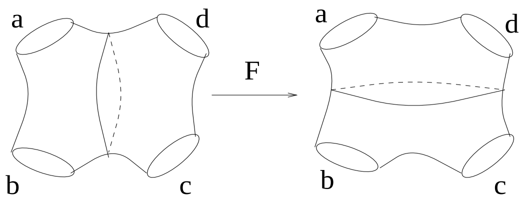

Figure 8.

To the moves A and D from Fig. 10 we associate the maps $A : V_{mnp} \otimes V_{pp} \to V_{mnp}$, $\beta_{mnp} \otimes \beta_{pp} \to \beta_{mnp}$ and $D : V_{mm0} \otimes V_0 \to V_{mm0}$, $\beta_{mm0} \otimes \beta_0 \to \beta_{mm0}$. They will be called contractions of annuli, respectively disks, and their inverses expansions of annuli and disks. Finally, the map $C = V(id, 1)$ is the multiplication by $exp(3(r-2)\pi i/4r)$.

We will need as a main device the following result, whose proof can be found in [3].

Theorem 1.1. Let $(M_1, D_1, 0)$, $(M_2, D_2, 0)$ and $(M_3, D_3, 0)$ be three extended three manifolds that are obtained by gluing to the same extended three manifold the three handlebodies from Fig. 11 via the same gluing map along the union of the two pairs

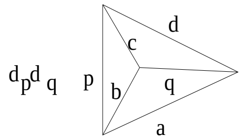

Figure 9.

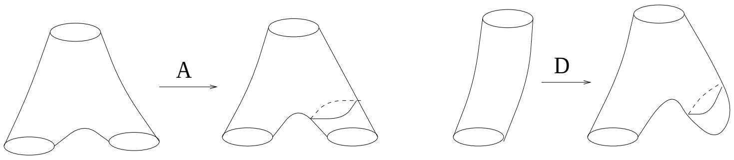

Figure 10.

of pants on the outside. Let V_1, V_2 and V_3 be the subspaces of the vector spaces of their boundaries that correspond to the labeling of the boundaries of the interior annuli by 1, and let v_1, v_2 and v_3 the vector components of $Z(M_1, D_1, 0)$, $Z(M_2, D_2, 0)$ and $Z(M_3, D_3, 0)$ lying in these vector spaces. Then V_1, V_2 and V_3 are canonically isomorphic and under this identification v_1, v_2 and v_3 satisfy

$$v_1 = tv_2 + t^{-1}v_3$$

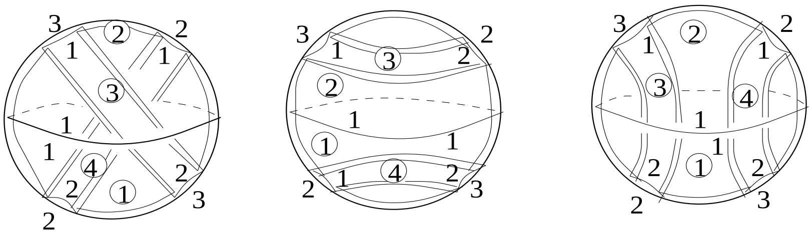

Figure 11.

2. The formula for the invariant of the complement of a graph

We now proceed with the proof of the formula for the invariant of a 3-manifold with boundary. First we discuss the case of the complement of a regular neighborhood of a graph in S^3, and afterwards the case of arbitrary manifolds.

<u>Definition.</u> A graph diagram consists of

- a finite set $\mathcal{V}$ of points in the plane,

- an immersion of a compact 1-manifold $\mathcal{D}$ in the plane such that the image of $\partial\mathcal{D}$ is $\mathcal{V}$, each point of $\mathcal{V}$ is the image of exactly 3 points in $\partial\mathcal{D}$ and in a neighborhood of that point the immersion looks like in Fig. 12,

- the only singularities lying in $int\mathcal{D}$ are transverse double points for which "under" and "over" information has been recorded.

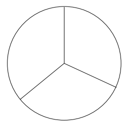

Figure 12.

An example of a graph diagram is shown in Fig. 13. In the language of [7] a graph diagram is called a ribbon diagram.

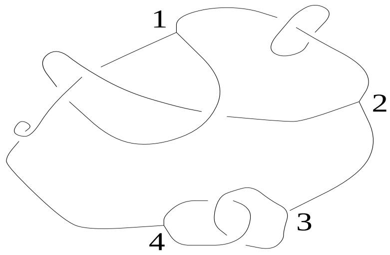

Figure 13.

Two graph diagrams are considered identical if one can pass from one of them to the other by a sequence of transformations from Fig. 14, or their inverses.

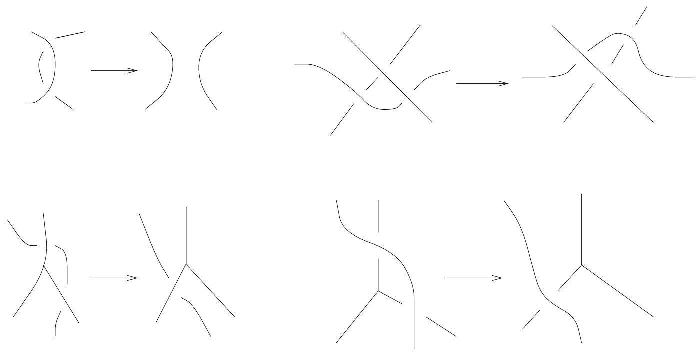

Figure 14.

In what follows, the images of segments from $\mathcal{D}$ will be called edges, and the images of circles will be called link components. Both edges and link components will be called strands.

Let Γ be a graph diagram. For each coloring γ of the strands by elements of $\mathcal{L}$ we let

$$d(\gamma) = \Pi_{E:edge} d_{\gamma(E)}$$
$$\beta_V(\gamma) = \otimes_{V:vertex} \beta_{\gamma(V)}$$

$$\beta_l(\gamma) = \otimes_{l:link\ component}\beta_{\gamma(l)\gamma(l)}$$

where $\beta_{\gamma(V)} = \beta_{\gamma(E_1)\gamma(E_2)\gamma(E_3)}$, E_1, E_2, E_3 being the incoming edges at V taken in clockwise order. We also denote by $\Gamma(\gamma)$ the value in the skein space of the plane of the diagram obtained by inserting on each link component l the Jones-Wenzl idempotent $f^{(\gamma(l))}$ and at each vertex V the normalized Kauffman triad $\beta_{\gamma(V)}$.

To each graph diagram Γ we associate an e-3-manifold $(M(\Gamma), D(\Gamma), 0)$ in the following way. Consider S^3 to be the compactification of $\mathbf{R}^3$ and let $\mathbf{R}^2 \times \{0\} \subset \mathbf{R}^3$ be the plane of the diagram. At each crossing push upward the "over" strand. Define $M(\Gamma)$ to be the complement of a regular neighborhood in S^3 of this graph. Put on $M(\Gamma)$ the orientation induced by S^3 and on $\partial M(\Gamma)$ the compatible orientation under the rule "first out".

Each strand has a meridinal circle, which is obtained by intersecting $\partial M(\Gamma)$ with a plane perpendicular to that strand. The decomposition curves are these circles. They should be parametrized such that the point $1 \in S^1$ lies above the edge. The seams of $D(\Gamma)$ are parallel to the blackboard framing. An example is shown in Fig. 15.

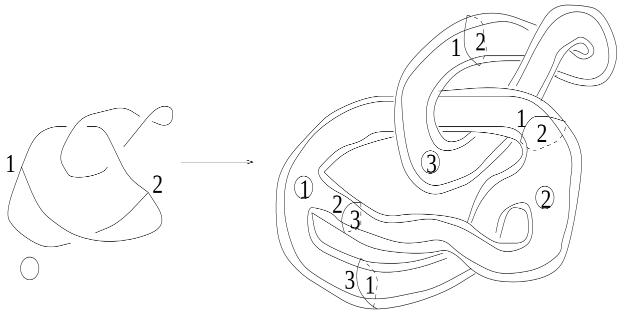

Figure 15.

<u>Proposition 2.1.</u> The quantum invariant of the e-3manifold $(M(\Gamma), D(\Gamma), 0)$ is given by

$$Z(M(\Gamma), D(\Gamma), 0) = X^{-n-1} \sum_{\gamma} d(\gamma)\Gamma(\gamma)\beta_V(\gamma) \otimes \beta_l(\gamma)$$

where n is half the number of vertices of Γ and the sum is over all colorings with the property that at each vertex we have an admissible triple.

<u>Proof:</u> Let us first note that, by Proposition 5.2 in [3], the property holds for link diagrams. Let us show that it holds for graph diagrams of the type described in Fig. 16. When we color this diagram, the only time when its value is nonzero is when the separating edges are colored by 0 (this follows from the properties of the Jones-Wenzl

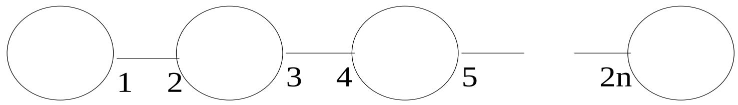

Figure 16.

idempotents [6]). On the other hand, the decomposition curves that correspond to these edges bound disks in $M(\Gamma)$, thus $Z(M(\Gamma), D(\Gamma), 0)$ lies in the vector space that corresponds to the labeling of all these edges by 0. Since β_{0kk} is equal to $(1/d_k)f^{(k)}$ as an element in the skein space of the disk, the value of this diagram for a certain coloring is $d_{k_1}d_{k_{n+1}}$ where k_1 is the color of the first loop and k_{n+1} is the color of the last loop. Let

$$Z(M(\Gamma), D(\Gamma), 0) = X^{-n-1} \sum_{d_{k_i}} c_{k_1,k_2,\cdots,k_{n+1}} \beta_{0k_1k_1} \otimes \beta_{0k_2k_2} \otimes \cdots \otimes \beta_{0k_{n+1}k_{n+1}}$$

By the gluing axiom (see [Ge], [8]), if we glue balls inside each tube that corresponds to a separating edge, for each of these balls the coefficients of the invariant get multiplied by $X/[1] = X$. Thus after gluing $n-1$ balls and contracting $n+1$ disks and $n-1$ annuli we get that the invariant of the new manifold is $X^{-1}\sum c_{k_1,k_2,\cdots,k_{n+1}}\beta_{k_1k_1} \otimes \beta_{k_2k_2} \otimes \cdots \otimes \beta_{k_{n+1}k_{n+1}}$, and since this manifold is the complement of a regular neighborhood of the trivial link, we get $c_{k_1,k_2,\cdots,k_{n+1}} = d_1^2d_2^2\cdots d_{n+1}^2$, thus the formula holds in this case.

The next step is to show that the formula holds for graph diagrams without crossings.

<u>Lemma 2.2</u> Any connected graph diagram without crossings can be reduced to one of the type described in Fig. 16 by a sequence of fusion moves (Fig. 17).

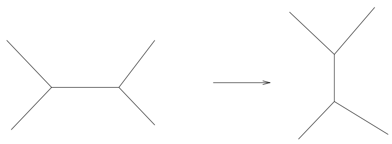

Figure 17.

<u>Proof:</u> Pick an outermost arc and by a sequence of fusion moves make its ends meet. Then proceed inductively.□

Returning to the proof of the theorem, recall that, by Theorem 4.1 in [3] the equality from Fig. 18 holds.

Thus the formula from the statement of the theorem and the invariant of the 3-manifold change the same way under fusion moves of Γ, which by Lemma 2.2 proves

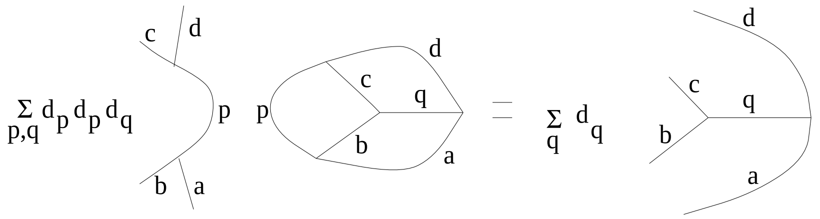

Figure 18.

the result for connected graph diagrams without crossings. Since a diagram with no crossings having several connected components can be obtained by taking connected sums of S^3's with connected diagrams in them, and since when taking the connected sum one takes the product of the invariants with an additional factor of X [8], the formula holds in this case too.

Let us now prove the result for arbitrary graphs. We do it by increasing the complexity of the diagram and decreasing the colors of *crossing* strands.

Let $Z(M(\Gamma), D(\Gamma), 0) = \sum_\gamma c(\gamma)\beta_V(\gamma) \otimes \beta_l(\gamma)$. We want to show that $c(\gamma) = X^{n-1}d(\gamma)\Gamma(\gamma)$.

If the coloring γ has the property that all strands that cross are labeled by 1, then the equality holds, since by Theorem 1.1 when we smooth the crossings both $c(\gamma)$ and $\Gamma(\gamma)$ change according to the Kauffman bracket skein relation, and the equality holds for diagrams without crossings. Moreover, if all strands that cross are labeled by 0 or by 1 the equality is still valid. This is because one can erase the strands labeled by 0 in the same manner as we did at the beginning of the proof.

For an arbitrary coloring γ, let $\mu(\gamma) = max_S\{\gamma(S), S$ strand crossing another strand $\}$, and $\nu(\gamma) = card\{S, \mu(S) = \gamma(S)$ and S crosses another strand$\}$. We will prove the theorem by induction on $(\mu(\gamma), \nu(\gamma))$, where the pairs are ordered lexicographically. As we saw above, the statement is true for $\mu(\gamma) = 1$. Suppose that the equality holds for all diagrams and all colorings with $(\mu(\gamma'), \nu(\gamma')) < (\mu(\gamma), \nu(\gamma))$ and let us prove it for γ.

Suppose that a strand S has $\gamma(S) = \mu(S) = m$, and crosses some other strands (Fig 19.a)). Double S in the region where it crosses other strands like in Fig. 19.b) to get the diagram Γ', then color the new edges by 1 and $m-1$ like in this figure. Let γ' be this coloring of the new diagram. We have $(\mu(\gamma'), \nu(\gamma')) < (\mu(\gamma), \nu(\gamma))$.

By using the identity from Fig 27 in [6] (note the different normalization of the Kauffman triad) we get $\Gamma'(\gamma') = 1/d_m^2\Gamma(\gamma)$. So by the induction hypothesis $c(\gamma') = X^{-n-2}d(\gamma')\Gamma'(\gamma')= X^{n-1}d(\gamma)d_m d_{m-1}d_1 \cdot d_m^{-2}\Gamma(\gamma) = X^{n-1}d(\gamma)d_{m-1}d_1 \cdot d_m^{-1}\Gamma(\gamma)$.

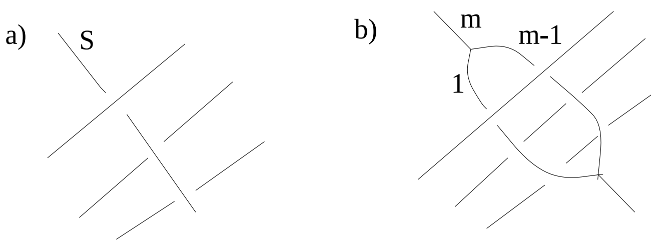

Figure 19.

On the other hand $(M(\Gamma'), D(\Gamma', 0)$ is obtained from $(M(\Gamma), D(\Gamma, 0)$ by expanding an annulus along the meridinal circle of S and gluing along it the mapping cylinder of a pair of pants. The vector component of the invariant of the mapping cylinder that contributes to $c(\gamma')$ is, by the mapping cylinder axiom ([3], [8]) $X^{-2}d_1 d_m d_{m-1}\beta_{1,m,m-1} \otimes \beta_{1,m-1,m}\otimes\beta_{mm}$ and the gluing introduces a factor X/d_m^2. Therefore $c(\gamma') = d_1 d_{m-1} d_m^{-1} X^{-1} c(\gamma)$ which gives the desired equality for $c(\gamma)$.□

3. The proof of the invariant formula

Now we will discuss the case of an arbitrary manifold. Following [7] we represent a 3-manifold with boundary by a graph diagram Γ in the plane, where some link components are marked with dots. An example is given in Fig. 20. The manifold $M(\Gamma)$ is the complement of a regular neighborhood of the subdiagram obtained by removing the dotted components, in the closed 3-manifold obtained by doing surgery along the marked link components by using the blackboard framing. Let $D(\Gamma)$ be the

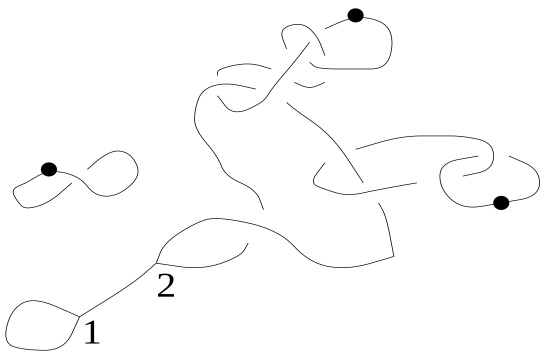

Figure 20.

DAP-decomposition of $\partial M(\Gamma)$ obtained by the procedure described at the beginning of the section.

Proposition 3.1. The quantum invariant of $(M(\Gamma), D(\Gamma), 0)$ is given by

$$Z(M(\Gamma),D(\Gamma),0)=X^{-n-m-1}C^{-\sigma}\sum_{\gamma}d(\gamma)(\prod_{l:marked}d^2_{\gamma(l)})\Gamma(\gamma)\beta_V(\gamma)\otimes(\otimes_{l:unmarked}\beta_{\gamma(l)\gamma(l)})$$

where n is half the number of vertices of Γ, m is the number of marked components and σ is the signature of the link formed by the marked components.

Proof: The result is a direct consequence of Proposition 2.1 and of the gluing axiom. Denote by Γ' the diagram that coincides with Γ except that none of the link components are marked.Then

$$Z(M(\Gamma'),D(\Gamma'),0)=X^{-n-1}\sum_{\gamma}d(\gamma)\Gamma(\gamma)\beta_V(\gamma)\otimes\beta_l(\gamma)$$

A solid torus with the DAP-decomposition as described in Fig. 21 has thein variant equal to $\sum d_p^2\beta_{pp}$. The manifold $M(\Gamma)$ can be obtained from $M(\Gamma')$ by gluing such

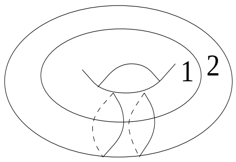

Figure 21.

tori along the marked link components such that the DAP-decompositions overlap. Of course in the gluing process the framing of the manifold will change. So we have

$$Z(M(\Gamma),D(\Gamma),\sigma)=X^{-n-m-1}\sum_{\gamma}d(\gamma)(\prod_{l:marked}d^2_{\gamma(l)})\Gamma(\gamma)\beta_V(\gamma)\otimes(\otimes_{l:unmarked}\beta_{\gamma(l)\gamma(l)})$$

for some σ.

Let us determine what σ is. For this let M_0 be the disjoint union of $M(\Gamma')$ and the m solid tori that we glue to it. Let $\Sigma_0=\partial M_0$ and let Σ be the part of $\Sigma 0$ that disappears in the gluing process. By the definition of the gluing of extended manifolds [8], $\sigma=-\sigma(L_1,L_2,L_3)$ where $\sigma(L_1,L_2,L_3)$ is Wall's nonadditivity function [9] applied to the three Lagrangian subspaces of $H_1(\Sigma)$ defined in the following way.

If we denote by J the subspace of $H_1(M_0)$ spanned by the decomposition curves in $\Sigma_0-\Sigma$ then $L_1=Ker(H_1(\Sigma)\rightarrow H_1(M_0)/J)$, L_2 is the subspace of $H_1(\Sigma)$ spanned by the decomposition curves and L_3 is the subspace spanned by elements of the form $(x,-f_*(x))$ with $x\in H_1(\partial M(\Gamma'))\cap H_1(\Sigma)$ and f the gluing map.

Denote $\Sigma_1=\Sigma\cap\partial M(\Gamma')$. Then by using the fact that Wall's function is computed by evaluating the intersection form on the vector space $L_1+(L_2\cap L_3)/(L_1\cap L_2+$

$L_1 \cap L_3$) we see that σ is in fact equal to $-\sigma(\mathrm{L}_1', L_2', L_3')$ where $L_1' = Ker(H_1(\Sigma_1)) \to H_1(M(\Gamma'))/(J \cap H_1(M(\Gamma')))$, L_2' is the subspace of $H_1(\Sigma_1)$ spanned by decomposition curves and L_3' is the subspace of $H_1(\Sigma_1)$ spanned by the curves that are parallel to the blackboard framing. Moreover, if at the beginning we erase the unmarked part of Γ we get the same spaces L_1', L_2', L_3'. Thus we can assume that the diagram that we started with was that of a link $\mathcal{L}$ on which we performed surgery, and $L_1' = Ker(H_1(\partial M(\mathcal{L})) \to H_1(M(\mathcal{L})))$, L_2' is spanned by meridians and L_3' is spanned by the blackboard framing. But then $-\sigma(L_1', L_2', L_3')$ is the signature of $\mathcal{L}$, and the theorem is proved.□

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