A Short Proof of a Result on Polynomials
by Răzvan Gelca

In this note we want to present a short proof of a result that appeared in [1]. For a polynomial 
\[ f(x) = \prod_{i=1}^{n} (x-x_i), \]
with distinct real roots \( x_1 < x_2 < \cdots < x_n \), we let \( d = \delta(f) = \min_{i} (x_{i+1}-x_{i}) \)
and \( g(x) = f'(x)/f(x) = \sum_{i=1}^{n} 1/(x-x_i) \). If \( k \) is a real number then the roots of the polynomial \( f' - kf \) are also real and distinct.

**PROPOSITION.** If for some \( j \), \( y_0 \) and \( y_1 \) satisfy \( y_0 < x_j < y_1 \leq y_0 + d \) then \( y_0 \) and \( y_1 \) are not zeros of \( f \) and \( g(y_0) < g(y_1) \).

**PROOF:** The hypothesis implies that for all \( i \), \( y_1 - y_0 \leq d \leq x_{i+1} - x_i \). Hence for \( 1 \leq i \leq j-1 \) we have \( y_0 - x_i \geq y_1 - x_{i+1} > 0 \) and so \( 1/(y_0 - x_i) \leq 1/(y_1 - x_{i+1}) \); similarly for \( j \leq i \leq n-1 \) we have \( y_1 - x_{i+1} \leq y_0 - x_i < 0 \) and again \( 1/(y_0 - x_i) \leq 1/(y_1 - x_{i+1}) \).

Finally \( y_0 - x_n < 0 < y_1 - x_1 \), so \( 1/(y_0 - x_n) < 0 < 1/(y_1 - x_1) \), and the result follows by addition of these inequalities.

**COROLLARY.** \( \delta(f' - kf) > \delta(f) \).

**PROOF:** If \( y_0 \) and \( y_1 \) are zeros of \( f' - kf \) with \( y_0 < y_1 \) then they are separated by a zero of \( f \) and satisfy \( g(y_0) = g(y_1) = k \). Hence from the proposition we can not have \( y_1 \leq y_0 + d \), so \( y_1 - y_0 > d \) as required.


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