REPRESENTATIONS OF THE KAUFFMAN BRACKET SKEIN ALGEBRA OF THE PUNCTURED TORUS

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ABSTRACT. In this paper we describe the action of the Kauffman bracket skein algebra on some vector spaces that arise as relative Kauffman bracket skein modules of tangles in the punctured torus. We show how this action determines the Reshetikhin-Turaev representation of the punctured torus. We rephrase our results to describe the quantum group quantization of the moduli space of flat $SU(2)$-connections on the punctured torus with fixed trace of the holonomy around the boundary.

1. Introduction

Skein modules and skein algebras [16], [19] have been studied intensively, especially in connection with the Witten-Reshetikhin-Turaev theory [1], [15], [6] and the Jones polynomial [10]. The skein algebras of surfaces and their action of skein modules of handlebodies have been studied in [4], [8], and [9] in relation to the quantization of moduli spaces of connections on surfaces, which is part of the Witten-Reshetikhin-Turaev theory. Extensive studies have been performed in the case of the torus.

In this note we go one step further and examine the punctured torus. We do this because of the fact that the basic data of the modular functor of the Reshetikhin-Turaev topological quantum field theory is contained entirely in the punctured torus and in the sphere with four punctures [6].

We describe below the representations of the Kauffman bracket skein algebra that are relevant to the quantization problem. Then we show how the Reshetikhin-Turaev representations of the mapping class group of the punctured torus, which form the basic data on the punctured torus, can be recovered explicitly from these representations.

2. The Kauffman bracket skein algebra of the punctured torus

In what follows we will use the notations and results from [12], with t standing for A. Let $\Sigma_{1,1}$ be the *punctured torus*, namely the torus with one open disk removed. for a curve that is a smooth embedding of a circle in $\Sigma_{1,1} \times [0,1]$, a framing consists of a smooth vector field along the curve always orthogonal to it. We are concerned with families of such curves,

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called framed links. The framing can always be made parallel to $\Sigma_{1,1}$, the blackboard framing. As such we will represent framed links by curves, the framing being self-understood.

It is standard [16] to define the Kauffman bracket skein module $K_t(\Sigma_{1,1}\times[0,1])$ as the quotient of the free $\mathbb{C}[t,t^{-1}]$-module with basis the set of isotopy classes of framed links in $\Sigma_{1,1}\times[0,1]$ by the Kauffman bracket skein relation shown in Figure 1. The links that show up in this figure are supposed to be equal except in an embedded ball. Also, every trivial link component is replaced by a factor of $-t^2-t^{-2}$.

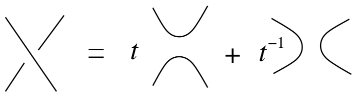

FIGURE 1

The topological operation of gluing one cylinder over the punctured torus on top of another induces a multiplication on $K_t(\Sigma_{1,1}\times[0,1])$, which turns it into an algebra. This is the *Kauffman bracket skein algebra of the punctured torus.*

Simple closed curves on the punctured torus are defined by pairs (p,q) of coprime integers with $p\geq 0$, with q/p being the slope of the curve, together with the curve ∂ that is parallel to the boundary. As shown in [2], the Kauffman bracket skein algebra of the punctured torus is generated by the curves $(1,0)$, $(0,1)$, $(1,1)$, depicted in Figure 2. Let ∂ denote the boundary

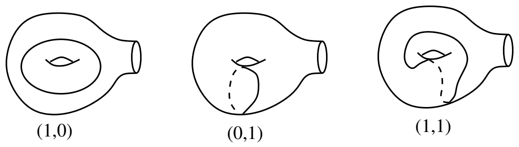

FIGURE 2

curve.

As a $\mathbb{C}[t,t^{-1}]$-module, $K_t(\Sigma_{1,1}\times[0,1])$ is free with basis $(p,q)_T\partial^k$, $p,k\geq 0$, $q\in\mathbb{Z}$, where ∂ is a curve parallel to the boundary and $(p,q)_T = T_n(p/n,q/n)$, with $n=gcd(p,q)$ and $T_n(x)$ the Chebyshev polynomial of first kind ($T_{n+1}(x)=xT_n(x)-T_{n-1}(x)$, $T_0(x)=2$, $T_1(x)=x$).

That multiplication of curves on the punctured torus is significantly different from that on the torus is illustrated by the second of the following formulas.

Proposition 2.1. *In* $K_t(\Sigma_{1,1}\times[0,1])$,

$$\begin{aligned}(2,0)_T(0,1)_T &= t^2(2,1)_T+t^{-2}(2,-1)_T\\ (2,1)_T(0,1)_T &= t^2(2,2)_T+t^{-2}(2,0)_T+\partial+t^2+t^{-2}.\end{aligned}$$

3. The representations

First, fix a positive integer r and let $t = \exp \frac{\pi i}{2r}$. Let also $[n] = \sin \frac{n\pi}{r} / \sin \frac{\pi}{r}$ be the quantized integer. The vector spaces on which we represent the Kauffman bracket skein algebra of the punctured torus are parametrized by the integers n with $0 \leq 2n \leq r-2$. These are the vector spaces $V_{r,n}$ which we will define in what follows.

Consider a solid torus with $2n$ disjoint marked points on the boundary, which should be numbered by $1, 2, \ldots, 2n$. For computational purposes, we can consider these points to lie, in increasing order, on a diameter of the puncturing disk. Consider the free $\mathbb{C}[t, t^{-1}]$-module with basis the set of isotopy classes of framed tangles with ends the $2n$ marked points. Such a tangle consists of several embedded circles together with n embedded arcs whose ends are the $2n$ points. The framing consists of a 1-dimensional continuous vector field on each of the components of the tangle, with the convention that the vectors that belong to the endpoints lie on the specified diameter of the puncturing disk. We will always draw our tangles in the blackboard framing, namely such that the vector field is parallel to the plane of the paper. An example for $n = 2$ is shown in Figure 3 a).

We define the Kauffman bracket skein module of the solid torus with $2n$ points on the boundary, $K_t(S^2 \times \mathbb{D}^2, 2n)$, to be the quotient of this free module by the Kauffman bracket skein relations.

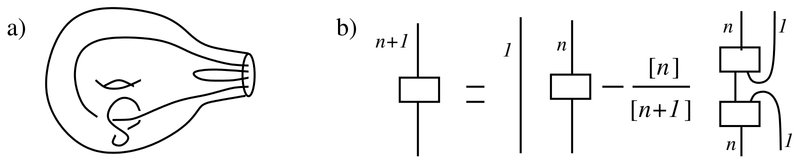

Figure 3

The topological operation of gluing the cylinder over the punctured torus $\Sigma_{1,1} \times [0,1]$ to the complement of the puncturing disk in the boundary of the solid torus gives rise to an action of $K_t(\Sigma_{1,1} \times [0,1])$ on $K_t(S^1 \times \mathbb{D}^2, 2n)$.

For $k < n$, we will define a family of inclusions of $K_t(S^1 \times \mathbb{D}^2, 2k)$ into $K_t(S^1 \times \mathbb{D}^2, 2n)$. To this end, let δ be the data consisting of a function $f : \{1, 2, \ldots, 2k\} \to \{1, 2, \ldots, 2n\}$ such that $f(i) - f(i-1)$ is odd for $i = 2, 3, \ldots, 2k$ and a pairing of the $2n - 2k$ numbers in the complement of $\mathrm{Im} f$ such that if (p, q) is a pair then p and q belong to the same interval $(f(i-1), f(i))$ and for every two pairs (p, q) and (r, s), $(r-p)(r-q)(s-p)(s-q) > 0$. For each such δ we define the inclusion

$$i_\delta : K_t(S^1 \times \mathbb{D}^2, 2k) \to K_t(S^1 \times \mathbb{D}^2, 2n)$$

by identifying the $2k$ boundary points of $K_t(S^1\times\mathbb{D}^2,2k)$ with the boundary points of $K_t(S^2\times\mathbb{D}^2,2n)$ indexed by $f(i)$, $i=1,2,\ldots,2k$, for each pair (p,q), connecting these points by an arc isotopic to the segment $[p,q]$.[1]

Next we factor $K_t(S^1\times\mathbb{D}^2,2n)$ by the skein relation $f^{r-1}=0$, where f^n, $n\geq 1$ are the Jones-Wenzl idempotents defined recursively in Figure 3 b), with f^1 being just one strand. The result of this factorization is a finite-dimensional space $K_{t,r}(S^1\times\mathbb{D}^2,2n)$ called the reduced Kauffman bracket skein module. The inclusions i_δ factor to inclusions between reduced Kauffman bracket skein modules. The action of $K_t(\Sigma_{1,1}\times[0,1])$ factors to an action of the same algebra on the reduced Kauffman bracket skein module.

For $n\leq m\leq r-2-n$, define the skein $v_{2n,m}$ as shown in Figure 4 a).

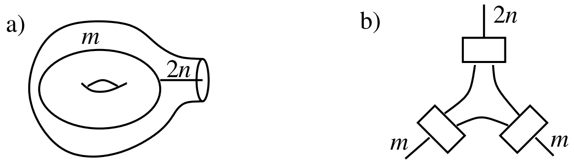

FIGURE 4

Here there are m respectively $2n$ parallel strands, as specified, with the corresponding Jones-Wenzl idempotents placed on them. At the trivalent vertex there is the Kauffman triad described in Figure 4 b) (see [12]).

Lemma 3.1. *The vector space $K_{t,r}(S^1\times\mathbb{D}^2,2n)$ is finite dimensional with basis $i_\delta(v_{2k,m})$, where $0\leq k\leq n$, $k\leq m\leq r-2-k$, and δ ranges over all possible sets of data defined above.*

Proof. There is a planar projection of the solid torus onto an annulus such that the projections of all these elements have no crossings. To see that they form a basis, note first that any skein in $K_{t,r}(S^1\times\mathbb{D}^2,2n)$ can be written as a linear combination of skeins of the form $i_\delta(\sigma)$, where σ is a skein in some $K_{t,r}(S^1\times\mathbb{D}^2,2k)$ which in a tubular neighborhood of the puncturing disk is just an $2k$-strand decorated by the $2k$th Jones-Wenzl idempotent. Consequently, the set $i_\delta(v_{2k,m})$, indexed by $0\leq k\leq m$, $k\leq m\leq r-2$ and δ, spans $K_{t,r}(S^1\times\mathbb{D}^2,2n)$. Next, we can apply the arguments from [14], to reduce everything to the case where the trivalent vertex has an admissible coloring, namely to $k\leq m\leq r-2-k$, and then conclude that these elements form a basis. □

Because ∂ is in the center of $K_t(\Sigma_{1,1}\times[0,1])$, the eigenspaces of ∂ are invariant subspaces of the above representation. The eigenvalues of ∂ are $-t^{4k+2}-t^{-4k-2}$, $k\leq n$. The eigenspace of the eigenvalue $-t^{4k+2}-t^{-4k-2}$ has a basis $i_\delta(v_{2k},m)$, for all δ and m, $k\leq m\leq r-2-k$. Among these we distiguish the ones with $k=n$. These are the spaces of interest to us since they correspond to quantizations of moduli spaces of flat connections

[1]If in a diagram a skein σ has not crossings, then $i_\delta(\sigma)$ has no crossing either.

on the punctured torus (see § 4 below). Let therefore $V_{r,n}$ be the eigenspace of ∂ corresponding to the eigenvalue $-t^{4n+2}-t^{-4n-2}$, $n=0,1,\dots,r-2$. A basis of this vector space consists of the vectors $v_{2n,m}$, $n\le m\le r-2-n$.

Theorem 3.2. *Let n be an integer such that $0\le n\le \frac{r-2}{2}$. The representation of $K_t(\Sigma_{1,1}\times[0,1])$ on $V_{r,n}$ is given by*

$$\begin{aligned}
(1,0)v_{2n,m} &= v_{2n,m+1}+\frac{[m-n][m+n+1]}{[m][m+1]}v_{2n,m-1}\\
(0,1)v_{2n,m} &= (-t^{2m+2}-t^{-2m-2})v_{2n,m}\\
(1,1)v_{2n,m} &= (-t^{-2m-3})v_{2n,m+1}+(-t^{2m+1})\frac{[m-n][m+n+1]}{[m][m+1]}v_{2n,m-1}
\end{aligned}$$

where $n\le m\le r-2-n$, with the convention that $v_{2n,n-1}=v_{2n,r-1-n}=0$.

Proof. The second relation is standard (see [14]). To compute $(1,0)v_{2n,m}$, we proceed as in Figure 5, with the notation from [12] where $\Delta_n=(-1)^{n+1}[n+1]$ and $\theta(m,n,p)$ is the quantum invariant of the trivalent graph of the letter θ with the edges colored by m,n,p. Here for the second equality we use the Recoupling Theorem (Ch. 7 in [12]) for one of the strands colored by m.

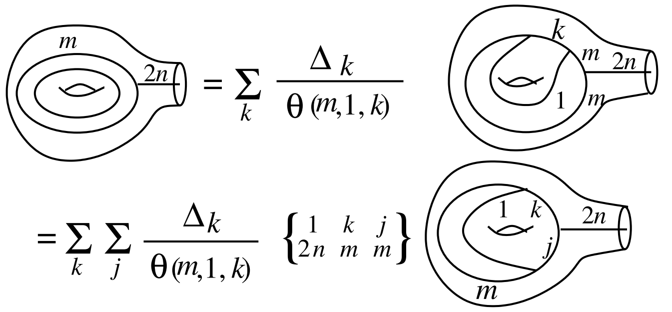

FIGURE 5

Next, by Lemma 7 Ch. 5 in [12], this is nonzero if and only if $j=k$, and in this case it is equal to the skein in Figure 6. Also, since the triple $(1,m,k)$

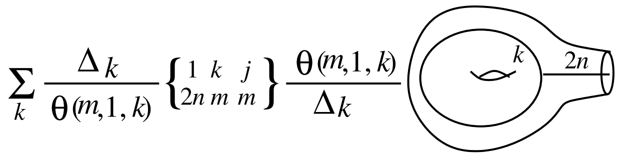

FIGURE 6

is admissible only if $k=m-1$ or $k=m+1$, it follows that the sum has only two terms. Substituting the values of the two quantum 6j-symbols we obtain the formula from the statement.

The action of $(1,1)$ is computed using

$$(1,1) = \frac{1}{t^2 - t^{-2}}[t(1,0)(0,1) - t^{-1}(0,1)(1,0)].$$

□

Remark 3.3. In [3] a proof was given based only on the definition of Jones-Wenzl idempotents and Kauffman triads.

In the case where $n = 0$, we obtain the representation of the Kauffman bracket skein algebra of the torus on the reduced skein module of the punctured torus from [4].

Corollary 3.4. *The representation of $K_t(\Sigma_{1,1} \times [0,1])$ on $V_{r,n}$ is irreducible.*

Proof. Let $v = \sum_m c_m v_{2n,m}$ be a vector in $V_{r,n}$. Because $v_{2n,m}$ are eigenvectors of $(0,1)$ with distinct eigenvalues, using linear combinations of elements of the form $(0,1)^n v$ we can generate any basis vector that appears with nonzero coefficient in the expansion of v. Applying $(1,0)$ repeatedly, we can furhter generate a vector that contains any other basis vector with nonzero coefficient. It follows that from v we can generated any basis vector. Hence v is cyclic, and therefore the representation is irreducible. □

4. The action of the mapping class group

In this section we will show how the Reshetikhin-Turaev representation of the mapping class group of the punctured torus, which appears in Reshetikhin-Turaev theory [17], can be computed from the representation of the Kauffman bracket skein algebra described in Theorem 3.2. The Reshetikhin-Turaev representation of the punctured torus is part of the basic data of the modular functor (see [20], [6], [5]). It is known that the representation of the skein algebra of the punctured torus determines this representation, see for example [18]; here we show how this with explicit formulas, the way it was done in [7] for the torus.

The mapping class group of the punctured torus, with fixed boundary, is generated by the maps S, T, and T_1, where S exchanges the curves $(1,0)$ and $(0,1)$, T is the positive Dehn twist along the curve $(0,1)$ and T_1 is the positive twist of the boundary.

4.1. **The S-matrix.** For a number λ and a sequence $(x_l)_{l\geq 1}$ define the sequence $P_n(\lambda, (x_l)_{l\geq 1})$ recursively by

$$P_{n+1}(\lambda, (x_l)_{l\geq 1}) = \lambda P_n(\lambda, (x_l)_{l\geq 1}) - x_n P_{n-1}(\lambda, (x_l)_{l\geq 1}),$$
$$P_0(\lambda, (x_l)_{l\geq 1}) = 1, \quad P_1(\lambda, (x_l)_{l\geq 1}) = \lambda.$$

Let $S = (a_{jk})$, $0 \leq j, k \leq r - 2n - 2$ be the S-matrix. The equations

$$(1,0)Sv_{2n,n+j} = S(0,1)v_{2n,n+j}$$
$$(0,1)Sv_{2n,n+j} = S(1,0)v_{2n,n+j}$$

yield respectively the recursive relations

$$a_{j-1,k} = (-t^{2n+2k+2} - t^{-2n-2k-2})a_{j,k} - \frac{[j+1][2n+j+2]}{[n+j+1][n+j+2]}a_{j+1,k}$$

$$a_{j,k-1} = (-t^{2n+2j+2} - t^{-2n-2j-2})a_{j,k} - \frac{[k+1][2n+k+2]}{[n+k+1][n+k+2]}a_{j+1,k}.$$

Thus a_{kj} can be obtained by a backwards double recursion. Normalizing by setting $a_{r-2n-2,r-2n-2} = 1$, we obtain

Proposition 4.1. *For* $0 \leq k, j \leq r - 2n - 2$,

$$a_{r-2n-2-j,r-2n-2-k} = P_j(\lambda_{r-n}, (x_l)_{l\geq 1}) \cdot P_k(\lambda_{r-n-j}, (x_l)_{l\geq 1}),$$

where

$$x_l = \frac{[r-n-1-l][n+r-l]}{[r-l-1][r-l]}, \quad l \geq 1,$$

$$\lambda_m = -t^{2m-2} - t^{-2m+2}, \quad m \geq 0.$$

Remark 4.2. This formula should be contrasted with those exhibited in [18] and [6].

4.2. **The T-matrix and the twist on the boundary.** Because $T(0,1) = (0,1)T$ and $(0,1)$ has 1-dimensional eigenspaces, it follows that T is diagonal, say $T = (b_{j,j})_j$, $0 \leq j \leq r - 2n - 2$. The equality

$$(1,0)Tv_{2n,n+j} = T(1,1)v_{2n,n+j}$$

yields the recursive relation $b_{j,j} = -t^{2n+2j+1}b_{j-1,j-1}$. As T is only projectively defined, we are free to choose $b_{1,1}$, and we set it equal to $(-1)^n t^{n^2-1}$. Then $b_{j,j} = (-1)^{n+j}t^{(n+j)^2-1}$, the well known formula.

The twist T_1 on the boundary commutes with all operators in $K_t(\Sigma_{1,1} \times [0,1])$. Because the representation of $K_t(\Sigma_{1,1} \times [0,1])$ on $V_{r,n}$ is irreducible, it follows that T_1 acts as multiplication by a scalar. We may choose this scalar to be $t^{(2n)^2-1}$.

5. The quantum group quantization of the moduli space of flat $su(2)$-connections on the punctured torus

As explained in [8] and [9], we are actually concerned with the quantum group setting, since it is our paradigm that the quantum group quantization of the moduli space of flat connections on a surface is an analogue of Weyl quantization.

The operators are quantizations of Wilson lines. Recall that for a simple closed curve γ on the punctured torus, the Wilson line W_γ is the function on the space of $su(2)$-connections on the punctured torus obtained by taking the trace of the holonomy of the connections along γ. The vector space has an orthogonal basis specified by the same diagrams as those for $v_{2n,m}$, this time with strands colored by the $m+1$- respectively $2n+1$-dimensional irreducible representations of the quantum group of $SU(2)$.

The quantum group quantization can also be done with skein modules (see [13], [9]); the skein associated to a Wilson line W_γ is γ itself. The formulas for this action are the same except that the minus sign in front of t is deleted in each of the formulas for $(0,1)$ and $(1,1)$.

For each γ, W_γ is real valued, so the associated operator $\mathrm{Op}(W_\gamma)$ must be self-adjoint. This shows that, while the vectors $v_{2n,m}$ are orthogonal, they are not unit vectors. We normalize them to

$$w_{2n+1,m+1} = \left(\prod_{j=n+1}^{m} \frac{[j-n][j+n+1]}{[j][j+1]}\right)^{-1/2} v_{2n,m},$$

which are now unit vectors. The shift in the indices is so that they agree with the dimensions of the corresponding irreducible representation of the quantum group. We thus have

Proposition 5.1. *The quantization at $\hbar = \frac{1}{2r}$ of the moduli space of flat $su(2)$-connections on the punctured torus with the trace of the holonomy on the boundary equal to $2\cos\frac{4\pi i n}{r}$, for $n \in \{0,1,\ldots,r-2\}$, has the Hilbert space $\mathcal{H}_{r,n}$ with orthonormal basis $w_{2n+1,m}$, $n+1 \le m \le r-1-n$, and with the algebra of quantum observables acting by*

$$Op(W_{(1,0)})w_{2n+1,m} = \sqrt{\frac{[m-n][m+n+1]}{[m][m+1]}}w_{2n+1,m+1}$$
$$+\sqrt{\frac{[m-1-n][m+n]}{[m-1][m]}}w_{2n+1,m-1}$$
$$Op(W_{(0,1)})w_{2n+1,m} = (t^{2m}+t^{-2m})w_{2n+1,m}$$
$$Op(W_{(1,1)})w_{2n+1,m} = t^{-2m-1}\sqrt{\frac{[m-n][m+n+1]}{[m][m+1]}}w_{2n+1,m+1}$$
$$+t^{2m-1}\sqrt{\frac{[m-1-n][m+n]}{[m-1][m]}}w_{2n+1,m-1}.$$

References

[1] Ch. Blanchet, N. Habegger, G. Masbaum, P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology, **34** (1995), no. 4, 883–927.

[2] D. Bullock, J.H. Przytycki, *Multiplicative structure of Kauffman bracket skein module quantizations*, Proc. Amer. Math. Soc., **128** (2000), 923–931.

[3] J.P. Cho, *The Kauffman bracket skein algebra of the punctured torus*, PhD Thesis.

[4] Ch. Frohman, R. Gelca, *Skein modules and the noncommutative torus*, Trans. Amer. Math. Soc., **352** (2000), 4877–4888.

[5] Ch. Frohman, J. Kania-Bartoszyńska, *$SO(3)$ topological quantum field theory*, *Commun. Anal.Geom.* **4**(1996), 589–679.

[6] R. Gelca, *Topological quantum field theory with corners based on the Kauffman bracket*, Comment. Math. Helv., **72**(1997), 216-243.

[7] R. Gelca, *On the holomorphic point of view in the theory of quantum knot invariants*, J. Geom. Phys., **56** (2006), 2163–2176.

[8] R. Gelca, A. Uribe, *The Weyl quantization and the quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus are the same*, Commun. Math. Phys., **233**(2003), 493–512.
[9] R. Gelca, A. Uribe, *Quantum mechanics and non-abelian theta function for the gauge group $SU(2)$*, **arXiv:1007.2010**.
[10] V.F.R. Jones, *Polynomial invariants of knots via von Neumann algebras*, Bull. Amer. Math. Soc., **12**(1995), 103–111.
[11] V.F.R. Jones, *Index of subfactors*, Inventiones Math., **72** (1983), 1–24.
[12] L. Kauffman, S. Lins, *Temperley-Lieb Recoupling Theory and Invariants of* 3-*Manifolds*, Princeton Univ. Press, 1994.
[13] R. Kirby, P. Melvin, *The 3-manifold invariants of Witten and Reshetikhin-Turaev for $sl(2,\mathbb{C})$*, Inventiones Math., **105** (1991), 473–545.
[14] W.B.R. Lickorish, *Skeins and handlebodies*, Pacific J. Math., **149** (1993), 337–349.
[15] W.B.R. Lickorish, *The skein method for* 3-*manifold invariants*, J. Knot Theor. Ramif., **2** (1993), 171-194.
[16] J.H. Przytycki, *Skein modules of 3-manifolds*, Bull. Pol. Acad. Sci. **39(1-2)** (1991) 91–100.
[17] N. Reshetikhin, V. Turaev, *Invariants of* 3-*manifolds via link polynomials and quantum groups*, Inventiones Math., **103** (1991), 547–597.
[18] J. D. Roberts, *Skeins and mapping class groups*, Math. Proc. Cam. Phil. Soc. **115** (1994), 53–77.
[19] V.G. Turaev, *Algebras of loops on surfaces, algebras of knots, and quantization*, Adv. Ser. in Math. Phys., **9** (1989), eds. C.N. Yang, M.L. Ge, 59–95.
[20] K. Walker, *On Witten's 3-manifold invariants*, preprint, 1991.
[21] H. Wenzl, *On sequences of projections* C.R. Math. Rep. Acad. Sci. R. Can. IX (1987), 5–9.
[22] E. Witten, *Quantum field theory and the Jones polynomial*, Commun. Math. Phys., **121** (1989), 351–399.

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