QUANTUM MECHANICS AND NON-ABELIAN THETA FUNCTIONS FOR THE GAUGE GROUP $SU(2)$

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Abstract. This paper outlines an approach to the non-abelian theta functions of the $SU(2)$-Chern-Simons theory with the methods used by A. Weil for studying classical theta functions. First we translate in knot theoretic language classical theta functions, the action of the finite Heisenberg group, and the discrete Fourier transform. Then we explain how the non-abelian counterparts of these arise in the framework of the quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface, in the guise of the non-abelian theta functions, the action of a skein algebra, and the Reshetikhin-Turaev representation of the mapping class group. We prove a Stone-von Neumann theorem on the moduli space of flat $SU(2)$-connections on the torus, and using it we deduce the existence and the formula for the Reshetikhin-Turaev representation on the torus from quantum mechanical considerations. We show how one can derive in a quantum mechanical setting the skein that allows handle slides, which is the main ingredient in the construction of quantum 3-manifold invariants.

Contents

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1. Introduction

This paper outlines a study of the non-abelian theta functions that arise in Chern-Simons theory by adapting the method used by André Weil for studying classical theta functions [44]. We discuss the case of the gauge group $SU(2)$, which is important because it corresponds to the Witten-Reshetikhin-Turaev topological quantum field theory, and hence is related to the Jones polynomial of knots [46], [35]. The methods can be applied to more general gauge groups, which will be done in subsequent work.

In Weil's approach, classical theta functions come with an action of the finite Heisenberg group and a projective representation of the mapping class group. By analogy, our point of view is that the theory of non-abelian theta functions consists of:

• the Hilbert space of non-abelian theta functions, namely the holomorphic sections of the Chern-Simons line bundle;

• an irreducible representation on the space of theta functions of the algebra generated by quantized Wilson lines (i.e. of the quantizations of traces of holonomies of simple closed curves);

• a projective representation of the mapping class group of the surface on the space of non-abelian theta functions.

The representation of the mapping class group intertwines the quantized Wilson lines; in this sense the two representations satisfy the exact Egorov identity.

Non-abelian theta functions, quantized Wilson lines, and the projective representations of the mapping class groups of surfaces have each been studied separately; we suggest that they should be studied together. One should recall one instance when these were considered together: the proof of the asymptotic faithfulness of the Reshetikhin-Turaev representation [11], [2].

Our prototype is the quantization of a one-dimensional particle. The paradigm is that *the quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface and the Reshetikhin-Turaev representation of the mapping class group are the analogues of the Schrödinger representation of the Heisenberg group and of the metaplectic representation.* The Schrödinger representation arises from the quantization of the position and the momentum of a one-dimensional free particle, and is a consequence of a fundamental postulate in quantum mechanics. It is a unitary irreducible representation of the Heisenberg group, and the Stone-von Neumann theorem shows that it is unique. This uniqueness implies that linear changes of coordinates (which act as outer automorphisms of the Heisenberg group) are also quantizable, and their quantization yields an infinite dimensional representation of the metaplectic group.

Weil [44] observed that a finite Heisenberg group acts on classical theta functions, and that the well known Hermite-Jacobi action of the modular group $SL(2,\mathbb{Z})$ is induced via a Stone-von Neumann theorem. Then it was noticed that classical theta functions, the action of the Heisenberg group, and of the modular group arise from the Weyl quantization of Jacobian varieties. As such, classical theta functions are the holomorphic sections of a line bundle over the moduli space of flat $u(1)$-connections on a surface, and by analogy, the holomorphic sections of the similar line bundle over the moduli space of flat $\mathfrak{g}$-connections over a surface (where $\mathfrak{g}$ is the Lie algebra of a compact simple Lie group) were called non-abelian theta functions. Witten [46] placed non-abelian theta functions in the context of Chern-Simons theory, related them to the Jones polynomial [18] and conformal field theory, and gave new methods for studying them. This had a great impact in the guise of the Verlinde formula which computes the dimension of the space of non-abelian theta functions. We explain how within Witten's theory one can find the non-abelian analogues of Weil's constructs. This is done for the group $SU(2)$.

Because the work joins methods from the theory of theta functions, quantum mechanics, representation theory, and low dimensional topology, we made it as self-contained as possible. At the heart of the paper lies a comparison between classical theta functions on the Jacobian variety of a complex torus and the non-abelian theta functions of the gauge group $SU(2)$.

We first review the prototype: the Schrödinger and metaplectic representations. Then, we recall the necessary facts about classical theta functions from [16], just for the case of the torus. We point out two properties of the finite Heisenberg group: the Stone-von Neumann theorem, and the fact that its group algebra is symmetric with respect to the action of the mapping class group of the torus. Each is responsible for the existence of the Hermite-Jacobi action by discrete Fourier transforms on theta functions. These properties have non-abelian counterparts, which we reveal later in the paper. We then rephrase theta functions, the Schrödinger representation, and the discrete Fourier transform in topological language following [16], using skein modules [41], [32].

The fact that skein modules can be used to describe classical theta functions is a corollary of Witten's Feynman integral approach to the Chern-Simons theory for the group $U(1)$. In fact, Witten has explained in [46] that the topological quantum field theory of any Lie group gives rise to skein relations, hence it should have a skein theoretic version. For example, the skein theoretic version for the gauge group $SU(2)$ was constructed in [6]. We showed in [16] that the skein modules of the abelian theory arise naturally from quantum mechanics, without relying on quantum field theory.

Next we recall the construction of non-abelian theta functions from the quantization of the moduli space of flat connections on a surface, and present in detail the case of the torus, where the moduli space is the pillow case. The paper continues with a description of the quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface, rephrased into the language of the skein modules. We deduce that the quantum group quantization is defined by the left action of a skein algebra on a quotient of itself. Because we are interested in the multiplicative structure, we use the skein relations of the Reshetikhin-Turaev invariant instead of those of the Kauffman bracket, since the latter introduce sign discrepancies. We recall our previous result [15] that on the pillow case, the quantum group and Weyl quantizations coincide. Then we prove a Stone-von Neumann theorem on the pillow case.

An application is to deduce the existence of the Reshetikhin-Turaev representation on the torus as a consequence of this Stone-von Neumann theorem. We show how the explicit formula for this representation can be computed from quantum mechanical considerations. Consequently we arrive at the element Ω by which one colors knots as to allow Kirby moves along them. This element is the fundamental building block in the construction of quantum 3-manifold invariants (see [35], [6], [26]), and we give a natural way to obtain it.

We conclude the paper by explaining how the Reshetikhin-Turaev representation of the mapping class group of a surface is a non-abelian analogue of the Hermite-Jacobi action given by discrete Fourier transforms.

2. The prototype

2.1. **The Schrödinger representation.** In this section we review briefly the Schrödinger and the metaplectic representations. For a detailed discussion we suggest [29] and [31].

In the canonical formalism, a classical mechanical system is described by a symplectic manifold (M^{2n}, ω), which is the phase space of the system. The classical observables are C^∞ functions on M. To each $f \in C^\infty(M^{2n}, \mathbb{R})$ one associates a Hamiltonian vector field X_f on M^{2n} by $df(\cdot) = \omega(X_f, \cdot)$.

This vector field defines a Hamiltonian flow on the manifold which preserves the form ω. The symplectic form defines a Poisson bracket by $\{f, g\} = \omega(X_f, X_g)$. There is a special observable H, called the Hamiltonian (total energy) of the system. The time evolution of an observable is described by the equation

$$\frac{df}{dt} = \{f, H\}.$$

Quantization replaces the symplectic manifold by a Hilbert space, real-valued observables f by self-adjoint operators $\mathrm{Op}(f)$ called quantum observables, Hamiltonian flows by one-parameter groups of unitary operators, and the Poisson bracket of $\{f, g\}$ by $\frac{\hbar}{i}[\mathrm{Op}(f), \mathrm{Op}(g)]$, where $\hbar$ is Planck's constant and $[\cdot,\cdot]$ is the commutator of operators. Dirac's conditions should hold: $\mathrm{Op}(1) = Id$ and

$$\mathrm{Op}(\{f, g\}) = \frac{i}{\hbar}[\mathrm{Op}(f), \mathrm{Op}(g)] + O(\hbar).$$

The second condition says that the quantization is performed in the direction of the Poisson bracket. The time evolution of a quantum observable is described by Schrödinger's equation

$$i\hbar \frac{d\mathrm{Op}(f)}{dt} = [\mathrm{Op}(f), \mathrm{Op}(H)].$$

A fundamental example is that of a particle in a 1-dimensional space, which we discuss in the case where Planck's constant is equal to 1. The phase space is $\mathbb{R}^2$, with coordinates the position x and the momentum y, and symplectic form $\omega = dx \wedge dy$. The associated Poisson bracket is

$$\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

The symplectic form ω induces a nondegenerate antisymmetric bilinear form on $\mathbb{R}^2$, also denoted by ω, given by $\omega((x, y), (x', y')) = xy' - x'y$.

The Lie algebra of observables has a subalgebra generated by $Q(x, y) = x$, $P(x, y) = y$, and $E(x, y) = 1$, called the Heisenberg Lie algebra. Abstractly, this algebra is defined by $[Q, P] = E$, $[P, E] = [Q, E] = 0$.

It is a postulate of quantum mechanics that the quantization of the position, the momentum, and the constant functions is the representation of the Heisenberg Lie algebra on $L^2(\mathbb{R}, dx)$ defined by

$$Q \to M_x, \quad P \to \frac{1}{i}\frac{d}{dx}, \quad E \to iId.$$

Here M_x denotes the operator of multiplication by the variable: $\phi(x) \to x\phi(x)$. This is the Schrödinger representation of the Heisenberg Lie algebra.

The Lie group of the Heisenberg Lie algebra is the Heisenberg group. It is defined as $\mathbb{R}^3$ with the multiplication rule

$$(x, y, t)(x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2}\omega((x, y), (x', y'))\right).$$

It is standard to denote $\exp(xQ+yP+tE)=(x,y,t)$. By exponentiating the Schrödinger representation of the Lie algebra one obtains the Schrödinger representation of the Heisenberg group:

$$\exp(Q)\to e^{2\pi i M_x},\quad \exp(P)\to e^{-\frac{d}{dx}},\quad \exp(E)\to e^{2\pi i Id}.$$

Specifically, for $\phi\in L^2(\mathbb{R},dx)$,

$$\exp(y_0P)\phi(x)=\phi(x-y_0),\quad \exp(x_0Q)\phi(x)=e^{2\pi i x x_0}\phi(x),$$
$$\exp(tE)\phi(x)=e^{2\pi i t}\phi(x),$$

meaning that $\exp(y_0P)$ acts as a translation, $\exp(x_0Q)$ acts as the multiplication by a character, and $\exp(tE)$ acts as the multiplication by a scalar. This is the rule for quantizing exponential functions. Specifically, $\exp(x_0Q+y_0P+tE)$ is the quantization of the function $f(x,y)=\exp(2\pi i(x_0x+y_0y+t))$.

Extending by linearity one obtains the quantization of the group ring of the Heisenberg group. This was further generalized by Hermann Weyl, who gave a method for quantizing all functions $f\in C^\infty(\mathbb{R}^2)$ by using the Fourier transform

$$\hat{f}(\xi,\eta)=\iint f(x,y)\exp(-2\pi i x\xi-2\pi i y\eta)dxdy$$

and then defining

$$\mathrm{Op}(f)=\iint \hat{f}(\xi,\eta)\exp 2\pi i(\xi Q+\eta P)d\xi d\eta,$$

where for $\exp(\xi Q+\eta P)$ he used the Schrödinger representation.

Theorem (Stone-von Neumann) The Schrödinger representation of the Heisenberg group is the unique irreducible unitary representation of this group such that $\exp(tE)$ acts as $e^{2\pi i t}Id$ for all $t\in\mathbb{R}$.

There are two other important realizations of the irreducible representation that this theorem characterizes. One comes from the quantization of the plane in a holomorphic polarization. The Hilbert space is the Bargmann space,

$$\mathrm{Bargmann}(\mathbb{C})=\left\{f:\mathbb{C}\to\mathbb{C}\text{ holomorphic },\int_{\mathbb{C}}|f(z)|^2e^{-2\pi|\mathrm{Im}\ z|^2}dz\wedge d\bar{z}<\infty\right\},$$

where the Heisenberg group acts by

$$\exp(x_0P)f(z)=f(z-x_0),\quad \exp(y_0Q)f(z)=e^{\pi(y_0^2-2iy_0z)}f(z+iy_0),$$
$$\exp(tE)f(z)=e^{2\pi i t}f(z).$$

For the other one has to choose a Lagrangian subspace $\mathbf{L}$ of $\mathbb{R}P+\mathbb{R}Q$ (which in this case is just a one-dimensional subspace). Then $\exp(\mathbf{L}+\mathbb{R}E)$ is a maximal abelian subgroup of the Heisenberg group. Consider the character of this subgroup defined by $\chi_{\mathbf{L}}(\exp(l+tE))=e^{2\pi i t}, l\in\mathbf{L}$. The Hilbert space of the quantization, $\mathcal{H}(\mathbf{L})$, is defined as the space of functions $\phi(u)$ on $\mathbf{H}(\mathbb{R})$ satisfying

$$\phi(uu') = \chi_L(u')^{-1}\phi(u) \text{ for all } u' \in \exp(\mathbf{L} + \mathbb{R}E)$$

and such that $u \to |\phi(u)|$ is a square integrable function on the left equivalence classes modulo $\exp(\mathbf{L} + \mathbb{R}E)$. The representation of the Heisenberg group is given by

$$u_0\phi(u) = \phi(u_0^{-1}u).$$

If we choose an algebraic complement $\mathbf{L}'$ of $\mathbf{L}$, meaning that we write $\mathbb{R}P + \mathbb{R}Q = \mathbf{L} + \mathbf{L}' = \mathbb{R} + \mathbb{R}$, then $\mathcal{H}(\mathbf{L})$ can be realized as $L^2(\mathbf{L}') \cong L^2(\mathbb{R})$. Under a natural isomorphism,

$$\begin{aligned}
&\exp(x_0)\phi(x) = \phi(x - x_0), \quad x, x_0 \in \mathbf{L}' \\
&\exp(y_0)\phi(x) = e^{2\pi i\omega(x,y_0)}\phi(x), \quad x \in \mathbf{L}', y_0 \in \mathbf{L} \\
&\exp(tE)\phi(x) = e^{2\pi it}\phi(x), \quad x \in \mathbf{L}'
\end{aligned}$$

where ω is the standard symplectic form on $\mathbb{R}P+\mathbb{R}Q$. For $\mathbf{L} = \mathbb{R}P$ and $\mathbf{L}' = \mathbb{R}Q$, one obtains the standard Schrödinger representation in the position representation. For $L = \mathbb{R}Q$ and $L' = \mathbb{R}P$, one obtains the Schrödinger representation in the momentum representation: $\exp(y_0P)\phi(x) = e^{-2\pi ixy_0}\phi(x)$, $\exp(x_0Q)\phi(x) = \phi(x - x_0)$, $\exp(tE)\phi(x) = e^{2\pi it}\phi(x)$.

2.2. **The metaplectic representation.** The Stone-von Neumann theorem implies that if we change coordinates by a linear symplectomorphism and then quantize, we obtain a unitary equivalent representation of the Heisenberg group. Hence linear symplectomorphisms can be quantized, giving rise to unitary operators, although they do not arise from Hamiltonian flows. Irreducibility implies, by Schur's lemma, that these operators are unique up to a multiplication by a constant. Hence we have a projective representation ρ of the linear symplectic group $SL(2,\mathbb{R})$ on $L^2(\mathbb{R})$. This can be made into a true representation by passing to the double cover of $SL(2,\mathbb{R})$, namely to the metaplectic group $Mp(2,\mathbb{R})$. The representation of the metaplectic group is known as the *metaplectic representation* or the Segal-Shale-Weil representation.

The fundamental symmetry that Weyl quantization has is that, if $h \in Mp(2,\mathbb{R})$, then

$$\mathrm{Op}(f \circ h^{-1}) = \rho(h)\mathrm{Op}(f)\rho(h)^{-1},$$

for every observable $f \in C^\infty(\mathbb{R}^2)$, where $\mathrm{Op}(f)$ is the operator associated to f through Weyl quantization. For other quantization models this relation holds only mod $O(\hbar)$, (*Egorov's theorem*). When it is satisfied with equality, as it is in our case, it is called the *exact* Egorov identity.

An elegant way to define the metaplectic representation is to use the third version of the Schrödinger representation from §2.1, which identifies the metaplectic representation as a Fourier transform (see [29]). Let h be a linear symplectomorphism of the plane, then let $\mathbf{L}_1$ be a Lagrangian subspace of

$\mathbb{R}P+\mathbb{R}Q$ and $\mathbf{L}_2 = h(L_1)$. Define the quantization of h as $\rho(h):\mathcal{H}(\mathbf{L}_1)\to\mathcal{H}(\mathbf{L}_2)$,

$$(\rho(h)\phi)(u) = \int_{\exp\mathbf{L}_2/\exp(\mathbf{L}_1\cap\mathbf{L}_2)} \phi(uu_2)\chi_{\mathbf{L}_2}(u_2)d\mu(u_2),$$

where $d\mu$ is the measure induced on the space of equivalence classes by the Haar measure on $\mathbf{H}(\mathbb{R})$.

To write explicit formulas for $\rho(h)$ one needs to choose the algebraic complements $\mathbf{L}_1'$ and $\mathbf{L}_2'$ of $\mathbf{L}_1$ and $\mathbf{L}_2$ and unfold the isomorphism $L^2(\mathbf{L}')\cong L^2(\mathbb{R})$. For example, for

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

if we set $\mathbf{L}_1 = \mathbb{R}P$ with variable y and $L_2 = S(\mathbf{L}_1) = \mathbb{R}Q$ with variable x and $\mathbf{L}_1' = \mathbf{L}_2$ and $\mathbf{L}_2' = S(\mathbf{L}_1') = \mathbf{L}_1$, then

$$\rho(S)f(x) = \int_{\mathbb{R}} f(y)e^{-2\pi ixy}dy,$$

the usual Fourier transform, which establishes the unitary equivalence between the position and the momentum representations. Similarly, for

$$T_a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

if we set $\mathbf{L}_1 = \mathbf{L}_2 = \mathbb{R}P =$, $\mathbf{L}_1' = \mathbb{R}Q$, and $\mathbf{L}_2' = \mathbb{R}(P+Q)$, then

$$\rho(T_a)f(x) = e^{2\pi i x^2 a}f(x)$$

which is the multiplication by the exponential of a quadratic function. These are the well known formulas that define the action of the metaplectic group on $L^2(\mathbb{R}, dx)$.

The cocycle of the projective representation of the symplectic group is

$$c_L(h', h) = e^{-\frac{i\pi}{4}\boldsymbol{\tau}(\mathbf{L}, h(\mathbf{L}), h'\circ h(\mathbf{L}))}$$

where $\boldsymbol{\tau}$ is the Maslov index. This means that

$$\rho(h'h) = c_{\mathbf{L}}(h', h)\rho(h')\rho(h)$$

for $h, h' \in SL(2,\mathbb{R})$.

3. Classical theta functions

3.1. Classical theta functions from the quantization of the torus. For an extensive treatment of theta functions the reader can consult [30], [29], [31]. We consider the simplest situation, that of theta functions on the Jacobian variety of a 2-dimensional complex torus $\mathbb{T}^2$. Our discussion is sketchy, details can be found, for general closed Riemann surfaces, in [16].

Given the complex torus and two simple closed curves a and b (see Figure 1) which define a canonical basis of $H_1(\mathbb{T}^2,\mathbb{R})$ (or equivalently of $\pi_1(\mathbb{T}^2)$), consider a holomorphic 1-form ζ such that $\int_a \zeta = 1$. Then the complex number $\tau = \int_b \zeta$, which depends on the complex structure, has positive imaginary part. The *Jacobian variety* associated to $\mathbb{T}^2$, denoted by $\mathcal{J}(\mathbb{T}^2)$, is a 2-dimensional torus with complex structure obtained by viewing τ as an element in its Teichmüller space. Equivalently,

$$\mathcal{J}(\mathbb{T}^2) = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau.$$

We introduce real coordinates (x, y) on $\mathcal{J}(\mathbb{T}^2)$ by setting $z = x+\tau y$. In these coordinates, $\mathcal{J}(\mathbb{T}^2)$ is the quotient of the plane by the integer lattice. The Jacobian variety is endowed with the canonical symplectic form $\omega = dx \wedge dy$, which is a generator of $H^2(\mathbb{T}^2, \mathbb{Z})$. $\mathcal{J}(\mathbb{T}^2)$ with its complex structure and this symplectic form is a Kähler manifold.

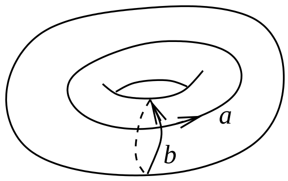

Figure 1

In short, classical theta functions and the action of the Heisenberg group on them can be obtained by applying Weyl quantization to $\mathcal{J}(\mathbb{T}^2)$ in the holomorphic polarization. To obtain theta functions, we apply the procedure of geometric quantization. We start by setting $\hbar = \frac{1}{N}$, with N a positive *even integer*.

The Hilbert space of the quantization consists of the *classical theta functions*, which are the holomorphic sections of a line bundle over the Jacobian variety. This line bundle is the tensor product of a line bundle of curvature $-2\pi i N\omega$ and a half-density. By pulling back the line bundle to $\mathbb{C}$, we can view these sections as entire functions satisfying certain periodicity conditions. The line bundle with curvature $2\pi i N\omega$ is unique up to tensoring with a flat bundle. Choosing the latter appropriately, we can ensure that the periodicity conditions are

$$f(z + m + n\tau) = e^{-2\pi i N(\tau n^2+2nz)} f(z).$$

An orthonormal basis of the space of classical theta functions is given by the *theta series*

$$\theta_j^\tau(z) = \sum\nolimits_{n\in\mathbb{Z}} e^{2\pi i N\left[\frac{\tau}{2}\left(\frac{j}{N}+n\right)^2+z\left(\frac{j}{N}+n\right)\right]}, \quad j = 0, 1, 2, \ldots, N-1. \tag{3.1}$$

It will be convenient to extend this definition to all indices j by the periodicity condition $\theta_{j+N}^\tau(z) = \theta_j^\tau(z)$, namely to take indices modulo N.

Given a complex torus one doesn't get automatically the theta series, for that one needs a pair of generators of the fundamental group. Here, the generators a and b of the fundamental group of the original torus give rise to the curves on the Jacobian variety that are the images of the curves in $\mathbb{C}$

from 0 to 1 respectively from 0 to τ. The complex structure and these two generators of $\pi_1(\mathcal{J}(\mathbb{T}^2))$ define a point in the Teichmüller space of $\mathcal{J}(\mathbb{T}^2)$, which is parametrized by the complex number τ.

Let us turn to the operators. The only exponentials on the plane that are double periodic, and therefore give rise to functions on the torus, are

$$f(x,y) = \exp 2\pi i(mx+ny), \quad m,n\in\mathbb{Z}.$$

Since the torus is a quotient of the plane by a discrete group, we can apply the Weyl quantization procedure. In the complex polarization Weyl quantization is defined as follows (see [10]): A fundamental domain of the torus is the unit square $[0,1]\times[0,1]$ (this is done in the (x,y) coordinates, in the complex plane it is actually a parallelogram). The value of a theta function is completely determined by its values on this unit square. The Hilbert space of classical theta functions can be isometrically embedded into $L^2([0,1]\times[0,1])$ with the inner product

$$\langle f,g\rangle = (-iN(\tau-\bar{\tau}))^{1/2}\int_0^1\int_0^1 f(x,y)\overline{g(x,y)}e^{iN(\tau-\bar{\tau})\pi y^2}dxdy.$$

The operator associated by Weyl quantization to a function f on the torus is the Toeplitz operator with symbol $e^{-\frac{\hbar\Delta_\tau}{4}}f$, where Δ_τ is the Laplacian, which in the (x,y) coordinates is given by the formula

$$\Delta_\tau \;=\frac{i}{\pi(\tau-\bar{\tau})}\left[\tau\bar{\tau}\frac{\partial^2}{\partial x^2}-(\tau+\bar{\tau})\frac{\partial^2}{\partial x\partial y}+\frac{\partial^2}{\partial y^2}\right]. \tag{3.2}$$

This means that the Weyl quantization of f maps a classical theta function g to the orthogonal projection onto the Hilbert space of classical theta functions of $(e^{-\frac{\Delta_\tau}{4N}}f)g$. The following result is standard; see [16] for a proof.

Proposition 3.1. *The Weyl quantization of the exponentials is given by*

$$Op\left(e^{2\pi i(px+qy)}\right)\theta_j^\tau(z) = e^{-\frac{\pi i}{N}pq-\frac{2\pi i}{N}jq}\theta_{j+p}^\tau(z).$$

The Weyl quantization of the exponentials gives rise to the Schrödinger representation of the Heisenberg group with integer entries $\mathbf{H}(\mathbb{Z})$ onto the space of theta functions. This Heisenberg group is

$$\mathbf{H}(\mathbb{Z}) = \{(p,q,k),\ p,q,k\in\mathbb{Z}\}$$

with multiplication

$$(p,q,k)(p',q',k') = (p+p',q+q',k+k'+(pq'-qp')).$$

The proposition implies that

$$(p,q,k)\mapsto \text{ the Weyl quantization of } e^{\frac{\pi i}{N}k}\exp 2\pi i(px+qy)$$

is a group morphism. This is the Schrödinger representation.

The Schrödinger representation of $\mathbf{H}(\mathbb{Z})$ is far from faithful. Because of this we factor it out by its kernel. The kernel is the subgroup consisting of the elements of the form $(p,q,k)^N$, with k even [16]. Let $\mathbf{H}(\mathbb{Z}_N)$ be the finite Heisenberg group obtained by factoring $\mathbf{H}(\mathbb{Z})$ by this subgroup, and let $\exp(pP+qQ+kE)$ be the image of (p,q,k) in it. Then

$$\exp(pP)\theta_j^\tau=\theta_{j+p}^\tau,\quad \exp(qQ)\theta_j^\tau=e^{-\frac{2\pi i}{N}qj}\theta_j^\tau,\quad \exp(kE)\theta_j^\tau=e^{\frac{\pi i}{N}k}\theta_j^\tau$$

for all $p,q,k,j\in\mathbb{Z}$.

The following is a well known result (see for example [16] for a proof).

Theorem 3.2. *(Stone-von Neumann) The Schrödinger representation of* $\mathbf{H}(\mathbb{Z}_N)$ *is the* unique *irreducible unitary representation of this group with the property that* $\exp(kE)$ *acts as* $e^{\frac{\pi i}{N}k}Id$ *for all* $k\in\mathbb{Z}$.

We must mention another important representation of the finite Heisenberg group, which by the Stone-von Neumann theorem is unitary equivalent to this one. It comes from the quantization of the torus in a real polarization. An orthonormal basis of the Hilbert space is given by Bohr-Sommerfeld leaves of the polarization, and so the Hilbert space can be identified with $L^2(\mathbb{Z}_N)$. If the polarization is given by "Q curves", the finite Heisenberg group acts by

$$\exp(pP)f(j)=f(j-p),\exp(qQ)f(j)=e^{-\frac{2\pi i}{N}qj}f(j),\exp(kE)f(j)=e^{\frac{\pi i}{N}k}f(j).$$

Thus $\exp(pP)$ acts as translation and $\exp(qQ)$ acts as a multiplication by a character of $\mathbb{Z}_N$. The characteristic functions of the singletons: $\delta_i(j)=\delta_{ij}$, $i=0,1,2,\ldots,N-1$ correspond to the theta series $\theta_0^\tau,\theta_1^\tau,\ldots,\theta_{N-1}^\tau$ through the unitary isomorphisms that identifies the two representations. Note that a left shift in the index corresponds to a right shift in the variable.

Remark 3.3. At close look, there is a sign discrepancy between these formulas and those from § 2.1, which shows up in the exponent from the action of $\exp(qQ)$. This is due to a disagreement between the standard notations in quantum mechanics and topology, in which the roles of the letters p and q are exchanged. Because Chern-Simons theory has been studied mainly by topologists, we use the convention for p and q from topology. The above formulas describe the action of the Heisenberg group in the the momentum representation.

The Schrödinger representation of the finite Heisenberg group can be extended by linearity to a representation of the group algebra with coefficients in $\mathbb{C}$ of the finite Heisenberg group, $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$. Since the elements of $\exp(\mathbb{Z}E)$ act as multiplications by constants, this is in fact a representation of the algebra $\mathcal{A}_N$ obtained by factoring $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N)]$ by the relations $\exp(kE)-e^{\frac{\pi i}{N}k}$ for all $k\in\mathbb{Z}$. By abuse of language, we will call this representation the Schrödinger representation as well. The Schrödinger representation of $\mathcal{A}_N$ defines the quantizations of trigonometric polynomials on the torus.

Proposition 3.4. *a) The algebra of Weyl quantizations of trigonometric polynomials is the algebra of all linear operators on the space of theta functions.*

b) The Schrödinger representation of the algebra $\mathcal{A}_N$ on theta functions is faithful.

Proof. For a proof of part a) see [16]. Part b) follows from the fact that $\exp(pP+qQ)$, $p, q = 0, 1, \ldots, N-1$, form a basis of $\mathcal{A}_N$ as a vector space. □

Because of this result, we may identify $\mathcal{A}_N$ with the algebra of Weyl quantizations of trigonometric polynomials. $\mathcal{A}_N$ can also be described in terms of the noncommutative torus. (The relevance of the noncommutative torus for Chern-Simons theory was first revealed in [12] for the gauge group $SU(2)$.)

The ring of trigonometric polynomials in the noncommutative torus is $\mathbb{C}_t[U^{\pm 1}, V^{\pm 1}]$, the ring of Laurent polynomials in the variables U and V subject to the noncommutation relation $UV = t^2VU$. The noncommutative torus itself is a C*-algebra in which $\mathbb{C}_t[U^{\pm 1}, V^{\pm 1}]$ is dense, viewed as a deformation quantization of the algebra of smooth functions on the torus [36], [9]. The group algebra with coefficients in $\mathbb{C}$ of the Heisenberg group $\mathbf{H}(\mathbb{Z})$ is isomorphic to the ring of trigonometric polynomials in the noncommutative torus, with the isomorphism given by

$$(p, q, k) \to t^{k-pq}\, U^p V^q, \text{ for } p, q, k \in \mathbb{Z}.$$

If we set $U^N = V^N = 1$ and $t = e^{\frac{\pi i}{N}}$, we obtain the noncommutative torus at a root of unity $\widetilde{\mathbb{C}_t}[U^{\pm 1}, V^{\pm 1}]$. To summarize:

Proposition 3.5. *The algebra, $\mathcal{A}_N$, of the Weyl quantizations of trigonometric polynomials on the torus is isomorphic to $\widetilde{\mathbb{C}_t}[U^{\pm 1}, V^{\pm 1}]$.*

As explained in [16], the Schrödinger representation can be described as the left regular action of the group algebra of the finite Heisenberg group on a quotient of itself. The construction is similar to that for the metaplectic representation in the abstract setting from §2.2.

3.2. **Classical theta functions from a topological perspective.** In [16] the theory of classical theta functions was shown to admit a reformulation in purely topological language. Let us recall the facts.

Let M be an orientable 3-dimensional manifold. A framed link in M is a smooth embedding of a disjoint union of finitely many annuli. We consider framed oriented links, where the orientation of a link component is an orientation of one of the boundary components of the annulus. We draw all diagrams in the blacboard framing, meaning that the annulus is parallel to the plane of the paper.

Consider the free $\mathbb{C}[t, t^{-1}]$-module with basis the set of isotopy classes of framed oriented links in M, including the empty link $\emptyset$. Factor it by all equalities of the form depicted in Figure 2. In each of these diagrams, the two links are identical except for an embedded ball in M, inside of which they look as shown. This means that in each link we are allowed to smoothen a crossing provided that we add a coefficient of t or t^{-1}, and any trivial link component can be ignored. These are called *skein relations*. For normalization reasons, we add the skein relation that identifies the trivial knot with $\emptyset$. The skein relations are considered for all possible embeddings of a ball. The result of the factorization is called the *linking number skein module of* M, denoted by $\mathcal{L}_t(M)$.

If M is a 3-dimensional sphere, then each link L is, as an element of $\mathcal{L}_t(S^3)$, equivalent to the empty link with the coefficient equal to t raised to the sum of the linking numbers of ordered pairs of components and the writhes of the components, hence the name.

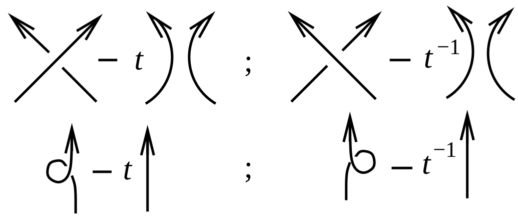

FIGURE 2

These skein modules were first introduced by Przytycki in [33]. He pointed out that they represent one-parameter deformations of the group ring of $H_1(M, \mathbb{Z})$ and computed them for all 3-dimensional manifolds.

For the fixed positive integer N we define the *reduced* linking number skein module of M, denoted by $\widetilde{\mathcal{L}}_t(M)$, as the quotient of $\mathcal{L}_t(M)$ obtained by setting $t = e^{\frac{i\pi}{N}}$ and $\gamma^N = \emptyset$ for every skein γ consisting of one link component, where γ^N denotes N parallel copies of γ. As a rule followed throughout the paper, in a skein module t is a free variable, while in a reduced skein module it is a root of unity.

If $M = \mathbb{T}^2 \times [0,1]$, the topological operation of gluing one cylinder on top of another induces a multiplication in $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ which turns $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ into an algebra, the *linking number skein algebra* of the cylinder over the torus. This multiplication descends to $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$. We want to explicate its structure.

For p and q coprime integers, orient the curve (p,q) by the vector that joins the origin to the point (p,q), and frame it so that the annulus is parallel to the torus. Call this the zero framing, or the *blackboard framing*. Any other framing of the curve (p,q) can be represented by an integer k, where $|k|$ is the number of full twists that are inserted on this curve, with k positive if the twists are positive, and k negative if the twists are negative. Note that in $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$, (p,q) with framing k is equivalent to $t^k(p,q)$.

If p and q are not coprime and n is their greatest common divisor, we denote by (p,q) the framed link consisting of n parallel copies of $(p/n, q/n)$, namely $(p,q) = (p/n, q/n)^n$. Finally, $\emptyset = (0,0)$ is the empty link, the multiplicative identity of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$.

Theorem 3.6. [16] *The algebra $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ is isomorphic to the group algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$, with the isomorphism induced by*

$$t^k(p,q) \to (p,q,k).$$

This map descends to an isomorphism between $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ and the algebra $\mathcal{A}_N$ of Weyl quantizations of trigonometric polynomials.

Remark 3.7. The determinant is the sum of the algebraic intersection numbers of the curves in (p,q,k) with the curves in (p',q',k'), so the multiplication rule of the Heisenberg group is defined using the algebraic intersection number of curves on the torus.

Identifying the group algebra of the Heisenberg group with integer entries with $\mathbb{C}_t[U^{\pm 1}, V^{\pm 1}]$, we obtain the following

Corollary 3.8. *The linking number skein algebra of the cylinder over the torus is isomorphic to the ring of trigonometric polynomials in the noncommutative torus.*

Let us look at the skein module of the solid torus $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$. Let α be the curve that is the core of the solid torus, with a certain choice of orientation and framing. The reduced linking number skein module $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ has basis α^j, $j = 0, 1, \ldots, N-1$.

Let h_0 be a homeomorphism of the torus to the boundary of the solid torus that maps the first generator of the fundamental group to a curve isotopic to α (a *longitude*) and the second generator to the curve on the boundary of the solid torus that bounds a disk in the solid torus (a *meridian*). The operation of gluing $\mathbb{T}^2 \times [0,1]$ to the boundary of $S^1 \times \mathbb{D}^2$ via h_0 induces a left action of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ onto $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$. This descends to a left action of $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ onto $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$.

It is important to observe that $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$ and $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ are quotients of $\mathcal{L}_t(\mathbb{T}^2 \times [0,1])$ respectively $\mathcal{L}_t(S^1 \times \mathbb{D}^2)$, with two framed curves equivalent on the torus if they are isotopic in the solid torus.

Theorem 3.9. [16] *There is an isomorphism that intertwines the action of the algebra of Weyl quantizations of trigonometric polynomials on the space of theta functions and the representation of $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$ onto $\widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$, and which maps the theta series $\theta_j^\tau(z)$ to α^j for all $j = 0, 1, \ldots, N-1$.*

Remark 3.10. The number

$$-qj = \begin{vmatrix} p & q \\ j & 0 \end{vmatrix}$$

is the sum of the linking numbers of the curves in the system (p,q) and those in the system α^j. So the Schrödinger representation is defined in terms of the linking number of curves.

Remark 3.11. The choice of generators of $\pi_1(\mathbb{T}^2)$ completely determines the homeomorphism h_0, allowing us to identify the Hilbert space of the quantization with the vector space with basis $\alpha^0 = \emptyset, \alpha, \ldots, \alpha^{N-1}$. As we have seen above, these basis elements are the theta series.

3.3. **The discrete Fourier transform for classical theta functions from a topological perspective.** The symmetries of classical theta functions are an instance of the Fourier-Mukai transform, the discrete Fourier transform. Following [16], we put them in a topological perspective.

An element

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z}) \tag{3.3}$$

of the mapping class group of the torus $\mathbb{T}^2$ induces the biholomorpic mapping between the Jacobian variety with complex structure defined by τ and the Jacobian variety with complex structure defined by $\tau' = \frac{a\tau+b}{c\tau+d}$. The mapping is $z' = \frac{z}{c\tau+d}$. Identifying the two tori by this mapping, we deduce that h induces a linear symplectomorphism of $\mathcal{J}(\mathbb{T}^2)$ (with the same matrix) that preserves the complex structure.

The map h induces an action of the mapping class group on the Weyl quantizations of exponentials given by

$$h \cdot \exp(pP + qQ + kE) = \exp[(ap + bq)P + (cp + dq)Q + kE].$$

This action is easy to describe in the skein theoretical setting, it just maps every framed link γ on the torus to $h(\gamma)$.

Theorem 3.12. *There is a projective representation ρ of the mapping class group of the torus on the space of theta functions that satisfies the exact Egorov identity*

$$h \cdot \exp(pP + qQ + kE) = \rho(h) \exp(pP + qQ + kE)\rho(h)^{-1}.$$

Moreover, for every h, $\rho(h)$ is unique up to multiplication by a constant.

Proof. We will exhibit two proofs of this well-known result, to which we will refer when discussing non-abelian Chern-Simons theory.

Proof 1: The map that associates to $\exp(pP + qQ + kE)$ the operator that acts on theta functions as

$$\theta_j^\tau \to \exp[(ap + bq)P + (cp + dq)Q + kE]\theta_j^\tau$$

is also a unitary irreducible representation of the finite Heisenberg group which maps $\exp(kE)$ to multiplication by $e^{\frac{i\pi}{N}}$. By the Stone-von Neumann theorem, this representation is unitary equivalent to the Schrödinger representation. This proves the existence of $\rho(h)$ satisfying the exact Egorov identity. By Schur's lemma, the map $\rho(h)$ is unique up to multiplication by a constant. Hence, if h and h' are two elements of the mapping class group, then $\rho(h' \circ h)$ is a constant multiple of $\rho(h')\rho(h)$. It follows that ρ defines a projective representation.

Proof 2: The map $\exp(pQ + qQ + kE) \to h \cdot \exp(pP + qQ + kE)$ extends to an automorphism of the algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$. Because the ideal by which we factor to obtain $\mathcal{A}_N$ is invariant under the action of the mapping class group, this automorphism induces an automorphism

$$\Phi : \mathcal{A}_N \to \mathcal{A}_N,$$

which maps each scalar multiple of the identity to itself. Since, by Proposition 3.4, $\mathcal{A}_N$ is the algebra of all linear operators on the N-dimensional space of theta functions, Φ is inner [42], meaning that there is $\rho(h) : \mathcal{A}_N \to \mathcal{A}_N$ such that $\Phi(x) = \rho(h)x\rho(h)^{-1}$. In particular

$$h \cdot \exp(pP + qQ + kE) = \rho(h)\exp(pP + qQ + kE)\rho(h)^{-1}.$$

The Schrödinger representation of $\mathcal{A}_N$ is obviously irreducible, so again we apply Schur's lemma and conclude that $\rho(h)$ is unique up to multiplication by a constant and $h \to \rho(h)$ is a projective representation. □

The representation ρ is the well-known Hermite-Jacobi action given by discrete Fourier transforms.

As a consequence of Proposition 3.4, for any element h of the mapping class group, the linear map $\rho(h)$ is in $\widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$, hence it can be represented by a skein $\mathcal{F}(h)$. This skein satisfies

$$h(\sigma)\mathcal{F}(h) = \mathcal{F}(h)\sigma$$

for all $\sigma \in \widetilde{\mathcal{L}}_t(\mathbb{T}^2 \times [0,1])$. Moreover $\mathcal{F}(h)$ is unique up to multiplication by a constant. We recall from [16] how to find a formula for $\mathcal{F}(h)$.

We start with the simpler case of

$$h = T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

the positive twist. Then $h((0,j)) = (0,j)$ for all j, and $h((1,0)) = (1,1)$. The equality $\mathcal{F}(T)(0,j) = (0,j)\mathcal{F}(T)$ for all j implies that we can write $\mathcal{F}(T) = \sum_{j=0}^{N-1} c_j(0,j)$ for some coefficients c_j. The equality

$$(1,1)\sum_{j=0}^{N-1} c_j(0,j) = \sum_{j=0}^{N-1} c_j(0,j)(1,0)$$

yields

$$\sum_{j=0}^{N-1} t^j c_j(1,j+1) = \sum_{j=0}^{N-1} t^{-j} c_j(1,j).$$

It follows that $t^j c_j = t^{-j-1}c_{j+1}$ for all $j = 0,1,\ldots,N-2$. Normalizing so that $c_0 > 0$ and $\mathcal{F}(T)$ defines a unitary map, we obtain $c_j = N^{-1/2}t^{1+3+\cdots+(2j-1)} = N^{-1/2}t^{j^2}$. We conclude that

$$\mathcal{F}(T) = N^{-1/2}\sum_{j=0}^{N-1} t^{j^2}(0,j).$$

To better understand this formula, we recall a few basic facts in low dimensional topology (see [37] for details).

Every 3-dimensional manifold is the boundary of a 4-dimensional manifold obtained by adding 2-handles $\mathbb{D}^2 \times \mathbb{D}^2$ to a 4-dimensional ball along the solid tori $\mathbb{D}^2 \times S^1$. On the boundary S^3 of the ball, the operation of adding handles gives rise to surgery on a framed link. Thus any 3-dimensional manifold can be obtained as follows. Start with framed link $L \subset S^3$. Take a regular neighborhood of L made out of disjoint solid tori, each with a framing curve on the boundary such that the core of the solid torus and this curve determine the framing of the corresponding link component. Remove these tori, then glue them back in so that the meridians are glued to the framing curves. The result is the desired 3-dimensional manifold.

The operation of sliding one 2-handle over another corresponds to sliding one link component along another using a Kirby band-sum move [23]. A *slide* of K_1 along K, denoted by $K_1\#K$, is obtained as by cutting open the two knots and then joining the ends along the opposite sides of an embedded rectangle. The result of sliding a trefoil knot along a figure-eight knot, both with the blackboard framing, is shown in Figure 3. For framed knots one should join the annuli along the opposite faces of an embedded cube (making sure that the result is an embedded annulus, not an embedded Möbius band). The band sum is not unique.

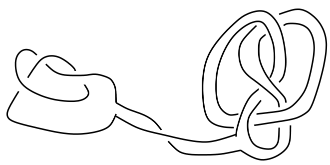

FIGURE 3

An element of the mapping class group of the torus can also be described by surgery. The twist T is obtained by surgery on the curve $(0,1)$ with framing 1. Explicitly, the mapping cylinder of T is obtained from $\mathbb{T}^2 \times [0,1]$ by removing a solid torus that is a regular neighborhood of $(0,1) \times \{\frac{1}{2}\}$ and gluing it back such that its meridian (the homologically trivial curve on the boundary) is mapped to the framing curve. The result is homeomorphic to $\mathbb{T}^2 \times [0,1]$, so that the restriction of the homeomorphism to $\mathbb{T}^2 \times \{0\}$ is the identity map and the restriction to $\mathbb{T}^2 \times \{1\}$ is T.

We introduce the element

$$\Omega_{U(1)} = N^{-1/2} \sum_{j=0}^{N-1} \alpha^j \in \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2),$$

Here $U(1)$ stands for the gauge group $U(1)$ (see § 4.1), which is related to classical theta functions by Chern-Simons theory. There is a well known analogue for the group $SU(2)$, to be discussed in § 6.1.

The skein $\mathcal{F}(T)$ is obtained by coloring the framed surgery curve of T by $\Omega_{U(1)}$. This means that we replace the surgery curve by the skein $\Omega_{U(1)}$ such that the curves α^j in the solid torus are parallel to the framing. In general, for a framed link L we denote by $\Omega_{U(1)}(L)$ the skein obtained by replacing every link component by $\Omega_{U(1)}$ such that α becomes the framing.

Using the fact that each element of the mapping class group is a product of twists [26], we obtain the following skein theoretic description of the discrete Fourier transform.

Theorem 3.13. [16] *Let h be an element of the mapping class group of the torus obtained by performing surgery on a framed link L_h in $\mathbb{T}^2 \times [0,1]$. The discrete Fourier transform $\rho(h) : \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2) \to \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is given by*

$$\rho(h)\beta = \Omega_{U(1)}(L_h)\beta.$$

Remark 3.14. For a framed curve γ on the torus, $h(\gamma)$ is the slide of γ along the components of L_h. The Egorov identity for $\Omega_{U(1)}(L_h)$ means in topological language that we are allowed to perform slides in the cylinder over the torus along curves colored by $\Omega_{U(1)}$. This points to a surgery formula for $U(1)$-quantum invariants of 3-manifolds (see [16]).

Let us recall the classical description of the Hermite-Jacobi action. For h as in (3.3),

$$\rho(h)\theta_j^\tau(z) = \exp\left(-\frac{\pi i N c z^2}{c\tau + d}\right)\theta_j^{\tau'}\left(\frac{z}{c\tau+d}\right).$$

The exponential factor is introduced to enforce the periodicity conditions of theta functions for the function on the right-hand side (see Capter I §7 in [30]). For those familiar with the subject, there are no parity restrictions on a, b, c, d because N being even, $SL_\theta(2,\mathbb{Z}) = SL(2,\mathbb{Z})$.

In particular, for the generators

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of $SL(2,\mathbb{Z})$ one has the Jacobi identities

$$\rho(S)\theta_j^\tau(z) = \left(\frac{-i\tau}{N}\right)^{1/2} \exp\left(\frac{z^2}{2\tau}\right)\theta_j^{-1/\tau}\left(\frac{z}{\tau}\right) = \left(\frac{i\tau}{N}\right)^{1/2}\sum_{k=0}^{N-1} e^{-\frac{2\pi i}{N}kj}\theta_k^\tau(z)$$

$$\rho(T)\theta_j^\tau(z) = \theta_j^{\tau+1}(z) = e^{\frac{2\pi i}{N}j^2}\theta_j^\tau(z).$$

We normalize $\rho(S)$ to a unitary operator dividing by $(-i\tau)^{1/2}$. Note that

$$\Omega_{U(1)} = S\emptyset \in \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2).$$

Alternatively, in the real polarization, S and T act on $L^2(\mathbb{Z}_N)$ by

$$(Sf)(j) = N^{-1/2}\sum_{k\in\mathbb{Z}_N} f(k)e^{-\frac{2\pi i}{N}jk} \text{ and } (Tf)(j) = e^{\frac{2\pi i}{N}k^2}f(j),$$

where the first is the discrete (or finite) Fourier transform and the second is interpreted as a partial discrete Fourier transform.

Like for the metaplectic representation, the Hermite-Jacobi representation can be made into a true representation by passing to an extension of the mapping class group of the torus. While a $\mathbb{Z}_2$-extension would suffice, we consider a $\mathbb{Z}$-extension instead, in order to show the similarity with the non-abelian theta functions.

Let $\mathbf{L}$ be a subspace of $H_1(\mathbb{T}^2, \mathbb{R})$ spanned by a simple closed curve. Define the $\mathbb{Z}$-extension of the mapping class group of the torus by the multiplication rule on $SL(2, \mathbb{Z}) \times \mathbb{Z}$,

$$(h', n') \circ (h, n) = (h' \circ h, n + n' - \boldsymbol{\tau}(\mathbf{L}, h(\mathbf{L}), h' \circ h(\mathbf{L})).$$

where $\boldsymbol{\tau}$ is the Maslov index [29]. Standard results in the theory of theta functions show that the Hermite-Jacobi action lifts to a representation of this group.

4. Non-abelian theta functions from geometric considerations

4.1. **Non-abelian theta functions from geometric quantization.** Let G be a compact simple Lie group, $\mathfrak{g}$ its Lie algebra, and Σ_g be a closed oriented surface of genus $g \geq 1$. Consider the moduli space of $\mathfrak{g}$-connections on Σ_g, which is the quotient of the affine space of all $\mathfrak{g}$-connections on Σ_g (or rather on the trivial principal G-bundle P on Σ_g) by the group $\mathcal{G}$ of gauge transformations $A \to \phi^{-1} A \phi + \phi^{-1} d\phi$, with $\phi : \Sigma_g \to G$ a smooth function. The space of all connections has a symplectic 2-form given by

$$\omega(A, B) = -\int_{\Sigma_g} \operatorname{tr}(A \wedge B),$$

where A and B are connection forms in its tangent space. According to [4], this induces a symplectic form, denoted also by ω, on the moduli space, which further defines a Poisson bracket. The group of gauge transformations acts on the space of all connections in a Hamiltonian fashion, with moment map the curvature. Thus the moduli space of *flat* $\mathfrak{g}$-connections

$$\mathcal{M}_g = \{A \,|\, A : \text{ flat } \mathfrak{g} - \text{connection}\}/\mathcal{G}.$$

arises as the symplectic reduction of the space of all connections by the group of gauge transformations. This space is the same as the moduli space of semi-stable G-bundles on Σ_g, and also the character variety of G-representations of the fundamental group of Σ_g. It is an affine algebraic set over the real numbers.

Each curve γ on the surface and each irreducible representation V of G define a classical observable on this space

$$A \to \operatorname{tr}_V \operatorname{hol}_\gamma(A),$$

called Wilson line, by taking the trace of the holonomy of the connection in the given irreducible representation of G. Wilson lines are regular functions on the moduli space. When $G = SU(2)$ we let the Wilson line for the n-dimensional irreducible representation be $W_{\gamma,n}$. When $n = 2$, we denote $W_{\gamma,2}$ by W_γ. The W_γ's span the algebra of regular functions on $\mathcal{M}_g$.

The form ω induces a Poisson bracket, which in the case of the gauge group $SU(2)$ was found by Goldman [17] to be

$$\{W_\alpha, W_\beta\} = \frac{1}{2}\sum_{x\in\alpha\cap\beta} \mathrm{sgn}(x)(W_{\alpha\beta_x^{-1}} - W_{\alpha\beta_x})$$

where $\alpha\beta_x$ and $\alpha\beta_x^{-1}$ are computed as elements of the fundamental group with base point x (see Figure 4), and $\mathrm{sgn}(x)$ is the signature of the crossing x; it is positive if the frame given by the tangent vectors to α respectively β is positively oriented (with respect to the orientation of Σ_g), and negative otherwise.

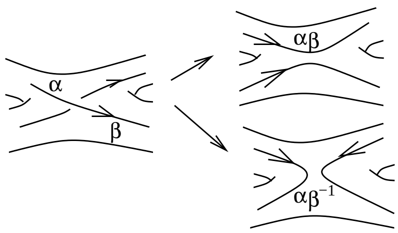

FIGURE 4

The moduli space $\mathcal{M}_g$, or rather the smooth part of it, can be quantized in the direction of Goldman's Poisson bracket as follows. First, set Planck's constant $\hbar = \frac{1}{N}$, where N is an even positive integer.

The Hilbert space can be obtained using the method of geometric quantization as the space of sections of a line bundle over $\mathcal{M}_g$. The general procedure is to obtain the line bundle as the tensor product of a line bundle with curvature $-2\pi i N\omega$ and a half-density [39]. The half-density is a square root of the canonical line bundle. Because the moduli space has a natural complex structure, it is customary to work in the complex polarization, in which case the Hilbert space consists of the holomorphic sections of the line bundle.

Let us briefly recall how each complex structure on the surface induces a complex structure on the moduli space. The tangent space to $\mathcal{M}_g$ at a nonsingular point A is the first cohomology group $H^1_A(\Sigma_g, \mathrm{ad}\ P)$ of the complex of $\mathfrak{g}$-valued forms

$$\Omega^0(\Sigma_g, \mathrm{ad}\ P) \stackrel{d_A}{\to} \Omega^1(\Sigma_g, \mathrm{ad}\ P) \stackrel{d_A}{\to} \Omega^2(\Sigma_g, \mathrm{ad}\ P).$$

Here P denotes the trivial principal G-bundle over Σ_g. Each complex structure on Σ_g induces a Hodge $*$-operator on the space of connections on Σ_g, hence a $*$-operator on $H^1_A(\Sigma_g, \mathrm{ad}\ P)$. The complex structure on $\mathcal{M}_g$ is $I : H^1_A(\Sigma_g, \mathrm{ad}\ P) \to H^1_A(\Sigma_g, \mathrm{ad}\ P)$, $IB = -*B$. For more details we refer the reader to [20]. This complex structure turns the smooth part of $\mathcal{M}_g$ into a complex manifold. It is important to point out that the complex structure is compatible with the symplectic form ω, in the sense that $\omega(B, IB) \geq 0$ for all B.

As said, the Hilbert space consists of the holomorphic sections of the line bundle of the quantization. These are the *non-abelian theta functions.*

The analogue of the group algebra of the finite Heisenberg group is the algebra of operators that are the quantizations of Wilson lines. They arise in the theory of the Jones polynomial [18] as outlined by Witten [46], being defined heuristically in the framework of quantum field theory. They are integral operators with kernel

$$< A_1|\mathrm{Op}(W_{\gamma,n})|A_2 >= \int_{\mathcal{M}_{A_1A_2}} e^{iNL(A)}W_{\gamma,n}(A)\mathcal{D}A,$$

where A_1, A_2 are conjugacy classes of flat connections on Σ_g modulo the gauge group, A is a conjugacy class under the action of the gauge group on $\Sigma_g \times [0,1]$ such that $A_{\Sigma_g\times\{0\}} = A_1$ and $A_{\Sigma_g\times\{1\}} = A_2$, and

$$L(A) = \frac{1}{4\pi}\int_{\Sigma_g\times[0,1]} \mathrm{tr}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)$$

is the Chern-Simons Lagrangian. The operator quantizing a Wilson line is defined by a Feynman path integral, which does not have a rigorous mathematical formulation. It is thought as an average of the Wilson line computed over all connections that interpolate between A_1 and A_2.

The skein theoretic approach to classical theta functions outlined in § 3.2 can be motivated by the Chern-Simons-Witten field theory point of view. Indeed, the Wilson lines can be quantized either by using one of the classical methods for quantizing the torus, or by using the Feynman path integrals as above. The Feyman path integral approach allows localizations of computations to small balls, in which a single crossing shows up. Witten [46] has explained that in each such ball skein relations hold, in this case the skein relations from Figure 2, which compute the linking number. As such the path integral quantization gives rise to the skein theoretic model.

On the other hand, Witten's quantization is symmetric under the action of the mapping class group of the torus, a property shared by Weyl quantization. And indeed, we have seen in § 3.2 that Weyl quantization and the skein theoretic quantization are the same. The relevance of Weyl quantization to Chern-Simons theory was first pointed out in [15] for the gauge group $SU(2)$. For the gauge group $U(1)$, it was noticed in [3]. The abelian Chern-Simons theory from the skein theoretic point of view was described in detail in [16].

4.2. **The Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus.** The moduli space $\mathcal{M}_1$ of flat $SU(2)$-connections on the torus is the same as the character variety of $SU(2)$-representations of the fundamental group of the torus. It is, therefore, parametrized by the set of pairs of matrices $(A, B) \in SU(2) \times SU(2)$ satisfying $AB = BA$, modulo conjugation. Since commuting matrices can be diagonalized simultaneously, and the two diagonal entries can be permuted simultaneously, the moduli space can be identified with the quotient of the torus $\{(e^{2\pi ix}, e^{2\pi iy}),\ x, y \in \mathbb{R}\}$ by the "antipodal" map $x \to -y$, $y \to -y$. This space is called the *pillow case*.

The pillow case is the quotient of $\mathbb{R}^2$ by horizontal and vertical integer translations and by the symmetry σ with respect to the origin. Except for 4 singularities, $\mathcal{M}_1$ is a symplectic manifold, with symplectic form $\omega = 2\pi i dx \wedge dy$. The associated Poisson bracket is

$$\{f, g\} = \frac{1}{2\pi i}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$$

The Weyl quantization of $\mathcal{M}_1$ in the complex polarization has been described in [15] for one particular complex structure. We do it now in general. Again Planck's constant is the reciprocal of an even integer $\hbar = \frac{1}{N} = \frac{1}{2r}$.

The tangent space at an arbitrary point on the pillow case is spanned by the vectors $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$. In the formalism of §4.1, these vectors are identified respectively with the cohomology classes of the $su(2)$-valued 1-forms

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dx \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} dy.$$

It follows that a complex structure on the original torus induces exactly the same complex structure on the pillow case. So we can think of the pillow case as the quotient of the complex plane by the group generated by $\mathbb{Z} + \mathbb{Z}\tau$ (Im $\tau > 0$) and the symmetry σ with respect to the origin. As before, we denote by (x, y) the coordinates in the basis $(1, \tau)$ and by $z = x + \tau y$ the complex variable. A fundamental domain for the group action in the (x, y)-coordinates is $\mathcal{D} = [0, \frac{1}{2}] \times [0, 1]$.

As seen in [15], a holomorphic line bundle $\mathcal{L}_1$ with curvature $4\pi i r dx \wedge dy$ on the pillow case is defined by the cocycle $\Lambda_1 : \mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*$,

$$\Lambda_1((x, y), (m, n)) = e^{4\pi i r(\frac{\tau}{2}n^2 - 2n(x+\tau y))} = e^{4\pi i r(\frac{\tau}{2}n^2 - 2nz)}$$
$$\Lambda_1((x, y), \sigma) = 1.$$

The square root of the canonical form is no longer the trivial line bundle, since for example the form dx is not defined globally on the pillow case. The obstruction for dx to be globally defined can be incorporated in a line bundle $\mathcal{L}_2$ defined by the cocycle $\Lambda_2 : \mathbb{R}^2 \times \mathbb{Z}^2 \to \mathbb{C}^*$,

$$\Lambda_2((x, y), (m, n)) = 1, \quad \Lambda_2((x, y), \sigma) = -1.$$

This line bundle can then be taken as the half-density.

The line bundle of the quantization is therefore $\mathcal{L}_1 \otimes \mathcal{L}_2$, defined by the cocycle $\Lambda_1\Lambda_2$. The Hilbert space $\mathcal{H}_r(\mathbb{T}^2)$ of non-abelian theta functions on the torus consists of the holomorphic sections of this line bundle. Hence the Hilbert space consists of the odd theta functions (this was discovered in [5]).

Because Weyl quantization of the pillow case is equivariant Weyl quantization of the torus, to specify a basis of $\mathcal{H}_r(\mathbb{T}^2)$ we need a pair of generators of the fundamental group. This complex structure and generators of $\pi_1(\mathbb{T}^2)$

determine a point in the Teichmüller space of the torus, specified by the complex number τ mentioned before. The orthonormal basis of the Hilbert space is

$$\zeta_j^\tau(z) = (\theta_j^\tau(z) - \theta_{-j}^\tau(z)), \quad j = 1, 2, \ldots, r-1,$$

where $\theta_j^\tau(z)$ are the theta series from §3.1. The definition of $\zeta_j^\tau(z)$ can be extended to all indices by the conditions $\zeta_{j+2r}^\tau(z) = \zeta_j^\tau(z)$, $\zeta_0^\tau(z) = 0$, and $\zeta_{r-j}^\tau(z) = -\zeta_{r+j}^\tau(z)$.

The space $\mathcal{H}_r(\mathbb{T}^2)$ can be embedded isometrically into $L^2(\mathcal{D})$, with the inner product

$$\langle f, g\rangle = 2(-2ir(\tau - \bar{\tau}))^{1/2} \iint_{\mathcal{D}} f(x,y)\overline{g(x,y)}e^{-2\pi i r(\tau-\bar{\tau})y^2} dxdy$$

The Laplacian is given by the formula (3.2) (with $N = 2r$).

The pillow case is the quotient of the plane by a discrete group, so again we can apply the Weyl quantization procedure. If p and q are coprime integers, then the Wilson line of the curve (p, q) of slope p/q on the torus for the 2-dimensional irreducible representation is

$$W_{(p,q)}(x,y) = \frac{\sin 4\pi(px+qy)}{\sin 2\pi(px+qy)} = 2\cos 2\pi(px+qy),$$

when viewing the pillow case as a quotient of the plane. This is because the character of the 2-dimensional irreducible representation is $\sin 2x/\sin x$. In general, if p and q are arbitrary integers, the function

$$f(x,y) = 2\cos 2\pi(px+qy)$$

is a linear combination of Wilson lines. Indeed, if $n = \gcd(p,q)$ then

$$2\cos 2\pi(px+qy) = \frac{\sin[2\pi(n+1)(\frac{p}{n}x + \frac{q}{n}y)]}{\sin 2\pi(\frac{p}{n}x+\frac{q}{n}y)} - \frac{\sin[2\pi(n-1)(\frac{p}{n}x+\frac{q}{n}y)]}{\sin 2\pi(\frac{p}{n}x+\frac{q}{n}y)},$$

so $2\cos 2\pi(px+qy) = W_{\gamma,n+1} - W_{\gamma,n-1}$ where γ is the curve of slope p/q on the torus. This formula also shows that Wilson lines are linear combinations of cosines, so it suffices to understand the quantization of the cosines.

Because

$$2\cos 2\pi(px+qy) = e^{2\pi i(px+qy)} + e^{-2\pi i(px+qy)},$$

the Weyl quantization of cosines can be obtained by taking the Schrödinger representation of the quantum observables that are invariant under the map $\exp P \to \exp(-P)$ and $\exp Q \to \exp(-Q)$, and restrict it to odd theta functions. We obtain the formula

$$\mathrm{Op}(2\cos 2\pi(px+qy))\zeta_j^\tau(z) = e^{-\frac{\pi i}{2r}pq}\left(e^{\frac{\pi i}{r}qj}\zeta_{j-p}^\tau(z) + e^{-\frac{\pi i}{r}qj}\zeta_{j+p}^\tau(z)\right).$$

In particular the ζ_j^τ's are the eigenvectors of $\mathrm{Op}(2\cos 2\pi y)$, corresponding to the holonomy along the curve which cuts the torus into an annulus. This shows that they are correctly identified as the analogues of the theta series.

5. Non-abelian theta functions from quantum groups

5.1. **A review of the quantum group $U_\hbar(sl(2,\mathbb{C}))$.** For the gauge group $SU(2)$, Reshetikhin and Turaev [35] constructed rigorously, by using quantum groups, a topological quantum field theory that fulfills Witten's programme. Within this theory, for each surface there is a vector space, an algebra of quantized Wilson lines, and a projective finite-dimensional representation of the mapping class group, namely the Reshetikhin-Turaev representation. The quantum group quantization has the advantage over geometric quantization that it does not depend on any additional structure, such as a polarization.

We set $\hbar = \frac{1}{N} = \frac{1}{2r}$, and furthermore $r > 1$. Let $t = e^{\frac{i\pi}{2r}}$ and, for an integer n, let $[n] = \frac{t^{2n}-t^{-2n}}{t^2-t^{-2}} = \sin\frac{n\pi}{r}/\sin\frac{\pi}{r}$, called the quantized integer.

The quantum group associated to $SU(2)$, denoted $U_\hbar(sl(2,\mathbb{C}))$ is obtained by passing to the complexification $SL(2,\mathbb{C})$ of $SU(2)$, taking the universal enveloping algebra of its Lie algebra, then deforming this algebra with respect to $\hbar$. It is the Hopf algebra over $\mathbb{C}$ with generators X, Y, K, K^{-1} subject to the relations

$$KK^{-1} = K^{-1}K = 1,\ KX = t^2XK,\ KY = t^{-2}YK,\ XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}}.$$

At the root of unity, namely when $N = 2r$, r integer, one has the additional factorization relations $X^r = Y^r = 0, K^{4r} = 1$[1].

As opposed to $SU(2)$, $U_\hbar(sl(2,\mathbb{C}))$ has only finitely many irreducible representations, among which we distinguish a certain family V^1, V^2, $\ldots$, V^{r-1} (for details see [35] or [24]). For each k, the space V^k has basis e_j, $j = -k_0, \ldots, k_0 - 1, k_0$, where $k_0 = \frac{k-1}{2}$, and the quantum group acts on it by

$$Xe_j = [k_0 + j + 1]e_{j+1}, \quad Ye_j = [k_0 - j + 1]e_{j-1}, \quad Ke_j = t^{2j}e_j.$$

The highest weight vector of this representation is e_{k_0}; it spans the kernel of X, is a cyclic vector for Y, and an eigenvector of K.

The Hopf algebra structure of $U_\hbar(sl(2,\mathbb{C}))$ makes duals and tensor products of representations be representations themselves. The quantum group acts on the dual V^{k*} of V^k by

$$Xe^j = -t^2[k_0 + j]e^{j-1}, \quad Ye^j = -t^{-2}[k_0 - j]e^{j+1}, \quad Ke^j = t^{-2j}e^j,$$

where $(e^j)_j$ is the basis dual to $(e_j)_j$. There is a (non-natural) isomorphism $D : V^{k*} \to V^k$,

$$D(e^j) = \frac{[k_0 - j][k_0 - j - 1]\cdots[1]}{[2k_0][2k_0 - 1]\cdots[k_0 - j + 1]}(-t^2)^j e_{-j}.$$

[1]In this case the quantum group is denoted by U_t in [35] and by $\mathcal{A}$ in [24].

A Clebsch-Gordan theorem holds,

$$V^m \otimes V^n = \bigoplus_p V^p \oplus B,$$

where p runs among all indices that satisfy $m+n+p$ odd and $|m-n|+1 \leq p \leq \min(m+n-1, 2r-1-m-n)$ and B is a representation that is ignored because it has no effect on computations.

A corollary of the Clebsch-Gordan theorem is the following formula

$$V^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \binom{n-j}{j} (V^2)^{n-2j} = S_{n-1}(V^2), \quad \text{for } n = 1, 2, \cdots, r-1.$$

Here $S_n(x)$ is the *Chebyshev polynomial of second kind* defined by

$$S_{n+1}(x) = xS_n(x) - S_{n-1}(x), \quad S_0(x) = 1, \quad S_1(x) = x.$$

We define the representation ring $R(U_\hbar(sl(2,\mathbb{C})))$ as the ring generated by V^j, $j = 1, 2, \ldots, r-1$ with multiplication $V^m \otimes V^n = \sum_p V^p$, where the sum is taken over all indices p that satisfy the conditions from the Clebsch-Gordan theorem.

Proposition 5.1. *The representation ring $R(U_\hbar(sl(2,\mathbb{C})))$ is isomorphic to $\mathbb{C}[V^2]/S_{r-1}(V^2)$. If we define $V^n = S_{n-1}(V^2)$ in this ring for all $n \geq 0$, then $V^{r+n} = -V^{r-n}$, $V^r = 0$, and $V^{n+2r} = V^n$ for all $n > 0$.*

5.2. **The quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface of genus greater than 1.** The definition of the quantization of the moduli space $\mathcal{M}_g$ of flat $SU(2)$-connections on a genus g surface Σ_g uses *ribbon graphs and framed links* embedded in 3-dimensional manifolds. A ribbon graph consists of the embeddings in the 3-dimensional manifold of finitely many connected components, each of which is homeomorphic to either an annulus or a tubular ϵ-neighborhood in the plane of a planar trivalent graph with small $\epsilon > 0$. As such, while the edges of a classical graph are 1-dimensional, those of a ribbon graph are 2-dimensional, an edge being homeomorphic to either a rectangle, or an annulus. Intuitively, edges are ribbons, hence the name. When embedding the ribbon graph in a 3-dimensional manifold, the framings keep track of the twistings of edges. A framed link is a particular case of a ribbon graph. The link components and the edges of ribbon graphs are oriented. All ribbon graphs used depicted below are taken with the "blackboard framing", meaning that the ϵ-neighborhood is in the plane of the paper.

With these conventions at hand, let us quantize the moduli space $\mathcal{M}_g$. The Hilbert space $\mathcal{H}_r(\Sigma_g)$ is defined by specifying a basis, the analogue of the theta series. Exactly how in order to specify a basis of the space of theta functions one needs a pair of generators of $\pi_1(\mathbb{T}^2)$, here one needs additional structure on Σ_g, which comes in the form of an *oriented rigid structure*.

A rigid structure on a surface is a collection of simple closed curves that decompose it into pairs of pants, together with "seams" that keep track of the

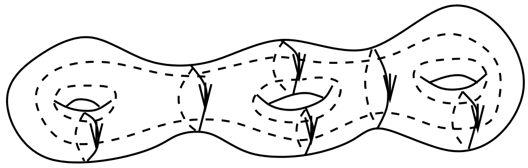

FIGURE 5

twistings. The seams are simple closed curves that, when restricted to any pair of pants, give 3 nonintersecting arcs that connect pairwise the boundary components. An oriented rigid structure is one in which the decomposing curves are oriented. An example is shown in Figure 5, with decomposing curves drawn with continuous line, and seams with dotted line.

Map Σ_g to the boundary of a handlebody H_g such that the decomposition curves bound disks in H_g. The disks cut H_g into balls. Consider the trivalent graph that is the core of H_g, with a vertex at the center of each ball and an edge drawn for each disk. The framing of edges should be parallel to the seams (more precisely, to the region of the surface that lies between the seems). The disks are oriented by the decomposition curves on the boundary, and the orientation of the edges of the graph should agree with that of the disks.

The vectors forming an orthogonal basis of $\mathcal{H}_r(\Sigma_g)$ consist of all possible colorings of this framed oriented trivalent graph by V^j's such that at each vertex the three indices satisfy the conditions from the Clebsch-Gordan theorem (note that the double inequality is invariant under permutations of m, n, p). Such a coloring is called *admissible*. In genus 3 and for the rigid structure from Figure 5, a basis element is shown in Figure 6. The inner product $\langle \cdot, \cdot \rangle$ is defined so that these basis elements are orthonormal.

This is a nice combinatorial description of the non-abelian theta functions for the Lie group $SU(2)$, which obscures their geometric properties and the origin of the name. The possibility to represent non-abelian theta functions as such graphs follows from the relationship found by Witten between theta functions and conformal field theory [46].

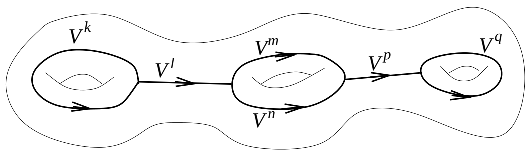

FIGURE 6

The matrix of the operator $\mathrm{Op}(W_{\gamma,n})$ associated to the Wilson line

$$W_{\gamma,n} : A \to \mathrm{tr}_{V^n} \mathrm{hol}_\gamma(A)$$

is computed as follows. First, let $1 \leq n \leq r-1$. Place the surface Σ_g in standard position in the 3-dimensional sphere so that it bounds a genus

g handlebody on each side. Draw the curve γ on the surface, give it the framing parallel to the surface, then color it by the representation V^n of $U_\hbar(sl(2,\mathbb{C})$. Add two basis elements e_p and e_q, viewed as admissible colorings by irreducible representation of the cores of the interior, respectively exterior handlebodies (see Figure 7). The interior and exterior handlebodies should be copies of the same handlebody with the same oriented rigid structure on the boundary (thus giving rise to the same non-abelian theta series), and these copies are glued along Σ_g by an orientation reversing homeomorphism as to obtain S^3.

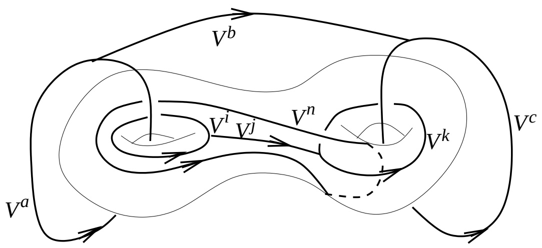

FIGURE 7

Erase the surface to obtain an oriented tangled ribbon graph in S^3 whose edges are decorated by irreducible representations of $U_\hbar(sl(2,\mathbb{C})$ (Figure 8). Project this graph onto a plane while keeping track of the crossings. The Reshetikhin-Turaev theory [35] gives a way of associating a number to this ribbon graph, which is independent of the particular projection and is called the Reshetikhin-Turaev invariant of the ribbon graph.

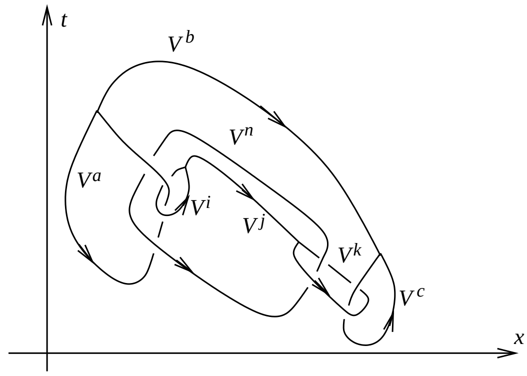

FIGURE 8

In short, the Reshetikhin-Turaev invariant is computed as follows. The ribbon graph should be mapped by an isotopy to one whose projection can be cut by finitely many horizontal lines into slices, each of which containing one of the phenomena from Figure 9 and some vertical strands. To each horizontal line that slices the graph one associates the tensor product of the representations that color the crossing strands, when pointing downwards, or their duals, when pointing upwards. To the phenomena from Figure 9 one associates, in order, the following operators:

- the flipped universal R-matrix $\check{R} : V^m \otimes V^n \to V^n \otimes V^m$ (obtained by composing the universal R-matrix with the flip $v \otimes w \to w \otimes v$),
- the inverse $\check{R}^{-1}$ of $\check{R}$,
- the projection operator $\beta_p^{mn} : V^m \otimes V^n \to V^p$, whose existence and uniqueness is guaranteed by the Clebsch-Gordan theorem,
- the inclusion $\beta_{mn}^p : V^p \to V^m \otimes V^n$,
- the contraction $E : V^{n*} \otimes V^n \to \mathbb{C}$, $E(f \otimes x) = f(x)$
- its dual $N : \mathbb{C} \to V^n \otimes V^{n*}$, $N(1) = \sum_j e_j \otimes e^j$,
- the isomorphism $D : V^{n*} \to V^n$,
- and its dual $D^* : V^n \to V^{n**} = V^n$ (see Lemma 3.18 in [24] for the precise identification of V^{n**} with V^n).

One then composes these operators from the bottom to the top of the diagram, to obtain a linear map from $\mathbb{C}$ to $\mathbb{C}$, which must be of the form $z \to \lambda z$. The number λ is the Reshetikhin-Turaev invariant of the ribbon graph. The blank coupons, i.e. the maps D, might be required in order to change the orientations of the three edges that meet at a vertex, to make them look as depicted in Figure 9. For example the map $V^{p*} \to V^m \otimes V^n$ is defined by identifying V^{p*} with V^p by the isomorphism D.

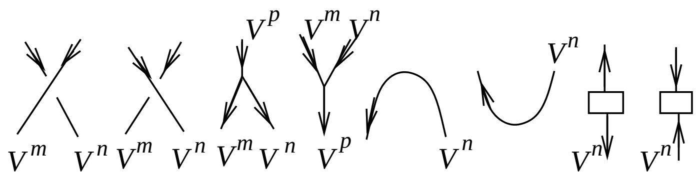

FIGURE 9

Returning to the quantization of Wilson lines, the Reshetikhin-Turaev invariant of the graph is equal to $[\mathrm{Op}(W_{\gamma,n})e_p, e_q]$, where $[\cdot,\cdot]$ is the nondegenerate bilinear pairing on $\mathcal{H}_r(\Sigma_g)$ defined in Figure 10. One can think of this as being the p, q-entry of the matrix of the operator, although this is not quite true because the bilinear pairing is not the inner product. But because the pairing is nondegenerate (see § 6.3), the above formula completely determines the operator associated to the Wilson line.

In view of Proposition 5.1, this definition of quantized Wilson lines is extended to arbitrary n by the conventions

$$\mathrm{Op}(W_{\gamma,r}) = 0, \quad \mathrm{Op}(W_{\gamma,r+n}) = -\mathrm{Op}(W_{\gamma,r-n}), \quad \mathrm{Op}(W_{\gamma,n+2r}) = -\mathrm{Op}(W_{\gamma,n})$$

It can be shown that this quantization is in the direction of Goldman's Poisson bracket [1].

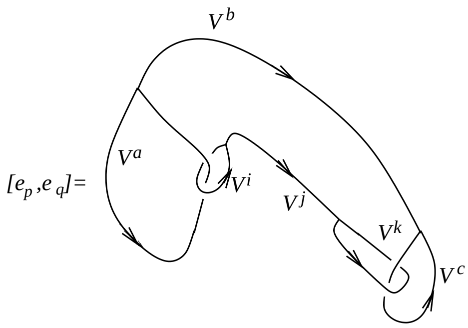

FIGURE 10

5.3. **Non-abelian theta functions from skein modules.** We rephrase the construction from § 5.2 in the language of skein modules. The goal is to express the quantum group quantization of Wilson lines as the left representation of a skein algebra on a quotient of itself, in the same way that the Schrödinger representation was described in § 3.2 as the left representation of the reduced linking number skein algebra of the cylinder over the torus on a quotient of itself.

One usually associates to $SU(2)$ Chern-Simons theory the skein modules of the Kauffman bracket. The Reshetikhin-Turaev topological quantum field theory has a Kauffman bracket analogue constructed in [6]. However, the Kauffman bracket skein relations introduce sign discrepancies in the computation of the desired left action. And since Theorem 5.8 in § 5.4 brings evidence that the quantum group quantization is the non-abelian analogue of Weyl quantization we will define our modules by the skein relations found by Kirby and Melvin in [24] for the Reshetikhin-Turaev version of the Jones polynomial.

We first replace the oriented framed ribbon graphs and links colored by irreducible representations of $U_\hbar(sl(2,\mathbb{C}))$ by formal sums of oriented framed links colored by the 2-dimensional irreducible representation. Two technical results are needed.

Lemma 5.2. *For all $n = 3, 4, \ldots, r-1$ the identity from Figure 11 holds.*

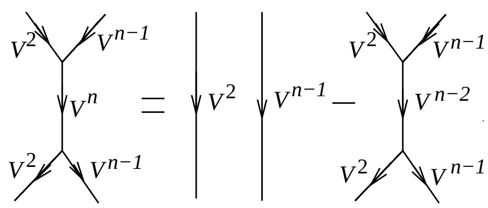

FIGURE 11

Proof. This is a corollary of the particular case of the Clebsch-Gordan theorem $V^n = V^2 \otimes V^{n-1} - V^{n-2}$. □

Lemma 5.3. *For any integers m, n, p satisfying $m+n+p$ odd and $|m-n|+1 \leq p \leq \min(m+n-1, 2r-1-m-n)$, the identity from Figure 12 holds.*

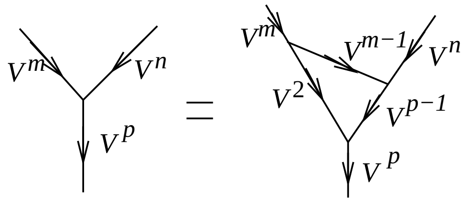

FIGURE 12

Proof. We assume familiarity with the proof of the quantum Clebsch-Gordan theorem in [35]. Set $m_0 = \frac{m-1}{2}$, $p_0 = \frac{p-1}{2}$. The morphism described by the diagram on the right is the composition of maps

$$V^p \xrightarrow{\beta^p_{2,p-1}} V^2 \otimes V^{p-1} \xrightarrow{1\otimes\beta^{p-1}_{mn}} V^2 \otimes \left(V^{m-1} \otimes V^n\right)$$
$$= \left(V^2 \otimes V^{m-1}\right) \otimes V^n \xrightarrow{\beta^{2,m-1}_m \otimes 1} V^m \otimes V^n.$$

Because of Schur's lemma and the quantum Clebsch-Gordan theorem, this composition is either the zero or the identity map. To show that is not the zero map, we look how the highest weight vector e_{p_0} in V^p transforms. We have

$$e_{p_0} \to e_{\frac{1}{2}} \otimes e_{p_0-\frac{1}{2}} \to e_{\frac{1}{2}} \otimes \sum_{i+j=p_0} c_{ij} e_i \otimes e_j$$
$$= \sum_{i+j=p_0} c_{ij} e_{\frac{1}{2}} \otimes e_i \otimes e_j \in V^2 \otimes V^{m-1} \otimes V^n.$$

The Clebsch-Gordan coefficients c_{ij} are nonzero, and in the sum there is a term $c_{m_0-\frac{1}{2},j} e_{\frac{1}{2}} \otimes e_{m_0-\frac{1}{2}} \otimes e_j$.

On the other hand, the inclusion $\beta^m_{2,m-1} : V^m \to V^2 \otimes V^{m-1}$ maps the highest weight vector e_{m_0} in V^m to $e_{\frac{1}{2}} \otimes e_{m_0-\frac{1}{2}}$, which is the product of the vectors of highest weights in V^2 respectively V^{m-1}. Hence if the $\frac{1}{2}, m_0 - \frac{1}{2}$-component of a vector v written in the basis $e_i \otimes e_j$ of $V^2 \otimes V^{m-1}$ is nonzero, then $\beta^m_{2,m-1} v$ is nonzero in V^m.

In particular, the above sum maps to a nonzero vector in $V^m \otimes V^n$. It follows that the diagram on the right of Figure 12 equals the inclusion $\beta^p_{mn} : V^p \to V^m \otimes V^n$, proving the identity. □

Proposition 5.4. *There is an algorithm for replacing each connected ribbon graph Γ colored by irreducible representations of $U_\hbar(sl(2,\mathbb{C}))$ by a sum of oriented framed links colored by V^2 that lie in an ϵ-neighborhood of the graph, such that if in any ribbon graph Γ' that has Γ as a connected component* *we replace* Γ *by this sum of links, we obtain a ribbon graph with the same Reshetikhin-Turaev invariant as* Γ'.

Proof. For framed knots, the property follows from the cabling formula given in Theorem 4.15 in [24]; a knot colored by V^n is replaced by $S_{n-1}(V^2)$.

If the connected ribbon graph has vertices, then by using the isomorphism D to identify irreducible representations of $U_\hbar(sl(2,\mathbb{C})$ with their duals, we can obtain the identity from Figure 12 with the arrows reversed. Also, by taking the adjoint of the map described by the diagram, we can turn it upside down, meaning that we can write a similar identity for β_p^{mn}.

Based on the two lemmas, the algorithm works as follows. First, use the identities from Figure 13 to remove all edges colored by V^1. Then apply repeatedly Lemma 5.3 until at each vertex of the newly obtained ribbon graph at least one of the three edges is colored by V^2. Next, use Lemma 5.2 to obtain a sum of graphs with the property that, at each vertex, two of the three edges are colored by V^2 and one is colored by V^1. Finally, use the identities from Figure 13 for $n = 2$ to transform everything into a sum of framed links whose edges are colored by V^2. Each of the links in the sum has and even number of blank coupons (representing the isomorphism D or its dual) on each component. Cancel the coupons on each link component in pairs, adding a factor of -1 each time the two coupons are separated by an odd number of maxima on the link component. The result is a formal sum of framed links with components colored by V^2. □

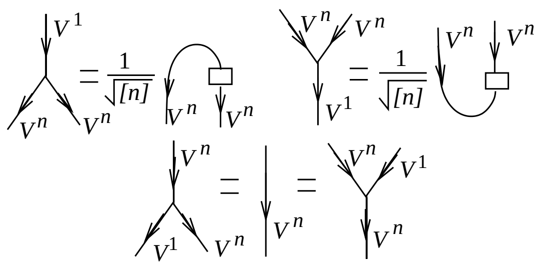

FIGURE 13

Theorem 4.3 in [24] allows us to compute the Reshetikhin-Turaev invariant of a framed link whose components are colored by V^2 using skein relations. First, forget about the orientation of links. Next, if three framed links L, H, V in S^3 colored by V^2 coincide except in a ball where they look like in Figure 14, then their Reshetikhin-Turaev invariants, denoted by J_L, J_H, and J_V satisfy

$$J_L = tJ_H + t^{-1}J_V \text{ or } J_L = \epsilon(tJ_H - t^{-1}J_V)$$

depending on whether the two crossing strands come from different components or not. Here ϵ is the sign of the crossing, obtained after orienting that link component (either orientation produces the same sign). Specifically, if

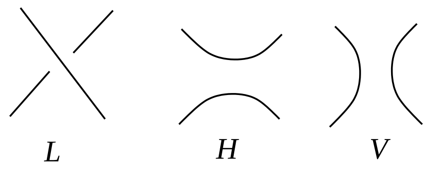

FIGURE 14

the tangent vectors to the over and under strand form a positive frame then the sign is positive, otherwise it is negative. Additionally if a link component bounds a disk inside a ball disjoint from the rest of the link, then it is replaced by a factor of $t^2 + t^{-2}$.

When $t = 1$, namely when $\hbar = 0$, it no longer matters whether one has an undercrossing or overcrossing, and both skein relations express the trace identity of $SU(2)$,

$$\mathrm{tr}(A)\mathrm{tr}(B) = \mathrm{tr}(AB) + \mathrm{tr}(A^{-1}B),$$

for Wilson lines

$$W_\alpha W_\beta = W_{\alpha\beta} + W_{\alpha^{-1}\beta}.$$

For arbitrary t, the skein relations are the trace identity for the quantum group $U_\hbar(sl(2,\mathbb{C}))$,

$$t\mathrm{tr}(AB) + t^{-1}\mathrm{tr}(S(A)B) = \sum_i \mathrm{tr}(s_iA)\mathrm{tr}(t_iB)$$

where $\sum_i s_i \otimes t_i$ is the universal R-matrix of $U_h(sl(2,\mathbb{C}))$ (see [7]). One should observe that these skein relations correspond to the trace identity, while the Kauffman bracket skein relations correspond to the trace identity for the negative of the trace.

This prompts us to introduce skein modules defined by these skein relations. Let for now t be an abstract variable, rather than the root of unity chosen at the beginning of § 5.2. For an orientable 3-dimensional manifold M, consider the free $\mathbb{C}[t, t^{-1}]$-module with basis the isotopy classes of framed links in M including the empty link. Factor this module by the skein relations

$$L = tH + t^{-1}V \text{ or } L = \epsilon(tH - t^{-1}V),$$

depending on whether the two crossing strands come from different components or not, where the links L, H, V are the same except in an embedded ball in M, inside of which they look as depicted in Figure 14. The same convention for ϵ is used, with the orientation of the crossing decided inside the ball. Additionally, replace any trivial link component that lies inside a ball disjoint from the rest of the link by a factor of $t^2 + t^{-2}$. We call the result of the factorization the *Reshetikhin-Turaev skein module* and denote it by $RT_t(M)$. One can show that $RT_t(M)$ is isomorphic to the Kauffman bracket skein module of M.

If $M = \Sigma_g \times [0,1]$ then the homeomorphism

$$\Sigma_g \times [0,1] \cup_{\Sigma_g} \Sigma_g \times [0,1] \approx \Sigma_g \times [0,1]$$

induces a multiplication on $R_t(\Sigma_g \times [0,1])$, turning it into an algebra, the *Reshetikhin-Turaev skein algebra.* This algebra is *not* isomorphic to the Kauffman bracket skein algebra except in genus one. In higher genus the multiplication rules are different, as can be seen by examining the product of a separating and a nonseparating curve that intersect.

The operation of gluing $\Sigma_g \times [0,1]$ to the boundary of a genus g handlebody H_g by a homeomorphism of the surface induces an $RT_t(\Sigma_g \times [0,1])$-module structure on $RT_t(H_g)$. Moreover, by gluing H_g with the empty skein inside to $\Sigma_g \times [0,1]$ we see that $RT_t(H_g)$ is the quotient of $RT_t(\Sigma_g \times [0,1])$ obtained by identifying the skeins in $\Sigma_g \times [0,1])$ that are isotopic in H_g.

In view of Lemma 5.2 and the identities from Figure 13, the irreducible representations V^n can be represented by skeins. Explicitly, $V^n = S_{n-1}(V^2) = f^{n-1}$, where f^n are defined recursively in Figure 15. These are the well-known Jones-Wenzl idempotents [19], [45].

The condition $S_{r-1}(V^2) = 0$ translates to the condition $f^{r-1} = 0$. This prompts us to define the *reduced Reshetikhin-Turaev skein module* $\widetilde{RT}_t(M)$ by factoring $RT_t(M)$ by $t = e^{\frac{i\pi}{2r}}$ and by the skein relation $f^{r-1} = 0$, taken in every possible embedded ball. This reduction is compatible with the multiplicative structure of $RT_t(\Sigma_g \times [0,1])$ and with its action on $RT_t(H_g)$.

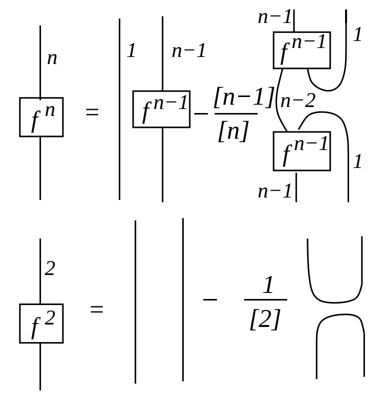

Figure 15

Proposition 5.5. *The quantum group quantization of the moduli space of flat $SU(2)$-connections on a surface Σ_g can be represented as the left multiplication of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ on $\widetilde{RT}_t(H_g)$.*

Proof. The proof is based on Proposition 5.4 and Proposition 5.1. Because each f^n involves n parallel strands, $RT_t(H_g)$ is a free $C[t, t^{-1}]$-module with basis the skeins obtained by

- replacing each edge of the core of H_g by a Jones-Wenzl idempotent in such a way that, if f^m, f^n, f^p meet at a vertex, then $m+n+p$ is even, $m+n \leq p$, $m+p \leq n$, $n+p \leq m$, and
- replacing the vertices by the unique collection of strands that lie in a disk neighborhood of the vertex and join the ends of the three Jones-Wenzl idempotents meeting there in such a way that there are no crossings.

Because of the Clebsch-Gordan theorem and Proposition 5.1, in the reduced skein module $\widetilde{RT}_t(H_g)$, only edges colored by f^n with $n \leq r-2$ need to be considered, and also if f^m, f^n, f^p meet at a vertex, then $m+1, n+1, p+1$ and their cyclic permutations should satisfy the double inequality from the Clebsch-Gordan theorem. Each element of this form comes from a basis element in the quantum group quantization. A more detailed explanation of this can be found, at least for the Kauffman bracket skein modules, in [28].

The computation from Figure 16, performed in the dotted annulus, shows that for a simple closed curve γ on the torus, $\mathrm{Op}(W_{\gamma,n})$ can be identified with the skein $S_{n-1}(\gamma) \in \widetilde{RT}_t(\Sigma_g \times [0,1])$. We conclude that the action of quantum observables on the Hilbert space is modeled by the action of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ on $\widetilde{RT}_t(H_g)$.

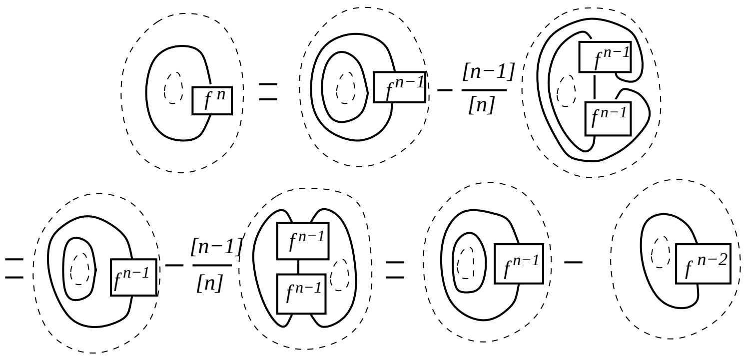

FIGURE 16

To identify the two quantization models, we also have to prove that the skeins associated to admissible colorings of the core of the handlebody form a basis, namely that they are linearly independent in $\widetilde{RT}_t(H_g)$.

The smooth part of $\mathcal{M}_g$ has real dimension $6g-6$. This smooth part is a completely integrable manifold in the Liouville sense. Indeed, the Wilson lines W_{α_i}, where α_i, $i = 1, 2, \ldots, 3g-3$, are the curves in Figure 17, form a maximal set of Poisson commuting functions (meaning that $\{W_{\alpha_i}, W_{\alpha_j}\} = 0$). The quantum group quantization of the moduli space of flat $SU(2)$-connections is thus a quantum integrable system, with the operators $\mathrm{Op}(W_{\alpha_1})$, $\mathrm{Op}(W_{\alpha_2})$, ..., $\mathrm{Op}(W_{\alpha_{3g-3}})$ satisfying the integrability condition.

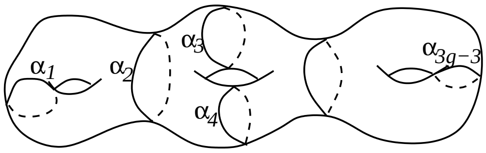

FIGURE 17

The identity from Figure 18, which holds for any choice of orientation of the strands, implies that the spectral decomposition of the commuting $(3g-3)$-tuple of self-adjoint operators

$$(\mathrm{Op}(W_{\alpha_1}), \mathrm{Op}(W_{\alpha_2}), ..., \mathrm{Op}(W_{\alpha_{3g-3}}))$$

has only 1-dimensional eigenspaces consisting precisely of the colorings of the edges following the specified rule. Indeed, the basis elements are as described in § 5.2 for the case where the curves that cut the surface into pairs of pants are $\alpha_1, \alpha_2, \cdots, \alpha_{3g-3}$, and the identity from Figure 18 shows that the eigenvalues of an e_j with respect to the $3g-3$ quantized Wilson lines completely determine the colors of its edges. This concludes the proof. □

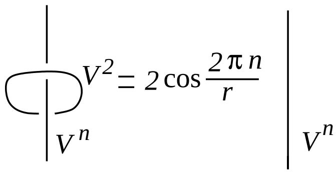

FIGURE 18

Remark 5.6. Proposition 5.5 should be compared to Theorem 3.9. Again the algebra of quantized observables is a skein algebra, the space of non-abelian theta functions is a quotient of this algebra, and the factorization relation is of topological nature; it is defined by gluing the cylinder over the surface to a handlebody via a homeomorphism. The skein modules $RT_t(\Sigma_g \times [0,1])$ and $\widetilde{RT}_t(\Sigma_g \times [0,1])$ are the analogues, for the gauge group $SU(2)$, of the algebras $\mathbb{C}[\mathbf{H}(\mathbb{Z})]$ and $\mathcal{A}_N$.

Since we have not yet proved that the pairing $[\,,\,]$ defined in § 5.2 is nondegenerate, we will take for the moment this representation of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ to actually be the quantum group quantization of the moduli space $\mathcal{M}_g$. We will prove the nondegeneracy in § 6.3.

The quantum group quantization is more natural than it seems. Quantum groups were introduced by Drinfeld to solve vertex models, as means of finding operators that satisfy the Yang-Baxter equation. They lead to the deformation quantization model for the quantization of $\mathcal{M}_g$ in [1]. This gives rise to the skein algebra of the surface, and by analogy with § 3.2 we are led to consider the skein module of the handlebody. The basis consisting of admissible colorings of the core of the handlebody appears when looking at the spectral decomposition of the system of commuting operators from the proof of Proposition 5.5.

5.4. **The quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus.** The quantum group quantization of $\mathcal{M}_1$ is a particular case of the construction in § 5.2 and has been described in [15]. A basis for the Hilbert space is specified by an oriented rigid structure on the torus. The curves a and b in Figure 1 define such a structure with a the seam and b the curve that cuts the torus into an annulus. Mapping the torus to the boundary of the solid torus such that b becomes null homologous and a the generator of the fundamental group, we obtain an orthonormal basis consisting of the vectors $V^1(\alpha), V^2(\alpha), \ldots, V^{r-1}(\alpha)$, which are the colorings of the core α of the solid torus by the irreducible representations $V^1, V^2, \ldots, V^{r-1}$ of $U_\hbar(sl(2,\mathbb{C}))$. These are the quantum group analogues of the ζ_j^τ's. Here, the orientation of the rigid structure, and hence of the core of the solid torus are irrelevant, reversing the orientation gives the same results in computations (orientation of link components is irrelevant [46]).

The operator associated to the function $f(x,y) = 2\cos 2\pi(px+qy)$ is computed like for higher genus surfaces. The bilinear form on the Hilbert space comes from the Hopf link and is $[V^j(\alpha), V^k(\alpha)] = [jk]$, $j, k = 1, 2, \ldots, r-1$. The value of $[\mathrm{Op}(2\cos 2\pi(px+qy))V^j(\alpha), V^k(\alpha)]$ is equal to the Reshetikhin-Turaev invariant of the three-component colored framed link consisting the curve of slope p/q on the torus embedded in standard position in S^3, colored by $V^{n+1} - V^{n-1}$ where n is the greatest common divisor of p and q, the core of the solid torus that lies on one side of the torus colored by V^j, and the core of the solid torus that lies on the other side colored by V^k. Coloring the curve by $V^{n+1} - V^{n-1}$ is the same as coloring it by

$$T_n(V^2) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (V^2)^{n-2j},$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind defined recursively by $T_0(x) = 2$, $T_1(x) = x$, $T_{n+1}(x) = xT_n(x) - T_{n-1}(x)$, for $n \geq 1$. Again, the quantum group quantization can be modeled by the action of the reduced Reshetikhin-Turaev skein algebra of the torus on the reduced Reshetikhin-Turaev skein module of the solid torus.

It has been shown in [15] that the quantum group quantization of the pillow case is unitary equivalent to Weyl quantization. However, that proof makes use of the Reshetikhin-Turaev representation of the mapping class group of the torus, and does not serve our purpose of showing *how* the Reshetikhin-Turaev representation arises from quantum mechanical considerations. For that reason we give a different proof of this result using the structure of the Reshetikhin-Turaev skein algebra of the torus.

For a pair of integers p, q, let n be their common divisor and define $(p,q)_T = T_n((p/n, q/n)) \in RT_t(\mathbb{T}^2 \times [0,1])$. The proof of the following result is identical to that of Theorem 4.1 in [12], which covers the case of the Kauffman bracket.

Theorem 5.7. *For any integers p, q, p', q' the following product-to-sum formula holds*

$$(p,q)_T(p',q')_T = t^{\left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|}(p+p', q+q')_T + t^{-\left|\begin{smallmatrix} p & q \\ p' & q' \end{smallmatrix}\right|}(p-p', q-q')_T.$$

As we can see, the Reshetikhin-Turaev and the Kauffman bracket skein algebras of the torus are isomorphic.

Theorem 5.8. [15] *The Weyl quantization and the quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus are unitary equivalent.*

Proof. We rephrase the quantum group quantization in terms of skein modules. The Hilbert space is $\widetilde{RT}_t(S^1 \times \mathbb{D}^2)$. Indeed, this skein module is spanned by the vectors $S_{j-1}(\alpha)$, $j = 1, 2, \ldots, r-1$, and these vectors are linearly independent because they are eigenvectors with different eigenvalues of the operator defined by $(0,1)$.

Considering the projection $\pi : RT_t(\mathbb{T}^2 \times [0,1]) \to RT_t(S^1 \times \mathbb{D}^2)$ defined by attaching the cylinder over the torus to the solid torus by the homeomorphism h_0 from § 3.2, and using Theorem 5.7 we deduce the recursive formula

$$\pi((p+1,q)_T) = t^{-q}\alpha\pi((p,q)_T) - t^{-2q}\pi((p-1,q)_T).$$

Also $\pi((0,q)_T) = t^{2q} + t^{-2q}$, and $\pi((1,q)_T) = t^{-2q}\alpha$. Solving the recurrence we obtain

$$\pi((p,q)_T) = t^{-pq}(t^{2q}S_p(\alpha) - t^{-2q}S_{p-2}(\alpha)).$$

Using again Theorem 5.7 we have

$$\begin{aligned}(p,q)_T T_j(\alpha) &= \pi[(p,q)_T(j,0)_T] = \pi[t^{-jq}(p+j,q)_T + t^{jq}(p-j,q)_T] \\ &= t^{-pq}[t^{-(2j-2)q}S_{p+j}(\alpha) + t^{(2j+2)q}S_{p-j}(\alpha) - t^{-(2j-2)q}S_{p+j-2}(\alpha) \\ &\qquad - t^{(2j-2)q}S_{p-j-2}(\alpha).\end{aligned}$$

Since $T_n(x) = S_n(x) - S_{n-2}(x)$ for all n, we have

$$(p,q)_T S_{j-1}(\alpha) = t^{-pq}(t^{-2qj}S_{p+j-1}(\alpha) + t^{2qj}S_{p-j+1}(\alpha), \text{ for } j > 0.$$

Reducing to the relative skein modules and using the fact that $S_{j-1}(\alpha) = V^j(\alpha)$, we obtain

$$\begin{aligned}\text{Op}(2\cos 2\pi(px+qy))V^j(\alpha) = e^{-\frac{\pi i}{2r}pq}\Big(&e^{\frac{\pi i}{r}qj}V^{j-p}(\alpha) \\ &+e^{-\frac{\pi i}{r}qj}V^{j+p}(\alpha)\Big),\end{aligned} \tag{5.1}$$

with the conventions $V^r(\alpha)=0$, $V^{j+2r}(\alpha)=V^j(\alpha)$, $V^{r+j}(\alpha)=-V^{r-j}(\alpha)$ if the indices leave the range $1,2,\ldots,r-1$. This is the formula for the Weyl quantization of the pillow case given in § 4.2, which proves the theorem. □

5.5. **A Stone–von Neumann theorem on the moduli space of flat $SU(2)$-connections on the torus.** Weyl quantization yields a representation of $\widetilde{RT}_t(\mathbb{T}\times[0,1])$ such that t acts as multiplication by $e^{\frac{i\pi}{2r}}$ and every simple closed curve on the torus acts as a self-adjoint operator. The algebra $\widetilde{RT}_t(\mathbb{T}\times[0,1])$ is a non-abelian analogue of the group algebra of the finite Heisenberg group. A Stone-von Neumann theorem holds also in this case.

Theorem 5.9. *The representation of the reduced Reshetikhin-Turaev skein algebra of the torus defined by the Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus is the* unique *irreducible representation of this algebra that maps simple closed curves to self-adjoint operators and t to multiplication by $e^{\frac{\pi i}{2r}}$. Moreover, quantized Wilson lines span the algebra of all linear operators on the Hilbert space of the quantization.*

Proof. We prove irreducibility by showing that any vector is cyclic. Because the eigenspaces of each quantized Wilson line are 1-dimensional, in particular those of $\mathrm{Op}(2\cos 2\pi y)$, it suffices to check this property for the eigenvectors of this operator, namely for ζ_j^τ, $j=1,2,\ldots,r-1$. And because

$$\begin{aligned}\mathrm{Op}(2\cos 2\pi x)\zeta_j^\tau &= \zeta_{j-1}^\tau+\zeta_{j+1}^\tau\\ \mathrm{Op}(2\cos 2\pi(x+y))\zeta_j^\tau &= t^{-1}(t^2\zeta_{j-1}^\tau+t^{-2}\zeta_{j+1}^\tau),\end{aligned}$$

by taking linear combinations we see that from ζ_j^τ we can generate both ζ_{j+1}^τ and ζ_{j-1}^τ. Repeating, we can generate the entire basis. This shows that ζ_j^τ is cyclic for each $j=1,2,\ldots,r-1$, hence the representation is irreducible.

To prove uniqueness, consider an irreducible representation of $\widetilde{RT}_t(\mathbb{T}^2\times[0,1])$ with the required properties. The condition $S_{r-1}(\gamma)=0$ for any simple closed curve γ on the torus implies, by the spectral mapping theorem, that the eigenvalues of the operator associated to γ are among the numbers $2\cos\frac{k\pi}{r}$, $k=1,2,\ldots,r-1$.

We write for generators $X=(1,0)$, $Y=(0,1)$, and $Z=(1,1)$ of $\widetilde{RT}_t(\mathbb{T}^2\times[0,1])$ the relations

$$\begin{aligned}&tXY-t^{-1}YX=(t^2-t^{-2})Z\\ &tYZ-t^{-1}ZY=(t^2-t^{-2})X\\ &tZX-t^{-1}XZ=(t^2-t^{-2})Y\\ &t^2X^2+t^{-2}Y^2+t^2Z^2-tXYZ-2t^2-2t^{-2}=0,\end{aligned}$$

by analogy with the presentation of the Kauffman bracket skein algebra of the torus found by Bullock and Przytycki in [8].

In fact $\widetilde{RT}_t(\mathbb{T}^2 \times [0,1])$ is generated by just X and Y, since we can substitute Z from the first equation. This gives the equivalent presentation

$$\begin{aligned}
&(t^2+t^{-2})YXY-(XY^2+Y^2X)=(t^4+t^{-4}-2)X\\
&(t^2+t^{-2})XYX-(YX^2+X^2Y)=(t^4+t^{-4}-2)Y\\
&(t^6+t^{-2}-2t^2)X^2+(t^{-6}+t^2-2t^{-2})Y^2+XYXY+YXYX\\
&\quad -t^2YX^2Y-t^{-2}XY^2X=2(t^6+t^{-6}-t^2-t^{-2}).
\end{aligned} \tag{5.2}$$

Setting $t=e^{\frac{i\pi}{2r}}$ the first equation in (5.2) becomes

$$2\cos\frac{\pi}{r}YXY-(XY^2+Y^2X)=4\sin^2\frac{\pi}{r}Y.$$

Let v_k be an eigenvector of Y with eigenvalue $2\cos\frac{k\pi}{r}$ for some $k\in\{1,2,\ldots,r-1\}$. We wish to generate a basis of the representation by acting repeatedly on v_k by X. For this set $Xv_k=w$. The above relation yields

$$2\cos\frac{\pi}{r}\cdot 2\cos\frac{k\pi}{r}Yw-4\cos^2\frac{k\pi}{r}w-Y^2w=4\sin^2\frac{\pi}{r}w.$$

Rewrite this as

$$\left[Y^2-4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}Y-4\left(\sin^2\frac{\pi}{r}+\cos^2\frac{k\pi}{r}\right)\right]w=0.$$

It follows that either $w=0$ or w is in the kernel of the operator

$$Y^2-4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}Y-4\left(\sin^2\frac{\pi}{r}+\cos^2\frac{k\pi}{r}\right)Id. \tag{5.3}$$

The second equation in (5.2) shows that if $Xv_k=w=0$ then $Yv_k=0$. This is impossible because of the third relation in (5.2). Hence $w\neq 0$, so w lies in the kernel of the operator from (5.3). Note that if λ is an eigenvalue of Y which satisfies

$$\lambda^2-4\cos\frac{k\pi}{r}\cos\frac{\pi}{r}\lambda-4\left(\sin^2\frac{\pi}{r}+4\cos^2\frac{k\pi}{r}\right)=0,$$

then necessarily $\lambda=2\cos\frac{(k\pm1)\pi}{r}$. It follows that

$$Xv_k=v_{k+1}+v_{k-1},$$

where $Yv_{k\pm1}=2\cos\frac{(k\pm1)\pi}{4}v_{k\pm1}$, and v_{k+1} and v_{k-1} are not simultaneously equal to zero. We wish to enforce v_k, v_{k+1}, and v_{k-1} to be elements of a basis. For that we need to check that v_{k+1} and v_{k-1} are nonzero, and we also need to understand the action of X on them.

Set $Xv_{k+1}=\alpha v_k+v_{k+2}$ and $Xv_{k-1}=\beta v_k+v_{k-2}$, where $Yv_{k\pm2}=2\cos\frac{(k\pm2)\pi}{r}v_{k\pm2}$. It might be possible that the scalars α and β are zero. The vectors v_{k+2}, v_{k-2} might as well be zero; if they are not zero, then they are eigenvectors of Y, and their respective eigenvalues are as specified (which can be seen by repeating the above argument).

Applying both sides of the second equation in (5.2) to v_k and comparing the v_k coordinate of the results we obtain

$$\cos\frac{\pi}{r}\cos\frac{(k+1)\pi}{r}\alpha+\cos\frac{\pi}{r}\cos\frac{(k-1)\pi}{r}\beta-\cos\frac{k\pi}{r}(\alpha+\beta)$$
$$=\cos\frac{2\pi}{r}\cos\frac{k\pi}{r}-\cos\frac{k\pi}{r}.$$

This is equivalent to

$$\left(\cos\frac{(k+2)\pi}{r}+\cos\frac{k\pi}{r}\right)(\alpha-1)+\left(\cos\frac{(k-2)\pi}{r}+\cos\frac{k\pi}{r}\right)(\beta-1)=0$$

that is

$$\sin\frac{(k+1)\pi}{r}(\alpha-1)+\sin\frac{(k-1)\pi}{r}(\beta-1)=0.$$

For further use, we write this as

$$(t^{4k+4}-1)(\alpha-1)+(t^{4k}-t^4)(\beta-1)=0. \tag{5.4}$$

Applying the two sides of the last equation in (5.2) to v_k and comparing the v_k coordinate of the results we obtain

$$\begin{aligned}&(t^6+t^{-2}-2t^2)(\alpha+\beta)+(t^{-6}+t^2-2t^{-2})4\cos^2\frac{k\pi}{r}\\&+8\cos\frac{k\pi}{r}\cos\frac{(k+1)\pi}{r}\alpha+8\cos\frac{k\pi}{r}\cos\frac{(k-1)\pi}{r}\beta-4t^2\cos^2\frac{k\pi}{r}(\alpha+\beta)\\&-4t^{-2}\cos^2\frac{(k+1)\pi}{r}\alpha-4t^{-2}\cos^2\frac{(k-1)\pi}{r}\beta=2(t^6+t^{-6}-t^2-t^{-2}).\end{aligned}$$

This can be rewritten as

$$\begin{aligned}&(t^6+t^{-2}-2t^2)(\alpha+\beta)+4\cos\frac{(2k+1)\pi}{r}\alpha+4\cos\frac{\pi}{r}\alpha+4\cos\frac{(2k-1)\pi}{r}\beta\\&+4\cos\frac{\pi}{r}\beta-2t^2\cos\frac{2k\pi}{r}\alpha-2t^2\cos\frac{2k\pi}{r}\beta-2t^2\alpha-2t^2\beta\\&-2t^{-2}\cos\frac{(2k+2)\pi}{r}\alpha-2t^{-2}\alpha-2t^{-2}\cos\frac{(2k-2)\pi}{r}\beta-2t^{-2}\beta\\&=2(t^6+t^{-6}-t^2-t^{-2})-2(t^{-6}+t^2-2t^{-2})-2(t^{-6}+t^2-2t^{-2})\cos\frac{2k\pi}{r}.\end{aligned}$$

Using the fact that $t=\cos\frac{k\pi}{2r}+i\sin\frac{k\pi}{2r}$ we can transform this into

$$\begin{aligned}&(t^{-4k-6}+t^{-4k+2}+2t^2-2t^{-4k-2}-t^6-t^{-2})(\alpha-1)\\&+(t^{4k-6}+t^{4k+2}+2t^2-2t^{4k-2}-t^6-t^{-2})(\beta-1)=0.\end{aligned}$$

Dividing through by $t^{-6}+t^2-2t^{-2}$ we obtain

$$(t^{-4k}-t^4)(\alpha-1)+(t^{4k}-t^4)(\beta-1)=0.$$

Combining this with (5.4), we obtain the system

$$(t^{4k+4}-1)u+(t^{4k}-t^4)v=0$$
$$(t^{-4k}-t^4)u+(t^{4k}-t^4)v=0$$

in the unknowns $u=\alpha-1$ and $v=\beta-1$. Recall that $t=e^{\frac{i\pi}{2r}}$.

The coefficient of v equals zero if and only if $k=1$, in which case we are forced to have $\beta=0$, because 0 is not an eigenvalue of Y. The coefficient of u in one of the equations is equal to zero if and only if $k=r-1$, in which case we are forced to have $\alpha=0$, because -1 is not an eigenvalue of Y.

In any other situation, by subtracting the equations we obtain

$$(t^4-t^{-4k})(t^{4k}+1)u=0.$$

This can happen only if $t^{4k}=-1$, namely if $2k=r$.

So, if $k\neq\frac{r}{2}$, then $Xv_k=v_{k+1}+v_{k-1}$ with v_{k+1} and v_{k-1} eigenvectors of Y with eigenvalues $2\cos\frac{(k+1)\pi}{r}$ respectively $2\cos\frac{(k-1)\pi}{r}$, and $Xv_{k\pm1}=v_k+v_{k\pm2}$, where $v_{k\pm2}$ lie in the eigenspaces of Y of the eigenvalues $2\cos\frac{(k\pm2)\pi}{r}$.

What if $k=\frac{r}{2}$? One of v_{k+1} and v_{k-1} is not zero, say v_{k+1}. Applying the above considerations to v_{k+1} we have $Xv_{k+1}=\alpha v_k+v_{k+2}$ and $X\alpha v_k=v_{k+1}+v'_{k-1}$, for some v'_{k-1} in the eigenspace of Y of the eigenvalue $2\cos\frac{(k-1)\pi}{r}$. Then on the one hand $Xv_k=v_{k+1}+v_{k-1}$ and on the other $\alpha Xv_k=v_{k+1}+v'_{k-1}$. This shows that $\alpha=1$, and because $(\alpha-1)+(\beta-1)=0$, it follows that $\beta=1$ as well. A similar conclusion is reached if $v_{k-1}\neq0$.

Repeating the argument we conclude that the irreducible representation, which must be the span of $X^mY^nv_k$ for $m,n\geq0$, has the basis $v_1,v_2,\ldots,v_{r-1}$, and X and Y act on these vectors by

$$Xv_j=v_{j+1}+v_{j-1},\quad Yv_j=2\cos\frac{j\pi}{r},$$

with the convention $v_0=v_r=0$. And we recognize the representation defined by the Weyl quantization of the moduli space of flat $SU(2)$-connections on the torus.

The fact that the algebra of all quantized Wilson lines is the algebra of all linear operators on the Hilbert space of the quantization is a corollary of Theorem 6.1 in [14]. □

6. The Reshetikhin-Turaev representation as a Fourier transform for non-abelian theta functions

6.1. **The Reshetikhin-Turaev representation of the mapping class group of the torus.** In this section we deduce the existence of the Reshetikhin-Turaev projective representation of the mapping class group of the torus from quantum mechanical considerations, and show that it can be computed explicitly *from these considerations.* This should be compared with the computations in § 3.3.

There is an action of the mapping class group of the torus on the ring of functions on the pillow case, given by

$$h \cdot f(A) = f(h_*^{-1}A),$$

where $h_*^{-1}A$ denotes the pull-back of the connection A by h. In particular the Wilson line of a curve γ is mapped to the Wilson line of the curve $h(\gamma)$. The action of the mapping class group on functions on the pillow case induces an action on the quantum observables by

$$h \cdot \mathrm{Op}(f(A)) = \mathrm{Op}(f(h_*^{-1}A)),$$

which for Wilson lines is

$$h \cdot \mathrm{Op}(W_\gamma) = \mathrm{Op}(W_{h(\gamma)}).$$

Theorem 6.1. *There exists a projective representation of the mapping class group of the torus that satisfies the exact Egorov identity*

$$Op(W_{h(\gamma)}) = \rho(h) Op(W_\gamma)\rho(h)^{-1}$$

with the quantum group quantization of Wilson lines. Moreover, $\rho(h)$ is unique up to multiplication by a constant.

Proof. We follow the first proof to Theorem 3.12. The bijective map $L \to h(L)$ on the set of isotopy classes of framed links in the cylinder over the torus induces an automorphism of the free $\mathbb{C}[t, t^{-1}]$-module with basis these isotopy classes of links. Because this map leaves invariant the ideal defined by the skein relations (for crossings and for the $r-1$st Jones-Wenzl idempotent), it defines an automorphism $\Phi : \widetilde{RT}_t(\mathbb{T}^2 \times [0,1]) \to \widetilde{RT}_t(\mathbb{T}^2 \times [0,1])$. The representation of $RT_t(\mathbb{T}^2 \times [0,1])$ given by $V^j(\alpha) \to \mathrm{Op}(W_{h(\gamma)})V^j(\alpha)$ is an irreducible representation of $\widetilde{RT}_t(\mathbb{T}^2 \times [0,1])$ which still maps t to multiplication by $e^{\frac{i\pi}{2r}}$ and simple closed curves to self-adjoint operators. In view of Theorem 5.9 this representation is equivalent to the standard representation. This proves the existence of the map $\rho(h)$ that satisfies the exact Egorov identity with quantizations of Wilson lines. Schur's lemma implies that $\rho(h)$ is unique up to multiplication by a constant and that ρ is a projective representation of the mapping class group. Let us mention that a computational proof of uniqueness was given in [14]. □

Also from Theorem 5.9 we deduce that for every h in the mapping class group, the map $\rho(h)$ can be represented as multiplication by a skein $\mathcal{F}(h) \in \widetilde{RT}_t(\mathbb{T}^2 \times [0,1])$. We want to find $\mathcal{F}(h)$ explicitly. We consider first the case of the positive twist T along the curve $(0,1)$. Since the twist leaves the curve $(0,1)$ invariant,

$$\mathcal{F}(T)(0,1)V^k(\alpha) = (0,1)\mathcal{F}(T)V^k(\alpha), \quad \text{for all } k.$$

And because the eigenspaces of $\mathrm{Op}(W_{(0,1)})$ are 1-dimensional, the linear operator defined by $\mathcal{F}(T)$ on the Hilbert space is a polynomial in $\mathrm{Op}(W_{(0,1)})$.

The polynomials $S_j(x)$, $0 \leq j \leq r-1$ form a basis for $\mathbb{C}[x]/S_{r-1}(x)$, so

$$\mathcal{F}(T) = \sum_{j=1}^{r-1} c_j S_{j-1}((0,1)), \quad c_j \in \mathbb{C}.$$

On the other hand,

$$(1,1)\mathcal{F}(T)V^k(\alpha) = \mathcal{F}(T)(1,0)V^k(\alpha). \tag{6.1}$$

Using (5.1) and the fact that $S_{j-1}((0,1))V^k(\alpha) = \frac{[jk]}{[k]}V^k(\alpha)$ for all j and k, we rewrite (6.1) as

$$\sum_j c_j \frac{[jk]}{k} t^{-1}(t^{-2k}V^{k+1}(\alpha) + t^{2k}V^{k-1}(\alpha)$$
$$= \sum_j c_j \left(\frac{[j(k+1)]}{[k+1]} V^{k+1}(\alpha) + \frac{[j(k-1)]}{[k-1]} V^{k-1}(\alpha) \right).$$

Setting the coefficients of V^{k+1} on both sides equal yields

$$\sum_{j=1}^{r-1} c_j[j(k+1)] = \frac{[k+1]}{[k]} t^{-2k-1} \sum_{j=1}^{r-1} c_j[jk].$$

Denoting $\sum_j c_{j=1}^{r-1}[j] = t^{-1}u$, we obtain the system of equations in c_j, $j = 1, 2, \ldots, r-1$,

$$\sum_{j=1}^{r-1} [kj]c_j = [k]t^{-k^2}u, \quad k = 1, 2, \ldots, r-1.$$

Recall that $[n] = \sin \frac{n\pi}{r} / \sin \frac{\pi}{r}$, so the coefficient matrix is a multiple of the matrix of the discrete sine transform. The square of the discrete sine transform is the identity map, so there is a constant C such that

$$c_j = C \sum_k [jk][k]t^{-k^2}.$$

Standard results in the theory of Gauss sums [25] show that $\sum_k [jk][k]t^{-k^2} = C'[j]t^{j^2}$ where C' is a constant independent of j. We conclude that $\mathcal{F}(T)$ is a multiple of $\sum_{j=1}^{r-1}[j]t^{j^2}S_{j-1}((0,1))$. We normalize $\mathcal{F}(T)$ to make it unitary by multiplying by $\eta = \sqrt{\frac{r}{2}} \sin \frac{\pi}{r} = (\sum_{j=1}^{r-1}[j]^2)^{-1/2}$, and for reasons that will become apparent in a moment we also multiply it by t^{-1}. In view of this formula, and by analogy with § 3.3, we define

$$\Omega_{SU(2)} = \eta \sum_{j=1}^{r-1} [j]V^j(\alpha) = \eta \sum_{j=1}^{r-1} S_{r-1}(\alpha) \in \widetilde{RT}_t(S^1 \times \mathbb{D}^2).$$

For a framed link L, let $\Omega_{SU(2)}(L)$ be the skein obtained by replacing each component of L by $\Omega_{SU(2)}$ such that the curve $(1,0)$ on the boundary of the solid torus is mapped to the framing. Then $\mathcal{F}(T)$ is the coloring by

$\Omega_{SU(2)}$ of the surgery curve of T. This is true for any twist, and because on the one hand any element of the mapping class group is a product of twists, and on the other $\rho(h)$ is unique, we obtain a description of the map $\rho(h)$ in topological terms, as an the analogue of Theorem 3.13.

Theorem 6.2. *Let h be an element of the mapping class group of the torus obtained by performing surgery on a framed link L_h in $\mathbb{T}^2 \times [0,1]$. The map $\rho(h) : \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2) \to \widetilde{\mathcal{L}}_t(S^1 \times \mathbb{D}^2)$ is given by*

$$\rho(h)\beta = \Omega_{SU(2)}(L_h)\beta.$$

Remark 6.3. Everyone familiar with the Witten-Reshetikhin-Turaev theory has recognized the element $\Omega_{SU(2)}$, which is a fundamental building block of that theory. The exact Egorov identity in Theorem 6.1 implies that γ can be slid along L_h colored by $\Omega_{SU(2)}$. Once this is observed, it is natural to try slides over knots in general, and to deduce the Reshetikhin-Turaev formula for 3-manifold invariants [35].

The projective representation of the mapping class group from Proposition 6.2 is the Reshetikhin-Turaev representation. There is an alternative representation, with a more geometric flavor. Since odd and even theta functions are invariant under the Hermite-Jacobi action given by discrete Fourier transforms, for an element h of the mapping class group of the torus given by (3.3) we have the action on the ζ_j^τ's:

$$\rho_{HJ}(h)\zeta_j^\tau(z) = \exp\left(-\frac{\pi i N c z^2}{c\tau + d}\right)\zeta_j^{\tau'}\left(\frac{z}{c\tau+d}\right)$$

where $\tau' = \frac{a\tau+b}{c\tau+d}$.

We compare this representations to the Reshetikhin-Turaev representation. Modulo a multiplication by a positive normalization constant,

$$\rho_{HJ}(S)\zeta_j^\tau(z) = \rho(S)(\theta_j^\tau(z) - \theta_{-j}^\tau(z)) = \sum_{k=0}^{2r-1}\left(e^{-\frac{\pi i}{r}jk}\theta_k^\tau(z) - e^{\frac{\pi i}{r}jk}\theta_{-k}^\tau(z)\right)$$

$$= \sum_{k=1}^{r-1}\left(e^{-\frac{\pi i}{r}jk}\theta_k^\tau(z) - e^{\frac{\pi i}{r}jk}\theta_{-k}^\tau(z)\right) + \sum_{k=1}^{r-1}\left(e^{-\frac{\pi i}{r}j(2r-k)}\theta_{2r-k}^\tau(z)\right.$$

$$\left.-e^{\frac{\pi i}{r}j(2r-k)}\theta_k^\tau(z)\right) = 2\sum_{k=1}^{r-1}\left(e^{-\frac{\pi i}{r}jk} - e^{\frac{\pi i}{r}jk}\right)(\theta_k^\tau(z) - \theta_{-k}^\tau(z))$$

$$= -4i\sum_{k=1}^{r-1}\sin\frac{\pi jk}{r}\zeta_k^\tau(z),$$

$$\rho_{HJ}(T)\zeta_j^\tau(z) = \rho(T)(\theta_j^\tau - \theta_{-j}^\tau(z)) = e^{\frac{\pi i}{r}j^2}(\theta_j^\tau(z) - \theta_{-j}^\tau(z)) = e^{\frac{\pi i}{r}j^2}\zeta_j^\tau(z).$$

The matrix of $\rho(S)$ defined via quantum groups has the j,k-entry equal to the Reshetikhin-Turaev invariant of the Hopf link with components colored by V^j respectively V^k, which is $[jk]$. We normalize both $\rho_{HJ}(S)$ and $\rho(S)$

by multiplying them by η. The map $\rho(T)$ introduces a positive twist on each basis element, and as such it is the diagonal matrix with diagonal entries $e^{\frac{\pi i}{2r}(j^2-1)}$. We have

$$\rho(T) = e^{\frac{i\pi}{2r}}\rho_{HJ}(T).$$

This explains why although for Weyl quantization one can pass to a $\mathbb{Z}_2$-extension of $SL(2,\mathbb{Z})$ to remove projectiveness and obtain a true representation, for the Reshetikhin-Turaev theory one has to take a full $\mathbb{Z}$-extension.

The S-matrices satisfy

$$\rho(S) = i\rho_{HJ}(S).$$

We have $\rho(S)^2 = Id$ but $\rho_{HJ}(S)^2 = -Id$. This is reflected in the fact that the map $z \to -z$ does not change the basis element $V^j(\alpha)$, while $\zeta_j(-z) = -\zeta_j(z)$. So for the Weyl quantization the curve b in Figure 1 does have to be oriented, while for the quantum group quantization it does not. Note also the equality $\Omega_{SU(2)} = \rho(S)\phi$.

The relationship with classical theta functions allows us to adapt a formula of Kač and Peterson [21] to obtain an explicit formula for the Reshetikhin-Turaev representation of the mapping class group of the torus.

Theorem 6.4. *Let*

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

be an element of the mapping class group of the torus. Then there is a number $c(2r,h) \in \mathbb{C}$ *such that*

$$\rho(h)\zeta_j^\tau(z) = c(2r,h)\sum_k e^{\frac{\pi i}{2r}(cdk^2+abj^2)}[bckj]\zeta^\tau_{aj+ck}(z)$$

where the sum is taken over a family of $j \in \mathbb{Z}$ *that give all representatives of the classes* cj *modulo* $2r\mathbb{Z}$ *and the square brackets denote a quantized integer.*

Proof. Because $2r$ is an even integer, the group $SL_\theta(2,\mathbb{Z})$ is the whole $SL(2,\mathbb{Z})$. By Proposition 3.17 in [21], there is a constant $\nu(2r,h)$ such that

$$\theta_j^{\tau'}\left(\frac{z}{c\tau+d}\right) = \nu(2r,h)\exp\left(\frac{2\pi i r c z^2}{c\tau+d}\right)\sum_k e^{\frac{i\pi}{2r}(cdk^2+2bckj+abj^2)}\theta^\tau_{aj+ck}(z)$$

with the same summation convention as in the statement of the theorem, and with $\tau' = \frac{a\tau+b}{c\tau+d}$. The map

$$\theta_j^\tau(z) \to \sum_k e^{\frac{i\pi}{2r}(cdk^2+2bckj+abj^2)}\theta^\tau_{aj+ck}(z)$$

is, up to multiplication by a constant, the unique map that satisfies the exact Egorov identity with the representation of the Heisenberg group. It follows that

$$\zeta_j^\tau(z) \to \sum_k e^{\frac{\pi i}{2r}(cdk^2+abj^2)}[bcjk]\zeta_{aj+ck}^\tau(z)$$

satisfies the exact Egorov identity with the Weyl quantization of the pillow case. There is a unique map with this property, up to multiplication by a constant. Hence the conclusion. □

6.2. **The structure of the reduced Reshetikhin-Turaev skein algebra of the cylinder over a surface.** As mentioned above, the element $\Omega_{SU(2)}$ is the fundamental building block in the construction of the Witten-Reshetikhin-Turaev quantum invariants of 3-manifolds. Here are some of its well known properties that will be used in the sequel.

Proposition 6.5. *If M is an orientable compact 3-manifold, then in $\widetilde{RT}_t(M)$ the following hold:*

a) *if O is the framed unknot in M, then $\Omega_{SU(2)}(O) = \eta^{-1}\emptyset$,*

b) *if K and K' are framed knots in M, then in $\widetilde{RT}_t(M)$,*

$$K \cup \Omega_{SU(2)}(K) = (K\#K') \cup \Omega_{SU(2)}(K'),$$

(recall that $K\#K'$ denotes the slide of K along K', see § 3.3),

c) *for all skeins, the skein relation from Figure 19 holds.*

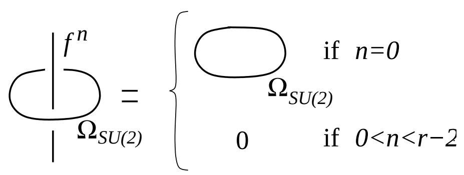

FIGURE 19

Lemma 6.6. *The quantum group quantizations of all Wilson lines on a surface generate the algebra of all linear operators on the Hilbert space of the quantization.*

Proof. The case of the torus was addressed in Theorem 5.9. For a higher genus surface Σ_g, the conclusion follows if we show that every nonzero vector is cyclic for the algebra generated by quantized Wilson lines.

Recall the curves α_i from Figure 20. Given a knot in the handlebody H_g, we can talk about the linking number of this knot with one of the curves α_i; just embed the handlebody in S^3 in standard position. We agree to take this with a positive sign. The linking number of a link L in H_g with the curve α_i is the sum of the linking numbers of the components. Associate to L the number $d(L)$ obtained by summing these for all $i = 1, 2, \dots, 3g-3$. Finally, for a skein $\sigma = \sum c_j L_j$, where L_j are links and $c_j \in \mathbb{C}$, let $d(\sigma) = \max_j d(L_j)$.

We claim that for each skein σ that is not a multiple of the empty link, there is a skein σ' such that $d(\sigma') < d(\sigma)$ and σ' is in the cyclic representation generated by σ.

To this end write σ in a basis of eigenvectors of the $3g-3$-tuple

$$(\mathrm{Op}(W_{\alpha_1}), \mathrm{Op}(W_{\alpha_2}), \ldots, \mathrm{Op}(W_{\alpha_{3g-3}}))$$

as $\sigma = \sum c_j e_j$. Because the spectral decomposition of this $3g-3$-tuple of operators has only 1-dimensional eigenspaces, each e_j with nonzero coefficient is in the cyclic representation generated by σ. For each such e_j, $d(e_j) \leq d(\sigma)$. If one of these inequalities is sharp, then the claim is proved. If not, we show that if e_k is not the empty link (i.e. the trivalent graph with all edges colored by V^1), then in the cyclic representation generated by e_k there is a skein σ' with $d(\sigma') < d(e_k)$.

After deleting all edges of e_k colored by the trivial representation V^1, the not necessarily connected graph obtained has an edge whose endpoints coincide, which is colored by some nontrivial representation V^n. Let β be a framed simple closed curve on σ_g that is isotopic to this edge and choose an α_i that intersects β as shown in Figure 20.

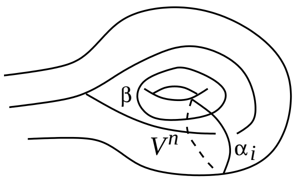

FIGURE 20

The recursive formula in Lemma 5.2 implies that $\mathrm{Op}(W_\beta)e_k$ is the sum of two skeins, σ' that has the edge linking α_i colored by V^{n-1} and σ'' that has the edge linking α_i colored by V^{n+1}. It is a standard fact that σ' is an eigenvector of $\mathrm{Op}(W_{\alpha_i})$ with eigenvalue $[2n-2]$, while, if it is nonzero, then σ'' is an eigenvector of $\mathrm{Op}(W_{\alpha_i})$ with eigenvalue $[2n+2]$. We can therefore conclude that σ' is in the cyclic representation generated by e_k, and therefore in the cyclic representation generated by σ.

Repeating, we eventually descend to the empty link. It remains to show that the empty link is cyclic. But this is obviously true, since each basis element can be represented as a framed link, hence is the image of a collection of nonintersecting framed simple closed curves on the boundary. This completes the proof. □

Theorem 6.7. *Given a genus g surface Σ_g, $g \geq 1$, the representation of $\widetilde{RT}_t(\Sigma_g \times [0,1])$ on $\widetilde{RT}_t(H_g)$ is faithful. Moreover, $\widetilde{RT}_t(\Sigma_g \times [0,1])$ is the algebra of all linear operators on $\widetilde{RT}_t(H_g)$.*

Proof. In view of Lemma 6.6, it suffices to show that the dimension of the vector space $\widetilde{RT}_t(\Sigma_g \times [0,1])$ equals the square of the dimension of $\widetilde{RT}_t(H_g)$.

This fact, at least for the Kauffman bracket, goes back to unpublished work of J. Roberts. In the case of the Kauffman bracket, part of the proof can be found in [38] and the case where r is an odd prime can be found in [13].

For two compact, orientable 3-dimensional manifolds M and N, we denote by $M\#N$ their connected sum, obtained by removing a 3-dimensional ball from both M and N and then gluing the resulting manifolds along the newly obtained boundary spheres. In $M\#N$, the manifolds M and N are separated by a sphere S_{sep}. In particular, by turning the handlebody H_g inside out, we see that $H_g\#H_g$ is an S^3 with two handlebodies removed.

Lemma 6.8. *Given the* 3*-dimensional manifolds M and N, the map*

$$\widetilde{RT}_t(M)\otimes\widetilde{RT}_t(N)\to\widetilde{RT}_t(M\#N)$$

defined by $(L,L')\to L\cup L'$, where L and L' are framed links in M respectively N, is an isomorphism of vector spaces.

Proof. The proof of this lemma was inspired by [34]. Any skein in $\widetilde{RT}_t(M\#N)$ can be written as $\sum_{j=1}^k c_j\sigma_j$, where c_j's are complex coefficients and each σ_j is a skein that intersects S_{sep} along the jth Jones-Wenzl idempotent. Taking a trivial knot colored by $\Omega_{SU(2)}$ and sliding it over S_{sep} we obtain, by using Proposition 6.5 c), the equality

$$\eta^{-1}\sum_{j=0}^{r-2}c_j\sigma_j=\eta^{-1}c_0.$$

Hence any skein is equal to a skein that is disjoint from S_{sep}. This proves that the map from the statement is an epimorphism. It is also a monomorphism because the skein module of a regular neighborhood of S_{sep} is trivial. Hence it is an isomorphism. □

From here we continue as in [38]. The manifold $H_g\times[0,1]$ can be obtained from $H_g\#H_g$ by surgery on a g-component framed link L_g as shown in Figure 21. Let $N_1\subset H_g\#H_g$ be a regular neighborhood of L_g, which consists of g solid tori. Let $N_2\subset H_g\times[0,1]$ be the union of the g surgery tori and L_g' the framed link in $H_g\times[0,1]$ consisting of the cores of these tori. Then $H_g\#H_g$ can be obtained from $H_g\times[0,1]$ by performing surgery on L_g'.

Every skein in $H_g\#H_g$ respectively $H_g\times[0,1]$ can be isotoped as to miss N_1 respectively N_2. The homeomorphism $\phi:(H_g\#H_g)\backslash N_1\to(H_g\times[0,1])\backslash N_2$ induces an isomorphism of skein modules

$$\phi:\widetilde{RT}_t((H_g\#H_g)\backslash N_1)\to\widetilde{RT}_t((H_g\times[0,1])\backslash N_2).$$

Unfortunately ϕ does not induce an isomorphism between $\widetilde{RT}_t(H_g\#H_g)$ and $\widetilde{RT}_t(H_g\times[0,1])$, it does not even give a well defined map because a skein in $H_g\#H_g$ can be pushed in many ways off N_1, while the images of these push-offs through ϕ are not isotopic.

Sikora's idea was to define $F_1:\widetilde{RT}_t(H_g\times[0,1])\to\widetilde{RT}_t(H_g\#H_g)$ by $F_1(\sigma)=\phi(\sigma)\cup\Omega_{SU(2)}(L_g')$. By Proposition 6.5 b), we are allowed to slide

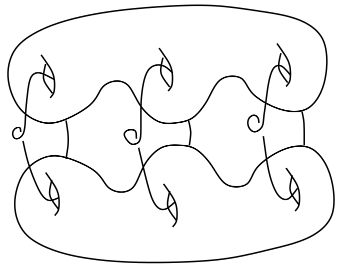

FIGURE 21

$\phi(\sigma)$ along L_g', which implies that this map is well defined. Its inverse is $F_2(\sigma) = \phi^{-1}(\sigma) \cup \Omega_{SU(2)}(L_g)$. Indeed, to see that these maps are the inverse of each other, push L_g' off N_2 in the direction of the framing of L_g'. Then each of the components of $\phi^{-1}(L_g')$ is the meridian of the surgery torus, and it surrounds once the corresponding component of L_g. By Proposition 6.5 c), $\Omega_{SU(2)}(L_g) \cup \Omega_{SU(2)}(\phi^{-1}(L_g')) = \emptyset \in \widetilde{RT}_t(H_g \# H_g)$. Hence $F_2 \circ F_1 = Id$. Similarly we check that $F_1 \circ F_2 = Id$, and the theorem is proved. □

Remark 6.9. This result is the non-abelian analogue of Proposition 3.4.

6.3. **The quantization of Wilson lines determines the Reshetikhin-Turaev representation.** Like for the torus, there is an action of the mapping class group of a surface Σ_g on the ring of regular functions on the moduli space $\mathcal{M}_g$ of flat $SU(2)$-connections on Σ_g given by $h \cdot f(A) = f(h_*^{-1}A)$. The action of the mapping class group on regular functions on $\mathcal{M}_g$ induces an action on the quantum observables, which for Wilson lines is given by

$$h \cdot \mathrm{Op}(W_\gamma) = \mathrm{Op}(W_{h(\gamma)}).$$

Theorem 6.10. *There is a projective representation ρ of the mapping class group of a closed surface that satisfies the exact Egorov identity*

$$Op(W_{h(\gamma)}) = \rho(h)\, Op(W_\gamma)\rho(h)^{-1}$$

with the quantum group quantization of Wilson lines. Moreover, for every h, $\rho(h)$ is unique up to multiplication by a constant.

Proof. We mimic the second proof to Theorem 3.12. The bijective map $L \to h(L)$ on the set of isotopy classes of framed links in the cylinder over the torus induces an automorphism of the free $\mathbb{C}[t, t^{-1}]$-module with basis these isotopy classes of links. Because the ideal defined by the skein relations is invariant under this map, the map defines an automorphism

$$\Phi : \widetilde{RT}_t(\Sigma_g \times [0,1]) \to \widetilde{RT}_t(\Sigma_g \times [0,1]).$$

By Proposition 6.7 the algebra $\widetilde{RT}_t(\Sigma_g \times [0,1])$ is the algebra of all linear operators on $\widetilde{RT}_t(H_g)$, so the automorphism Φ is inner [42]. This proves the existence of $\rho(h)$. The fact that ρ is a representation and the uniqueness are consequences of Schur's lemma. □

Each element of the mapping class group preserves the Atiyah-Bott symplectic form, so it induces a symplectomorphism of the $\mathcal{M}_g$. The theorem proves that the symplectomorphisms of $\mathcal{M}_g$ that arise from elements of the mapping class group can be quantized. Their quantization plays the role of the Fourier transform for non-abelian theta functions. Of course ρ is (up to multiplication by constants) the Reshetikhin-Turaev representation. This result shows that *all the information about the Reshetikhin-Turaev representation of the mapping class group is contained in the quantum group quantization of Wilson lines.*

Once we know that the element $\Omega_{SU(2)}$ allows handle slides, discovering the formula for $\rho(h)$ is easy. However if we were able to prove a Stone-von Neumann theorem in higher genus, or to prove Theorem 6.7 without using links colored by $\Omega_{SU(2)}$, then the formula for $\rho(h)$ could be deduced like in the case of the torus. The idea is to write $\rho(h)$ as a composition of twists along nonseparating curves using the Lickorish twist theorem, then examine each twist separately.

If γ is such a curve, with corresponding twist T_γ, and if $\rho(T_\gamma)$ is represented by a skein $\mathcal{F}(T_\gamma)$, then $\mathcal{F}(T_\gamma)$ commutes with all skeins that do not intersect γ. One can show that on each eigenspace of $\mathrm{Op}(W_\gamma)$ these skeins span the algebra of all linear operators. Hence the skeins commuting with $\mathcal{F}(T_\gamma)$ span the algebra of all operators that commute with $\mathrm{Op}(W_\gamma)$. Consequently, $\mathcal{F}(T_\gamma)$ is a polynomial in γ. Next we can restrict ourselves to a solid torus containing γ and follow the steps from the computation in § 6.1 to deduce the formula for $\mathcal{F}(T_\gamma)$.

As explained in [40] and [43], the projective representation of the mapping class group can be made into a true representation by passing to a $\mathbb{Z}$-extension of the mapping class group. Like for classical theta functions, the extension can be defined in terms of either the Maslov index.

For this, fix a rigid structure on the surface Σ_g and consider the subspace $\mathbf{L}$ of $H_1(\Sigma_g, \mathbb{R})$ spanned by the curves that dissect the surface into pairs of pants. Then $\mathbf{L}$ is a Lagrangian subspace of $H_1(\Sigma_g, \mathbb{R})$ with respect to the intersection form. The composition of extended homeomorphisms is defined by

$$(h', n') \circ (h, n) = (h' \circ h, n + n' - \boldsymbol{\tau}(\mathbf{L}, h(\mathbf{L}), h' \circ h(\mathbf{L})),$$

where $\boldsymbol{\tau}$ is the Maslov index with respect to the intersection pairing.

For completeness, we conclude our discussion with the proof of the following result mentioned in § 5.2 and whose importance was addressed at the end of § 5.3.

Proposition 6.11. *The bilinear pairing used in the definition of the quantum group quantization from § 5.2 is nondegenerate.*

Proof. We first give a description of the inner product on the Hilbert space $\mathcal{H}_r(\Sigma_g) = \widetilde{RT}_t(H_g)$ by diagrams, following an idea in [13].

The handlebody H_g has a natural orientation reversing symmetry s that leaves its core invariant. Glue two copies of H_g along their boundaries by the restriction of s to the boundary to obtain a connected sum of g copies of $S^1 \times S^2$, denoted $\#_g S^1 \times S^2$. This induces a pairing

$$\langle \ , \ \rangle_0 : \widetilde{RT}_t(H_g) \times \widetilde{RT}_t(H_g) \to \widetilde{RT}_t(\#_g S^1 \times S^2).$$

The manifold $\#_g S^2 \times S^2$ is obtained from S^3 by performing surgery on the trivial link with g components. Identifying $\widetilde{RT}_t(\#_g S^1 \times S^2)$ with $\widetilde{RT}_t(S^3)$ via Sikora's isomorphism as in the proof of Theorem 6.7, we deduce that the pairing $\langle \ , \ \rangle_0$ takes values in $\mathbb{C}$.

The pairing of two basis elements is given by the Reshetikhin-Turaev invariant of a graph like the one in Figure 22. We argue on this figure, but one should keep in mind that there are many different graphs that can be the cores of the same handlebody. By Proposition 6.5 c), in order for this Reshetikhin-Turaev invariant to be nonzero, in each pair of edges linked by a circle colored by $\Omega_{SU(2)}$ the colors must be equal. This is because in order for the tensor product $V^{j_i} \otimes V^{k_i}$ to contain a copy of V^1, the dimensions of the two irreducible representations must be equal. Note also that because we work in S^3, the pairs of edges like the V^{j_3} and V^{k_3} are also linked by a circle colored by $\Omega_{SU(2)}$, namely the circle that links V^{j_4} and V^{k_4}. In general, the edges corresponding to decomposition circles that do not disconnect the surface fall in this category.

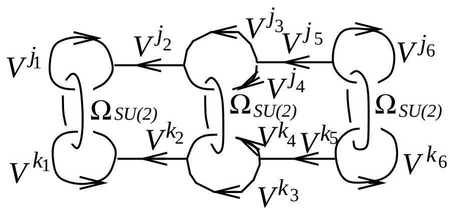

FIGURE 22

Let us examine next the pairs of edges that are not linked by surgery circles, such as those colored by V^{j_2}, V^{k_2} in the figure. In general, the edges that come from decomposition circles that disconnect the surface fall in this category. Rotating the graph by 90° and evaluating the Reshetikhin-Turaev invariant by the rules we obtain a homomorphism $\mathbb{C} = V^1 \to V^{j_2} \otimes V^{k_2}$. This homomorphism is nonzero if and only if $j_2 = k_2$. We conclude that the pairing of two distinct basis elements is zero. On the other hand, computing the pairing of a basis element with itself we can trace a V^1 from the bottom to the top, and the value of the pairing is $\Omega_{SU(2)}(O) = \eta^{-g}$. Hence $\langle \ , \ \rangle_0 = \eta^{-g} \langle \ , \ \rangle$, where $\langle \ , \ \rangle$ is the inner product.

The bilinear pairing $[\ , \ ]$ from § 5.2 is defined by gluing two copies of H_g along an orientation reversing homeomorphism as to obtain S^3. The homeomorphism is of the form $s \circ h$, so $[e_i, e_j] = \langle e_i, \rho(h) e_j \rangle$. Because $\rho(h)$

is an automorphism of the Hilbert space of the quantization, the pairing is nondegenerate. □

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