COMPACT PERTURBATIONS OF FREDHOLM n-TUPLES*

Răzvan Gelca[†]

April 7, 2006

Abstract

Let T be an operator on a Hilbert space.We show that the pair (T,T) can be perturbed to an invertible pair if and only if T is Fredholm of index zero.We also exhibit a large class of Fredholm n-tuples acting on a Banach space which cannot be perturbed by finite rank operators to invertible ones.

INTRODUCTION

It is well known that a Fredholm operator of index zero on a Banach space can be perturbed by a finite rank operator to an invertible one.In [2] it is asked if the same property remains true for commuting pairs of operators, or at least if one can perturb a pair of index zero with *compact* operators to get an invertible one.

*AMS(MOS)Subj.Classif.: 47A53, 47A55, 47A60, 47B05. Words and Phrases: Fredholm n-tuple, compact perturbation, index zero.

[†]Research partially supported by NSF.

There are several properties of the index that are preserved when we pass from one operator to commuting n-tuples of operators, for example the index is invariant under small-norm perturbations, or under perturbations with operators in the norm-closure of finite rank ones; in the Hilbert space case these are exactly the compact operators [1], [4].It has been proved in [3] that the Koszul complex of a Fredholm n-tuple of index zero has a finite dimensional perturbation to an exact complex, but the new complex is usually not a Koszul complex of a commuting n-tuple.

In the first part of the present paper we shall prove that on an infinite dimensional Banach space there exists a large class of Fredholm n-tuples of index zero that cannot be perturbed with finite rank operators to invertible ones. Using the same idea we shall prove in the second part that on an infinite dimensional Hilbert space no pair of the form $(T, 0)$ with T Fredholm and $ind(T) \neq 0$ can be perturbed with compact operators to an invertible pair, this will give a negative answer to the question raised by Raúl Curto in [2].

1. FINITE RANK PERTURBATIONS

In this section we consider the case of (bounded) linear operators acting on an infinite dimensional Banach space $\mathcal{X}$.

To each commuting n-tuple $T = (T_1, T_2, \cdots, T_n)$ of operators on $\mathcal{X}$, we attach a complex of Banach spaces, called the Koszul complex [5], as follows. Let $\Lambda^p = \Lambda^p[e_1, e_2, \cdots, e_n]$ be the p-forms on $\mathbf{C}^n$.Define the operator $D_T : \mathcal{X} \otimes \Lambda^p \to \mathcal{X} \otimes \Lambda^{p+1}$ by $D_T := T_1 \otimes E_1 + T_2 \otimes E_2 + \cdots + T_n \otimes E_n$, where $E_i\omega := e_i\omega, i = 1, \cdots, n$.

The Koszul complex is

$$0 \to \mathcal{X} \bigotimes \Lambda^0 \overset{D_T}{\to} \mathcal{X} \bigotimes \Lambda^1 \overset{D_T}{\to} \cdots \overset{D_T}{\to} \mathcal{X} \bigotimes \Lambda^n \to 0. \tag{1}$$

Let $H^p(T)$ be its cohomology spaces. The n-tuple T is called invertible if $H^p(T) = 0, 0 \leq p \leq n$, and Fredholm if $dimH^p(T) < \infty, 0 \leq p \leq n$, in which case we define its index to be $indT := \sum_{p=0}^{n}(-1)^p dimH^p(T)$.

The Taylor spectrum of T, denoted by $\sigma(T)$, is the set of all $z = (z_1, z_2, \cdots, z_n)$ in $\mathbf{C}^n$ such that $z - T = (z_1 - T_1, z_2 - T_2, \cdots, z_n - T_n)$ is not invertible. It is known that $\sigma(T)$ is a compact nonvoid set. For any holomorphic map $f : U \to \mathbf{C}^m$ on a neighborhood U of $\sigma(T)$ one can define $f(T)$ [6]; this functional calculus extends the polynomial calculus. By the spectral mapping theorem [6], $f(\sigma(T)) = \sigma(f(T))$.

If $T = (T_1, T_2, \cdots, T_n)$ and $T' = (T_1, T_2, \cdots, T_n, S)$ are commuting tuples, then we have a long exact sequence in cohomology

$$0 \to H^0(T') \to H^0(T) \xrightarrow{\hat{S}} H^0(T) \to H^1(T') \to H^1(T) \to \cdots$$

$$H^{p-1}(T) \to H^p(T') \to H^p(T) \xrightarrow{\hat{S}} H^p(T) \to \cdots \tag{2}$$

where $\hat{S}$ is the operator induced by $S \bigotimes 1 : \mathcal{X} \bigotimes \Lambda^p \to \mathcal{X} \bigotimes \Lambda^p, 0 \leq p \leq n$. If T' is invertible then $\hat{S}$ is an isomorphism. As a consequence of the long exact sequence, if T is Fredholm then T' is Fredholm of index zero. We shall prove that there exists a class of (n+1)-tuples of index zero of the form $(T_1, T_2, \cdots, T_n, S)$ with $(T_1, T_2, \cdots, T_n)$ Fredholm and ind$(T_1, T_2, \cdots, T_n) \neq 0$ that cannot be made invertible by finite rank perturbations.

Lemma 1.1. Let $(S_1, S_2, \cdots, S_n)$ be an invertible commuting n-tuple and let $f : \mathbf{C}^n \to \mathbf{C}^m$ be a holomorphic function with $f^{-1}(0) = \{0\}$. Then $f(S_1, S_2, \cdots, S_n)$ is invertible.

Proof. Straightforward from the spectral mapping theorem by noticing the fact that $f^{-1}(0) \bigcap \sigma(S_1, S_2, \cdots, S_n) = \emptyset$.

Lemma 1.2. Let $(S_1, S_2, \cdots, S_n)$ be a Fredholm commuting n-tuple, with the property that $ind(S_1, S_2, \cdots, S_n) \neq 0$. Then there exists a sequence of positive integers $\{m_k\}_k$ and $0 \leq p_0 \leq n$ such that $dimH^{p_0}(S_1^{m_k}, S_2, \cdots, S_n) \to \infty$ for $k \to \infty$.

Proof. By Corollary 3.8 in [4] $ind(S_1^m, S_2, \cdots, S_n) = m{\cdot}ind(S_1, S_2, \cdots, S_n)$, so $dimH^p(S_1^m, S_2, \cdot\cdot$ cannot all remain bounded.

Theorem 1.1. Let $(T_1, T_2, \cdots, T_n)$ be a Fredholm commuting n-tuple with $ind(T_1, T_2, \cdots, T_n) \neq 0$, and $p \in \mathbf{C}[z_1, z_2, \cdots, z_n]$ with $p(0) = 0$. Define the operator $T_{n+1} = p(T_1, T_2, \cdots, T_n)$. Then there do not exist finite rank operators $R_1, R_2, \cdots,\ R_n, R_{n+1}$ such that $(T_1 + R_1, T_2 + R_2, \cdots, T_{n+1} + R_{n+1})$ is an invertible comuting (n+1)-tuple.

Proof. Suppose that such finite rank operators exist and let $S_i = T_i + R_i, 1 \leq i \leq n+1$. Applying Lemma 1.1 to the function $f : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}, f(z_1, z_2, \cdots, z_n, z_{n+1}) = (z_1, z_2, \cdots, z_n, z_{n+1} - p(z_1, z_2, \cdots, z_n))$ we get that $(S_1, S_2, \cdots, S_n, R)$ must be invertible, where $R = S_{n+1} - p(S_1, S_2, \cdots, S_n)$. Clearly, R is a finite rank operator. By applying Lemma 1.1 to the function $\phi : \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}, \phi(z_1, z_2, \cdots, z_n, z_{n+1}) = (z_1^m, z_2, \cdots, z_n, z_{n+1})$ we get that $(S_1^m, S_2, \cdots, S_n, R)$ is also invertible, for every positive integer m.

Let $\{m_k\}_k$ and p_0 be the numbers obtained by applying Lemma 1.2 to the n-tuple $(S_1, S_2, \cdots, S_n)$; let $\hat{R} = \hat{R}(m_k, p_0)$ be the operator induced by R on $H^{p_0}(S_1^{m_k}, S_2, \cdots, S_n)$. Because $(S_1^{m_k}, S_2, \cdots, S_n)$ is invertible, $\hat{R}$ must be an isomorphism for every m_k. But this is impossible since $dimH^{p_0}(S_1^{m_k}, S_2, \cdots, S_n) \to \infty$ and $rank(\hat{R}) \leq \binom{n}{p_0} \cdot rank(R)$. This proves the theorem.

2. THE MAIN EXAMPLE

In what follows we shall restrict ourselves to bounded linear operators on an infinite dimensional Hilbert space $\mathcal{H}$. We shall start with a result about the structure of a Fredholm operator of positive index.

Lemma 2.1. Let T be a Fredholm operator with $indT > 0$.

Define $\mathcal{H}_n = kerT^n \ominus kerT^{n-1}$. Then $\mathcal{H}_n \neq (0), n \geq 2$. Let $T_n = T|kerT^n$.

$$T_n : \mathcal{H}_n \bigoplus kerT^{n-1} \to \mathcal{H}_{n-1} \bigoplus kerT^{n-2}; T_n = \begin{bmatrix} A_n & 0 \\ B_n & C_n \end{bmatrix}. \tag{3}$$

Then there exists n_0 such that, for $n > n_0$, A_n is an isomorphism.

Proof. Suppose that for some n $\mathcal{H}_n = (0)$. Then $kerT^n = kerT^{n-1}$. Hence $kerT^{n+k} = kerT^n, \forall k \geq 0$. But this contradicts the fact that $\lim_{n\to\infty} indT^{n+k} = \infty$.

Since $\mathcal{H}_n \perp kerT_{n-1}$, $T|\mathcal{H}_n$ is injective and $T\mathcal{H}_n \bigcap kerT_{n-2} = (0)$. This shows that A_n is injective. But then the sequence $\{dim\mathcal{H}_n\}_n$ is decreasing so it becomes stationary. Let n_0 be such that for $n > n_0$, $dim\mathcal{H}_n = dim\mathcal{H}_{n-1}$. Then for $n > n_0$, A_n is an injective operator between finite dimensional spaces of same dimension so it is an isomorphism.

Lemma 2.2. Let T and $\mathcal{H}_n, n \geq 2$, be as in the statement of previous lemma. If S is an operator that commutes with T, then for all $n \geq 1$, $kerT^n$ is an invariant subspace for S. Let $S_n = S|kerT^n$,

$$S_n : \mathcal{H}_n \bigoplus kerT^{n-1} \to \mathcal{H}_n \bigoplus kerT^{n-1}; S_n = \begin{bmatrix} X_n & 0 \\ Y_n & Z_n \end{bmatrix}. \tag{4}$$

Then there is n_0 such that for $n \geq n_0$, X_n is similar to X_{n_0}.

Proof. The fact that $kerT^n$ is invariant for S follows from the commutativity. Let A_n and n_0 be as in Lemma 2.1. Then $ST = TS$ implies $S_{n-1}T_n = T_nS_n, n \geq 2$. Therefore, $X_{n-1}A_n = A_nX_n, n \geq 2$. For $n > n_0$ A_n is an isomorphism hence X_n is similar to X_{n-1}. This proves the lemma.

Lemma 2.3. Let (T, S) be an invertible commuting pair. Then for any n, $S|kerT^n$ is an isomorphism of $kerT^n$.

Proof. Applying Lemma 1.1 to (T,S) and $f:\mathbf{C}^2\to\mathbf{C}^2$, $f(z_1,z_2)=(z_1^n,z_2)$ we get that (T^n,S) is invertible for any n. By the remarks made at the beginning of the first section, $\hat{S}:H^0(T^n)\to H^0(T^n)$ is an isomorphism. But $H^0(T^n)=kerT^n$, and the lemma is proved.

Theorem 2.1. Let T be a Fredholm operator with $indT\neq 0$. Then there do not exist compact operators K_1 and K_2 such that $(T+K_1,K_2)$ is an invertible commuting pair.

Proof. Suppose such K_1 and K_2 exist. Without loss of generality we may assume $indT>0$, otherwise we take T^* instead of T. We can also assume that $K_1=0$, otherwise we can denote $T+K_1$ by T, and let $K_2=K$.

Consider the spaces $\mathcal{H}_n, n\geq 2$, obtained by applying Lemma 2.1 to T, and let $K_n=K|kerT^n$. By Lemma 2.2,

$$K_n:\mathcal{H}_n\bigoplus kerT^{n-1}\to\mathcal{H}_n\bigoplus kerT^{n-1}; K_n=\begin{bmatrix} X_n & 0 \\ Y_n & Z_n\end{bmatrix}. \tag{5}$$

have the property that X_n similar to X_{n_0} for some n_0 and $n\geq n_0$. Applying Lemma 2.3 we get that the operators $X_n, n\geq 2$ are isomorphisms. If we denote by r the spectral radius of X_{n_0}, then $r>0$. From the fact that X_n is similar to X_{n_0} for $n\geq n_0$, (so all X_n's have the same spectral radius), it follows that $\|X_n\|\geq r$.

But $\|K|\mathcal{H}_n\|=\|K_n|\mathcal{H}_n\|\geq\|X_n\|\geq r$ for $n\geq n_0$. Because $\mathcal{H}_n\perp\mathcal{H}_m, n\neq m$, and $\mathcal{H}_n\neq(0)$ for any n, it follows that K is not compact, a contradiction. Therefore such K_1 and K_2 cannot exist.

Corollary 2.1. The pair (T,T) can be perturbed by compacts to an invertible commuting pair if and only if T can be perturbed by a compact to an invertible operator.

Bibliography

[1] Ambrozie, C.,*On Fredholm index in Banach spaces*, preprint, 1990;

[2] Curto, R., *Problems in multivariable operator theory*, Contemporary Math., 120 (1991), 15-17;

[3] Curto, R., in *Surveys of some recent results in operator theory*, Conway, J. and Morrel, B., eds., vol. II, Pitman Res. Notes in Math., Ser. 192, Longman Publ. Co., London, 1988, 25-90;

[4] Putinar, M., *Some invariants for semi-Fredholm systems of essentially commuting operators*, JOT 8(1982), 65-90;

[5] Taylor, J.,L., *A joint spectrum for several commuting operators*, J.Funct.Anal. 6 (1970), 172-191;

[6] Taylor, J.,L., *The analytic functional calculus for several commuting operators*, Acta Math. 125 (1970), 1-38.

Department of Mathematics, The University of Iowa, Iowa City, IA 52242.

E-mail: rgelca@math.uiowa.edu

and

Institute of Mathematics of Romanian Academy, P.O.Box 1-764, 70700 Bucharest, Romania.