NON-ABELIAN THETA FUNCTIONS A LA ANDRÉ WEIL

Răzvan Gelca        Alejandro Uribe
Texas Tech University  University of Michigan

WE WILL SHOW THAT THE CONSTRUCTS OF THE THEORY OF CLASSICAL THETA FUNCTIONS IN THE REPRESENTATION THEORETIC POINT OF VIEW OF ANDRÉ WEIL HAVE ANALOGUES FOR THE NON-ABELIAN THETA FUNCTIONS OF THE WITTEN-RESHETIKHIN-TURAEV THEORY
THE PROTOTYPE:

- the Schrödinger representation
- the metaplectic representation

They arise when quantizing a free 1-dimensional particle ($\hbar = 1$).

Phase space has coordinates $q$: position, $p$: momentum.

Quantization: phase space $\mapsto L^2(\mathbb{R})$,

\[ q \mapsto Q = \text{multiplication by } q, \]
\[ p \mapsto P = \frac{1}{i} \frac{d}{dq}. \]

Add to this the operator $E = Id$.

Heisenberg uncertainty principle:

\[ QP - PQ = i\hbar E \]
Weyl quantization:

\[ \hat{f}(\xi, \eta) = \int \int f(x, y) \exp(-2\pi i x \xi - 2\pi i y \eta) \, dx \, dy \]

and then defining

\[ \text{Op}(f) = \int \int \hat{f}(\xi, \eta) \exp 2\pi i (\xi Q + \eta P) \, d\xi \, d\eta. \]

Notation:

\[ \exp(xP + yQ + tE) = e^{2\pi i (xP + yQ + tE)}. \]

EXAMPLE:

\[ \exp(x_0 P)\phi(x) = \phi(x + x_0), \]
\[ \exp(y_0 Q)\phi(x) = e^{2\pi i xy_0} \phi(x). \]
The elements \( \exp(xP + yQ + tE) \), \( x, y, t \in \mathbb{R} \) form the Heisenberg group with real entries \( \mathbf{H}(\mathbb{R}) \):

\[
\exp(xP + yQ + tE) \exp(x'P + y'Q + t'E) = \exp[(x + x')P + (y + y')Q + (t + t' + \frac{1}{2}(xy' - yx'))E]
\]

The action of \( \mathbf{H}(\mathbb{R}) \) on \( L^2(\mathbb{R}) \) is called the Schrödinger representation.

**THEOREM (Stone-von Neumann)** The Schrödinger representation is the unique irreducible unitary representation of \( \mathbf{H}(\mathbb{R}) \) that maps \( \exp(tE) \) to multiplication by \( e^{2\pi it} \) for all \( t \in \mathbb{R} \).

**COROLLARY** Linear (symplectic) changes of coordinates can be quantized.
Recall that the linear maps that preserve the classical mechanics are

\[ SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad - bc = 1 \right\} . \]

If \( h \in SL(2, \mathbb{R}) \), and \( h(x, y) = (x', y') \) then

\[
\exp(xP + yQ + tE) \circ \phi = \exp(x'P + y'Q + tE)\phi
\]

is another representation, which by the Stone-von Neumann theorem is unitary equivalent to the Schrödinger representation.

Hence there is a unitary map \( \rho(h) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) such that the exact Egorov identity is satisfied:

\[
\exp(x'P + y'Q + tE) = \rho(h) \exp(xP + yQ + tE)\rho(h)^{-1}.
\]

The map \( h \to \rho(h) \) is a projective representation of \( SL(2, \mathbb{R}) \) on \( L^2(\mathbb{R}) \). One can make this into a true representation by passing to a double cover \( M(2, \mathbb{R}) \) of \( SL(2, \mathbb{R}) \). This is the metaplectic representation.
EXAMPLE: If

\[ S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \]

then

\[ \rho(S)\phi(x) = \int_{\mathbb{R}} \phi(y)e^{-2\pi i xy} dy \]

\[ \rho(T)\phi(x) = e^{2\pi ix^2 a} \phi(x). \]

The metaplectic representation can be interpreted as a general Fourier transform. It is a Fourier-Mukai transform.

In general, a Heisenberg group is a $U(1)$ (or cyclic) extension of a locally compact abelian group. It has an associated Schrödinger representation and Fourier-Mukai transform. The two are related by the exact Egorov identity.
A. Weil’s representation theoretic point of view:

Weyl quantization of a particle with periodic position and momentum ($\hbar = 1/N$, $N$ an even integer).

Quantization: phase space $\mapsto$ space of theta functions which has an orthonormal basis consisting of the theta series

$$\theta_j(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i N \left[ \frac{i}{2} \left( \frac{j}{N} + n \right)^2 + z \left( \frac{j}{N} + n \right) \right]}, \quad j = 0, 1, 2, \ldots, N - 1.$$  

The quantizations of the exponential functions on the torus generate a finite Heisenberg group $\textbf{H}(\mathbb{Z}_N)$ which is a $\mathbb{Z}_{2N}$-extension of $\mathbb{Z}_N \times \mathbb{Z}_N$:

$$\exp(pP + qQ + kE)\theta_j = e^{-\frac{\pi i}{N}pq} e^{\frac{2\pi i}{N} jq} e^{\frac{\pi i}{N} k} \theta_{j+p}, \quad p, q, k \in \mathbb{Z}.$$
**THEOREM (Stone-von Neumann)** The Schrödinger representation of $H(\mathbb{Z}_N)$ is the unique irreducible unitary representation of this group with the property that $\exp(kE)$ acts as $e^{\frac{\pi i}{N}k} \text{Id}$ for all $k \in \mathbb{Z}$.

The linear maps on the torus that preserve classical mechanics are

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

i.e. the mapping class group of the torus.

**COROLLARY** There is a projective representation $\rho$ of the mapping class group of the torus on the space of theta functions that satisfies the exact Egorov identity

$$\exp(p' P + q' Q + kE) = \rho(h) \exp(pP + qQ + kE) \rho(h)^{-1}$$

where $(p', q') = h(p, q)$. Moreover, for every $h$, $\rho(h)$ is unique up to multiplication by a constant.

This action of the mapping class group is called the Hermite-Jacobi action.
Non-abelian theta functions for the group $SU(2)$

Arise from the quantization of the moduli space of flat $su(2)$-connections on a genus $g$ surface $\Sigma$ ($\hbar = \frac{1}{2N}$, $N = 2r$, $r$ an integer).

$$\mathcal{M}_g^{SU(2)} = \{ A \mid A : su(2) - \text{connection} \} / \mathcal{G}$$
$$= \{ \rho : \pi_1(\Sigma) \longrightarrow SU(2) \} / \text{conjugation}$$

Quantization: $\mathcal{M}_g^{SU(2)} \mapsto$ space of non-abelian theta functions.

Non-abelian theta series can be parametrized by coloring the core of a genus $g$ handlebody by irreducible representations of $U_\hbar(SL(2, \mathbb{C}))$, the quantum group of $SU(2)$.

$U_\hbar(SL(2, \mathbb{C}))$ has irreducible representations $V^1, V^2, \ldots, V^{r-1}$. They satisfy a Clebsch-Gordan theorem

$$V^m \otimes V^n = \bigoplus_p V^p$$

where $|m - n| + 1 \leq p \leq \min(m + n - 1, 2r - 2 - m - n)$. 
The theta series are the colorings of the core by $V^1, V^2, \ldots, V^{r-1}$ such that at each vertex the conditions from the Clebsch-Gordan theorem are satisfied.

The analogues of the exponentials are the Wilson lines

$$W_{\gamma,n}(A) = \text{tr}_{V_n \text{hol}_\gamma}(A)$$

where $\gamma$ is a simple closed curve on the surface.

The operator associated to $W_{\gamma,n}$, denoted by $\text{op}(W_{\gamma,n})$, has a matrix whose “entries” are the Reshetikhin-Turaev invariants of diagrams of the form:
The Reshetikhin-Turaev theory yields a projective representation $\rho$ of the mapping class group of the surface on the space of non-abelian theta functions.

This projective representation satisfies the following relation with the quantizations of Wilson lines

$$\text{op}(W_{h(\gamma)},n) = \rho(h)\text{Op}(W_{\gamma},n)\rho(h)^{-1}$$

which is an exact Egorov identity.
Skein modules! There is an easy way to see these using skein modules.

The skein relations are those of the Reshetikhin-Turaev version of the Jones polynomial:

\[ L = tH + t^{-1}V \text{ or } L = \epsilon (tH - t^{-1}V) \]

depending on whether the two crossing strands come from different components or not, where \( \epsilon \) is the sign of the crossing. Set also the trivial knot equal to \( t^2 + t^{-2} \).

Let \( RT_t(M) \) be the Reshetikhin-Turaev skein module of the 3-manifold \( M \), obtained by factoring the free module with basis the isotopy classes of framed links in \( M \) by these relations.

As a module, \( RT_t(M) \) is isomorphic to the Kauffman bracket of the manifold, but...
If \( M = \Sigma \times [0, 1] \) then \( RT_t(\Sigma \times [0, 1]) \) is an algebra:

\[
\Sigma \times [0, 1] \cup \Sigma \times [0, 1] \approx \Sigma \times [0, 1].
\]

This algebra is not isomorphic to the Kauffman bracket skein algebra.

If \( M \) has a boundary, then \( RT_t(M) \) is a \( RT_t(\partial M \times [0, 1]) \)-module:

\[
\partial M \times [0, 1] \cup M \approx M.
\]

The reduced RT skein module of \( M \), denoted by \( \tilde{RT}_t(M) \) is obtained by factoring \( RT_t(M) \) by \( t = e^{i\pi/2r} \) and \( f_{r-1} = 0 \) where \( f_{r-1} \) is the \( r-1 \)st Jones-Wenzl idempotent.
The space of non-abelian theta functions of a genus $g$ surface $\Sigma_g$ is $\RT_t(H_g)$ where $H_g$ is the genus $g$ handlebody.

The skein theoretic versions of non-abelian theta series are obtained by replacing

$$V^k \rightarrow f_{k-1}.$$

Algebra generated by quantized Wilson lines is isomorphic to $\RT_t(\Sigma_g \times [0, 1])$ with the isomorphism given by

$$\text{Op}(W_{\gamma, 2}) \mapsto \gamma.$$

The action of operators on non-abelian theta functions coincides with the action of $\RT_t(\Sigma_g \times [0, 1])$ on $\RT_t(H_g)$. 
Our paradigm: There are the following analogies

At the level of the vector space

a. \( L^2(\mathbb{R}) \)

b. classical theta functions

c. non-abelian theta functions
Our paradigm: There are the following analogies

At the level of the vector space

a. $L^2(\mathbb{R})$

b. classical theta functions

c. non-abelian theta functions

At the level of quantum observables

a. The group algebra of the Heisenberg group $H(\mathbb{R})$

b. The group algebra of the finite Heisenberg group $H(\mathbb{Z}_N)$

c. The algebra generated by quantized Wilson lines $\mathfrak{op}(W_{\gamma,n})$
Our paradigm: There are the following analogies

At the level of the vector space

a. $L^2(\mathbb{R})$

b. classical theta functions
c. non-abelian theta functions

At the level of quantum observables

a. The group algebra of the Heisenberg group $H(\mathbb{R})$

b. The group algebra of the finite Heisenberg group $H(\mathbb{Z}_N)$
c. The algebra generated by quantized Wilson lines $\text{op}(W_{\gamma,n})$

At the level of quantized changes of coordinates

a. The metaplectic representation (i.e. the Fourier transform)

b. The Hermite-Jacobi action (i.e. the discrete Fourier transform)
c. The Reshetikhin-Turaev representation
For the exact Egorov identity

a. The exact Egorov identity for the metaplectic representation

\[ \exp(x' P + y' Q + tE) = \rho(h) \exp(x P + yQ + tE) \rho(h)^{-1}. \]

where \( h(x, y) = (x', y') \).

b. The exact Egorov identity for the Hermite-Jacobi action

\[ \exp(p' P + q' Q + kE) = \rho(h) \exp(p P + q Q + kE) \rho(h)^{-1} \]

where \( (p', q') = h(p, q) \).

c. The exact Egorov identity for the Reshetikhin-Turaev representation

\[ \text{op}(W_{h(\gamma)}, n) = \rho(h) \text{Op}(W_{\gamma}, n) \rho(h)^{-1} \]
Applications of this point of view:

1. The quantum group quantization of Wilson lines determines the Reshetikhin-Turaev representation.

3. **The case of the torus:**

The moduli space is the **pillow case**:

A basis of the space of non-abelian theta functions is

\[ \zeta_j(z) = (\theta_j(z) - \theta_{-j}(z)), \quad j = 1, 2, \ldots, r - 1, \]

where \( \theta_j \) are the theta series

\[ \theta_j(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i 2r \left( \frac{k}{2r} + n \right)^2 + z \left( \frac{j}{2r} + n \right)} \], \quad j = 0, 1, 2, \ldots, 2r - 1. \]

The \( \zeta_j \)'s can be represented graphically as
Fact: Just in genus one, the algebra of quantized Wilson lines, which is the reduced Reshetikhin-Turaev skein algebra, is isomorphic to the reduced Kauffman bracket skein algebra of the torus at $t = e^{i\pi/2r}$.

**THEOREM (Frohman-G.)** The Kauffman bracket skein algebra has the multiplication rule

$$(p_1, q_1)_T(p_2, q_2)_T = t^{p_1q_2-p_2q_1}(p_1 + p_2, q_1 + q_2)_T$$

$$+ t^{-(p_1q_2-p_2q_1)}(p_1 - p_2, q_1 - q_2)_T$$

where $(p, q)$ is the curve of slope $p/q$ on the torus if $\gcd(p, q) = 1$ and $(p, q) = T_n((p/n, q/n))$ if $n = \gcd(p, q) > 1$, $T_n$ being the Chebyshev polynomial of first kind.
A Stone-von Neumann theorem can be proved in this case:

**THEOREM (G.-Uribe)** The representation of the reduced Reshetikhin-Turaev skein algebra of the torus defined by the Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus is the unique irreducible representation of this algebra that maps simple closed curves to self-adjoint operators and $t$ to multiplication by $e^{\frac{\pi i}{2r}}$. Moreover, quantized Wilson lines span the algebra of all linear operators on the Hilbert space of the quantization.

This implies the existence of the Reshetikhin-Turaev representation

$$h \mapsto \rho(h)$$

of the mapping class group of the torus without apriori knowing it.

Because the skein algebra contains all linear operators, $\rho(h)$ can be represented as multiplication by a skein. Here is the computation...
Let $h = T$, the twist. Recall that $[n]$ denotes the quantized integer $\sin \frac{n\pi}{r} / \sin r$. The exact Egorov identity, in skein form, reads

$$h(\gamma) = \rho(h) \gamma \rho(h)^{-1}$$

where $\gamma$ is a simple closed curve.

From this identity we deduce that

$$\rho(h) = \sum_{j=1}^{r-1} \alpha_j (0, 1)^j.$$  

Rewrite this as

$$\rho(h) = \sum_{j=1}^{r-1} c_j S_{j-1}((0, 1))$$

where $S_n$ is the $n$th Chebyshev polynomial of second type.
Then

\[(1, 1) \rho(T) \zeta_k = \rho(T)(1, 0) \zeta_k.\]

\[
\sum_j c_j \frac{[j k]}{k} t^{-1}(t^{2k} \zeta_{k+1} + t^{2k} \zeta_{k-1})
= \sum_j c_j \left( \frac{[j(k + 1)]}{[k + 1]} \zeta_{k+1} + \frac{[j(k - 1)]}{[k - 1]} \zeta_{k-1} \right).
\]

Setting the coefficients of \( \zeta_{k+1} \) on both sides equal yields the system

\[
\frac{r-1}{j} \sum_{j=1}^{r-1} c_j [j(k + 1)] = \frac{[k + 1]}{[k]} t^{-2k-1} \sum_{j=1}^{r-1} c_j [j k].
\]

Solving we obtain

\[
\rho(h) = \sum_{j=1}^{r-1} [j] t^{j^2} S_{j-1}((0, 1)).
\]
We conclude that $\rho(h)$ is the skein obtained by coloring the surgery curve of $T$ by

$$\Omega = \sum_j [j]V^j$$

Using the fact that each element of the mapping class group is a composition of twists we obtain:

**THEOREM** Let $h$ be an element of the mapping class group of the torus defined by surgery on the framed link $L_h$ in $T^2 \times [0, 1]$. Then

$$\rho(h) : \widehat{RT}_t(S^1 \times D^2) \to \widehat{RT}_t(S^1 \times D^2)$$

is given by

$$\rho(h)\beta = \Omega(L_h)\beta$$

where $\Omega(L_h)$ is the skein obtained by coloring all components of $L$ by $\Omega$. 
The exact Egorov identity

\[ h(\gamma) = \rho(h)\gamma\rho(h)^{-1} \]

gives

The skein on the right is

Exact Egorov identity \implies handle slides in the cylinder over the torus.
In conclusion, by looking at non-abelian theta functions from André Weil’s point of view we can introduce in a natural way the element $\Omega$, which is the building block of the Reshetikhin-Turaev topological quantum field theory, and we arrive at slides along link components colored by $\Omega$, which is the main principle behind constructing quantum 3-manifold invariants.