SOME RESULTS ABOUT THE KAUFFMAN BRACKET SKEIN MODULE OF THE TWIST KNOT EXTERIOR

RĂZVAN GELCA AND FUMIKAZU NAGASATO

Abstract. In this paper, we list in explicit form the factoring relations of the Kauffman bracket skein module (KBSM for short) of a twist knot exterior. This is done using curves decorated by characters of irreducible $SL(2,\mathbb{C})$-representations. In the process, we exhibit a relation which holds in the KBSM of the knot exterior, called the minimal relation. In the final section we prove that, when specializing the variable of the Kauffman bracket at $t=-1$, the minimal relation becomes the defining polynomial of the $SL(2,\mathbb{C})$-character variety of the twist knot.

1. Motivation and background

1.1. **Motivation.** The Kauffman bracket skein module of a 3-manifold was introduced by J. Przyticki as a natural generalization of the Kauffman bracket to general 3-manifolds. It was later linked to the character variety of $SL(2,\mathbb{C})$-representations of the fundamental group of the manifold [B1], [PS]. In this perspective, Kauffman bracket skein modules were used to give an alternate description of the A-polynomial of Cooper, Culler, Gillet, Long, and Shalen, and to generalize it to a noncommutative setting [FGL]. The computation of the noncommutative generalization of the A-polynomial of a knot relies heavily on the good understanding of the skein module of the knot complement.

The noncommutative generalization of the A-polynomial was computed for the unknot in [FGL], trefoil knot in [G1], partially for $(2,2p+1)$-torus knots in [GS1], and for the figure-eight knot in [GS2]. In those papers, and also in [G2] this knot invariant was linked to the Jones polynomial. This relation led to new developments in the study of colored Jones polynomials [GL]. Let us also point out that such computations yield an alternative way of finding the classical A-polynomial, as it was discussed in [N].

The computation itself is done in three stages. The first stage consists of the understanding of the Kauffman bracket skein module of the knot complement. As knot complements can be obtained by attaching 2-handles to a handlebody, and such topological operations yield algebraic factorizations at the level of the skein module, it is necessary to write the factoring relations in *explicit* form. The second stage of the computation is concerned with determining the action of the Kauffman bracket skein algebra of the torus on the skein module of the knot complement, while the third stage is about finding the annihilator of the empty skein.

The present paper describes the first step in the computation of the noncommutative version of the A-polynomial for twist knots. Our convention is to count twists

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as in Figure 1 (which some authors call half-twists). We will also be concerned only with the case of positive twists. The Kauffman bracket skein module of the complement of a twist knot was determined in [BL]. Let us mention that recently T. Le extended this result to all 2-bridge knots [L]. However, none of the above mentioned works gives an explicit description of the factoring relations in terms of the basis of the skein module. This is the purpose of the present paper.

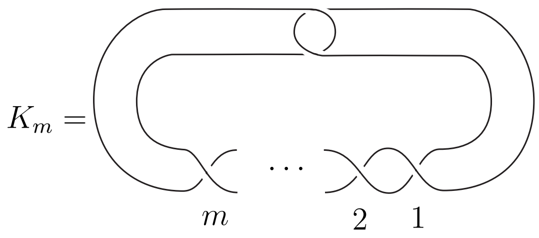


FIGURE 1. The m-twist knot K_m: the twists are ordered from the right to the left

1.2. **The Kauffman bracket skein module of the m-twist knot exterior.** Denote by K_m the m-twist knot ($m \geq 0$) in S^3, by E_{K_m} its exterior (i.e. the complement in S^3 of a regular neighborhood). Bullock and Lofaro have proved the following result.

Theorem 1 (Bullock, Lofaro [BL]). *The Kauffman bracket skein module of the complement of a regular neighborhood of the m-twist knot is the free $\mathbb{C}[t, t^{-1}]$-module with basis $x^i y^j$, $i \geq 0$, $0 \leq j \leq m$, where x and y are the curves described in Figure 2.*

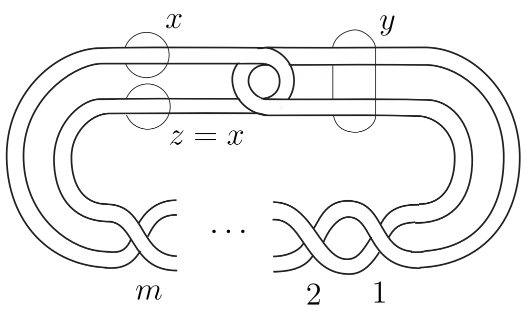


FIGURE 2. Skeins x, y and z in $\mathcal{K}_t(E_{K_m})$

Let us recall briefly the idea of the proof. Because the tunnel number of K_m is 1, the knot exterior E_{K_m} can be obtained by attaching a 2-handle to the genus 2 handlebody H_2.

The proof relies on the following theorems.

Theorem 2 (Przytycki [P2]). *The KBSM of the genus 2 handlebody is the free $\mathbb{C}[t, t^{-1}]$-module with basis $x^i y^j z^k$, $i, j, k \in \mathbb{Z}_{\geq 0}$, where x, y and z are the curves from Figure 3.*

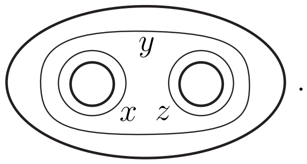


FIGURE 3. Skeins x, y and z in $\mathcal{K}_t(H_2)$

Theorem 3 (Przytycki [P1]).

$$\mathcal{K}_t(E_K) = \mathcal{K}_t(H_2)/J,$$

where J is the submodule of $\mathcal{K}_t(H_2)$ generated by

$$\{L - sl(L) | L : \text{any framed link in } H_2\}.$$

Here $sl(L)$ means the link obtained from L by an (arbitrary) handle slide through the attached 2-handle.

We see immediately that in $\mathcal{K}(E_{K_m})$ the skein z is identified with the skein x as seen in Figure 2 (via a handle slide relation).

The work of Bullock and Lofaro is concentrated on the elimination of the higher powers of y using handle-slides. In our paper we will give *explicit* formulas for these relations. These formulas are necessary for computing the action of the Kauffman bracket skein algebra of the boundary torus, and ultimately they should lead to computation of the noncommutative A-ideal of a twist knot.

2. The explicit form of the factoring relations

It is important to observe that, as it was the case with previously studied knots [G1] [GS1], the characters of irreducible $SL(2,\mathbb{C})$-representations play a special role. For this reason we will change the basis of the skein module of the knot exterior. Recall that $S_n(x)$, $n \geq 0$ are the polynomials defined recursively by $S_0(x) = 1$, $S_1(x) = x$, and $S_{n+1}(x) = xS_n(x) - S_{n-1}(x)$, definition extended for all integers n. These are the Chebyshev polynomials of second type.

The basis of the Kauffman bracket skein module of the twist knot exterior that we prefer is

$$\{S_i(x)S_j(y), \quad 0 \leq i, 0 \leq j \leq m\}.$$

Let us remark that for a curve c, $S_n(c)$ is the curve colored by the nth Jones-Wenzl idempotent.

2.1. The "minimal relation" in the KBSM of the twist knot exterior.

There is a special curve in the handlebody which can be used to obtain all the necessary factorization relations. Sliding this curve through the 2-handle yields a relation which we call *the minimal relation*.

Figure 4 describes the attaching curve. In this figure the handlebody is obtained by drilling through a ball two "tunnels" that twist around each other. A handle slide consists of a band sum with the attaching curve.

We now make an important convention. There is a handle-slide that identifies the curves x and z. We impose this condition in the skein module of the handlebody, i.e. we factor the skein module by the relation $x = z$. Everything below is done in this hypothesis.

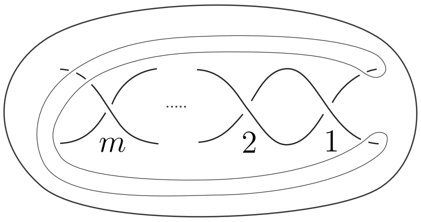
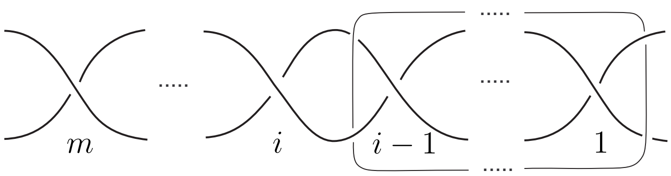


FIGURE 4. Attaching slope of $H_2 \cup$ (2-handle): Thick lines represent holes through a ball. The thin curve is the attaching slope; it lies on the boundary of H_2.

Consider the skeins X_i in $\mathcal{K}_t(H_2)$ defined as follows:

$X_i :=$



.

$$X_0 := (-t^2 - t^{-2})\phi,$$

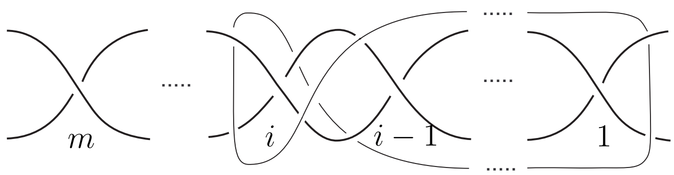
where ϕ denote the empty link. The skein X_{m+1} is the one that gives the minimal relation.

Lemma 1 (Recursive relation of X_i)**.** *The skein X_i, $(m+1 \geq i \geq 0)$, as an element in $\mathcal{K}_t(E_{K_m})$ satisfies the following recursive relation:*

$$X_{i+2} - t^2 y X_{i+1} + t^4 X_i + 2t^2 x^2 = 0, \ X_1 = -t^2 x^2 - t^4 y, \ X_0 = -t^2 - t^{-2}.$$

Proof. All of the operations below are done in the handlebody H_2. First, transform X_{i+1} as follows:

$(-t^{-3})$



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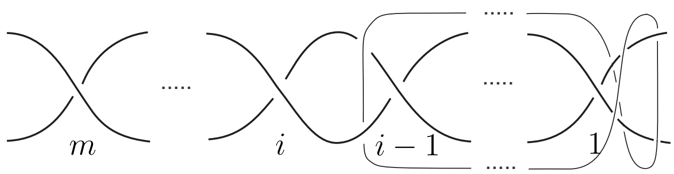
Then slide the kink to the right side as below:

$(-t^{-3})$



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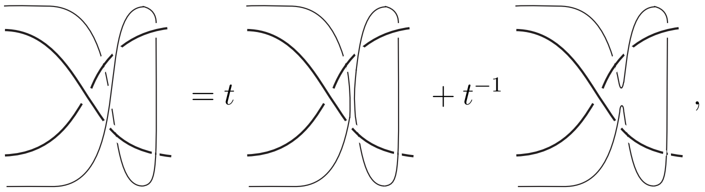
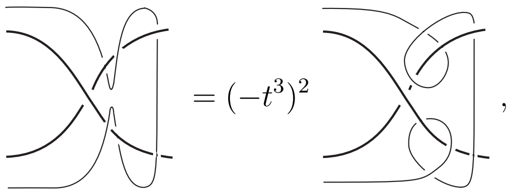
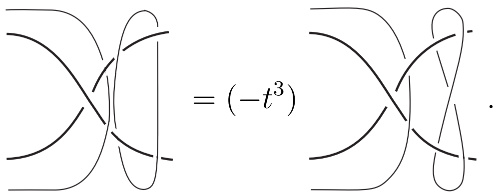
Here resolve the crossing by using the skein relation. For example,







Substituting $z = x$ (which is allowed according to our convention), we can get the recursive relation stated in Lemma 1. $\square$

Lemma 2 (General term)**.** *The skeins X_i, $(0 \leq i \leq m+1)$ can be written in terms of the basis of the skein module as*

$$X_i = -t^{2(i+1)} S_i(y) - t^{2i} S_{i-1}(y) x^2 + t^{2(i-1)} S_{i-2}(y) - 2t^{2(i-1)} x^2 \sum_{n=0}^{i-2} t^{2n} S_n(y)$$

$$= -t^{2(i+1)} S_i(y) - t^{2i} S_{i-1}(y) x^2 + t^{2(i-1)} S_{i-2}(y) - 2x^2 \frac{t^{2(i-1)} S_{i-1}(y) - t^{2i} S_{i-2}(y) - 1}{y - t^2 - t^{-2}},$$

There is no actual fraction in the second expression, as a polynomial in y, the numerator is divisible by the denominator. It is a matter of convenience to write it this way.

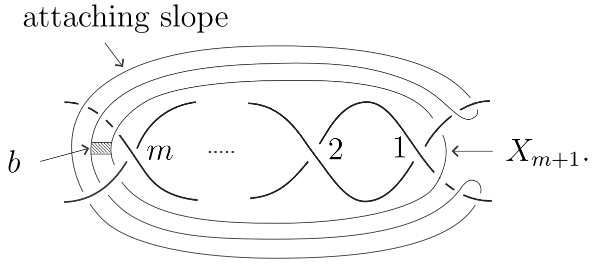
Proof. According to Lemma 1 the X_i's satisfy a second order nonhomogeneous recursive relation with constant coefficients. Set $a_i = t^{-2i} X_i$, so that the homogeneous part of the recursive relation looks like $a_{i+1} - y a_i + a_{i-1} = 0$, which is that of the Chebyshev polynomials. The general term is of the form $\alpha S_i(y) + \beta S_{i-1}(y)$, since $S_i(y)$ and $S_{i-1}(y)$ form a basis for the (2-dimensional) space of sequences satisfying this recursive relation.

On the other hand, the nonhomogeneous term is constant, so we can use the method of undetermined coefficients to find a particular solution, which is $-2x^2/(y - t^2 - t^{-2})$ (just formally). The coefficients α and β are determined from the initial condition. Initially they appear as fractions, but a routine computation with Chebyshev polynomials produce the formulas from the statement. $\square$

Lemma 3 (Handle-slide relation)**.** *The handle-slide of X_m yields the following relation*

$$X_{m+1} + t^{-4} X_m + t^{-2} x^2 = 0.$$

Proof. All of the operations below are done in the handlebody H_2 (modulo $x = z$). First, consider the following band sum of X_{m+1} and the attaching slope along a band b:



Let $sl_b(X_{m+1})$ be the resulting knot after the band sum. Then a relation $sl_b(X_{m+1}) - X_{m+1} = 0$ holds in $\mathcal{K}_t(E_{K_m})$. Resolving $sl_b(X_{m+1})$ and substituting $z = x$, we get,

$$sl_b(X_{m+1}) = -t^{-4}X_m - t^{-2}x^2.$$

This completes the proof. $\square$

Lemmas 2 and 3 give us the following theorem.

Theorem 4 (Minimal relation)**.** *Let*

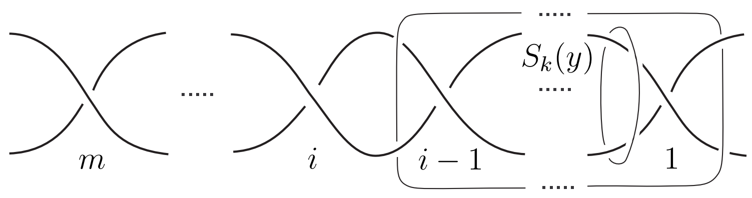
$$\begin{aligned}R_m(t) :=& S_{m+1}(y) + (t^{-6} - t^{-2}x^2)S_m(y) + ((2t^4 + t^{-8})x^2 - t^{-4})S_{m-1}(y) - t^{-10}S_{m-2}(y)\\ &+ 2x^2(t^{-2m-2} + t^{-2m-6})\sum_{i=0}^{m-2} t^{2i}S_i(y) - t^{-2m-6}x^2.\end{aligned}$$

Then in the complement of the m-twist knot $R_m(t) = 0$.

The equation $R_m(t) = 0$ is called the minimal relation.

2.2. **The other factorization relations (The action of $S_k(y)$ on X_i).** We now describe the relations that reduce $S_{m+k}(y)$, $k \geq 2$ in terms of polynomials of lower degree.

We concentrate on the skeins $X_i * S_k(y)$, $k \geq 0, 0 \leq i \leq m+1$, defined as

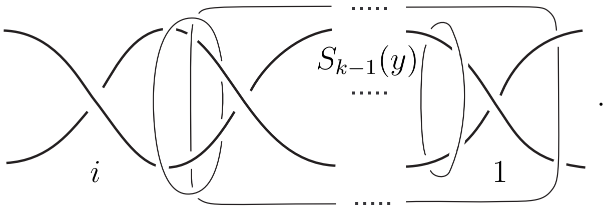
$X_i * S_k(y) :=$  ,

$$X_0 * S_k(y) := S_k(y)X_0.$$

Lemma 4. *For $k \geq 0$ and $i = 1, 2, \ldots, m+1$, as an element of $\mathcal{K}_t(E_{K_m})$, $X_i * S_k(y)$ satisfies the following relation:*

$$X_i * S_k(y) = t^4y(X_i * S_{k-1}(y)) + (-t^6 + t^{-2})(X_{i-1} * S_{k-1}(y)) - X_i * S_{k-2}(y) + 2(-t^4 + 1)x^2S_{k-1}(y).$$

Proof. First, consider the following configuration of $X_i * (yS_{k-1}(y))$.



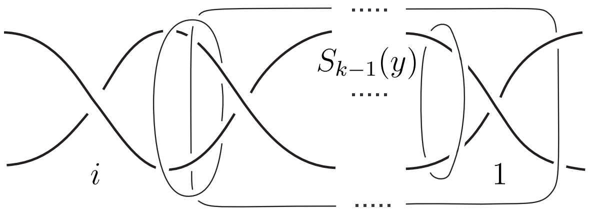
.

Calculate $X_i * (yS_{k-1}(y))$ as $(X_i * y) * S_{k-1}(y)$. Resolving $X_i * y$ we can reduce $(X_i * y) * S_{k-1}(y)$ to

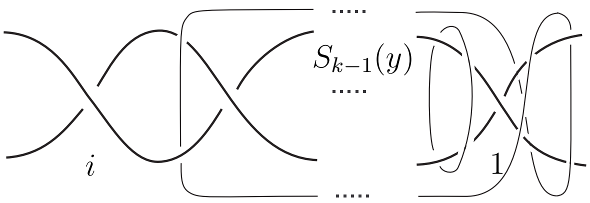
$$\{t^4 y X_i + (-t^6 + t^{-2})X_{i-1} + 2(-t^4 + 1)x^2\} * S_{k-1}(y).$$

Indeed,

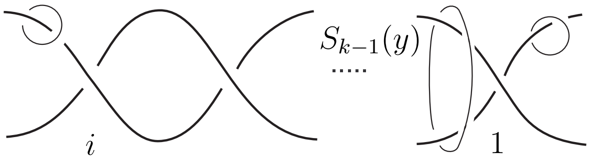
(1) $X_i * S_k(y) \ =$



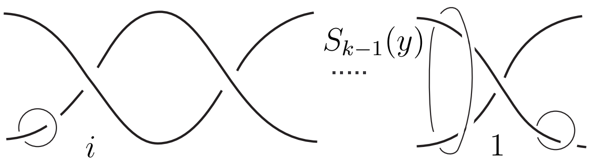
(2) $= \ -t^{-1}$



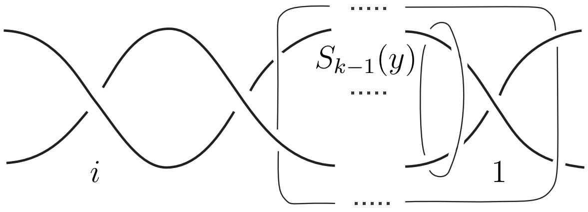
(3) $+$



(4) $+$



(5) $+ \ t^{-2}$



.

Note that we can calculate $(X_i * y) * S_{k-1}(y)$ separately as above, because there is no interaction between $X_i * y$ and $S_{k-1}(y)$. The term (2) can be calculated via the resolutions performed in the proof of Lemma 1 and is

$$t^4 y X_i * S_{k-1}(y) - t^6 X_{i-1} * S_{k-1}(y) - 2t^4 x^2 S_{k-1}(y).$$

Using the relation $yS_{k-1}(y) = S_k(y) + S_{k-2}(y)$, we obtain the desired recursive relation. □

As a consequence, we have

Lemma 5 (General term)**.**

$$\begin{aligned} t^{-2i} X_i * S_k(y) =& (y - t^2 - t^{-2})^{-1} \times [-t^{4k+2} S_{n+k+1}(y) + t^{-4k-2} S_{n-k-3}(y) \\ &+ t^{4k}(t^4 - S_2(x)) S_{n+k}(y) + t^{-4k}(-t^{-4} + S_2(x)) S_{n-k-2}(y) \\ &+ t^{4k-2}(-t^{-4} + (t^4-1) S_2(x)) S_{n+k-1}(y) \\ &+ t^{-4k+2}(t^4 - (t^{-4}-1) S_2(x)) S_{n-k}(y) \\ &+ 2x^2 t^{-2n} S_k(y) + \sum_{i=-k+1}^{k-1} t^{4i} \varphi_{i-k}(x) S_{n+i}(y)], \end{aligned}$$

where

$$\varphi_j(x) = \begin{cases} (t^4 - t^{-4}) S_2(x), & \text{if } j \text{ is odd} \\ t^6 - t^{-6} + (t^2 - t^{-2}) S_2(x), & \text{if } j \text{ is even} \end{cases}$$

Lemma 6 (Handle-slide relation)**.**

$$X_{m+1} * S_k(y) + t^{-4} X_m * S_k(y) + t^{-2} x^2 S_k(y) = 0.$$

Proof. The relation is obtained by noting that there is no interaction between the knots $sl_b(X_{m+1})$, defined in the proof of Lemma 3 and the skein $S_k(y)$ □

Combining the general term in Lemma 5 and the handle slide relation in Lemma 6, we obtain all the relations reducing $S_k(y)$, where $k \geq m+1$, to a skein with lower degree.

Theorem 5. *For* $k \geq 0$,

$$\begin{aligned} &-t^{4k+4} S_{m+k+2}(y) - t^{4k+2}(t^{-4} - t^4 + S_2(x)) S_{m+k+1}(y) \\ &-t^{4k}(t^{-4} - 1 + (t^{-4} + 1 - t^4) S_2(x)) S_{m+k}(y) \\ &-t^{4k-2}(t^{-8} + (2t^{-4} - 1 - t^4) S_2(x)) S_{m+k-1}(y) \\ &+t^{-4k}(t^8 + (2t^4 - 1 - t^{-4}) S_2(x)) S_{m-k}(y) \\ &+t^{-4k-2}(t^4 - 1 + (t^4 + 1 - t^{-4}) S_2(x)) S_{m-k-1}(y) \\ &+t^{-4k-4}(t^4 - t^{-4} + S_2(x)) S_{m-k-2}(y) + t^{-4k-6} S_{m-k-3}(y) \\ &+t^{-2m-2} x^2 S_{k+1}(y) + t^{-2m-2}(t^2 - t^{-2}) x^2 S_k(y) + t^{-2m-2} x^2 S_{k-1(y)} \\ &+t^2 \sum_{i=-k+1}^{k-2} \psi_{i-k}(x) t^{4i} S_{n+i}(y) = 0, \end{aligned}$$

where

$$\psi_j(x) = \begin{cases} t^4 - t^{-8} + (-2t^{-4} + t + t^4) S_2(x), & \text{if } j \text{ is odd} \\ t^6 - t^{-6} + (2t^2 - t^{-2} - t^{-6}) S_2(x), & \text{if } j \text{ is even.} \end{cases}$$

3. Application to the character variety of the m-twist knot

Recall the following theorem.

Theorem 6 (Bullock [B1], Przytycki-Sikora [PS])**.** *For any compact orientable 3-manifold M, there exists a surjective homomorphism Φ as algebra*

$$\Phi : \mathcal{K}_{-1}(M) \to \chi(\pi_1(M)),$$

defined by $\Phi(K) := -t_{[K]}$, $\Phi(K_1 \sqcup \cdots \sqcup K_i) := \prod_{j=1}^{i} \Phi(K_i)$, where $[K]$ is an element of $\pi_1(M)$ represented by the knot K with an unspecified orientation. Moreover the kernel of Φ is the nilradical $\sqrt{0}$.

According to the above theorem, $R_m(-1)$ has information of the defining polynomial of the character variety $X(\pi_1(E_{K_m}))$, that is.

$$\chi(\pi_1(E_{K_m})) = \mathcal{K}_{-1}(E_{K_m})/\sqrt{0} = \mathbb{C}[x,y]/\sqrt{\langle R_m(-1)\rangle}.$$

Note that taking the radical of a principal ideal corresponds to getting rid of the multiplicity of each irreducible factor in the irreducible decomposition of the generator. Hence the generator of $\sqrt{\langle R_m(-1)\rangle}$, which ideal is also principal, has the same zeros as $R_m(-1)$. In this sense we can consider the polynomial $R_m(-1)$ as the defining polynomial of $X(E_{K_m})$. Using Maple, we can find the following factorizations of $R_m(-1)$ over $\mathbb{Q}$:

$R_0(-1) = y + 2,$
$R_1(-1) = (y + 2)(y + x^2 - 1),$
$R_2(-1) = (y + 2)(y^2 + x^2y - y + x^2 - 1),$
$R_3(-1) = (y + 2)(y^3 + x^2y^2 - y^2 - 2y + x^2y + 1),$
$R_4(-1) = (y + 2)(y^4 + x^2y^3 - y^3 - 3y^2 + x^2y^2 - x^2y + 2y + 1),$
$R_5(-1) = (y + 2)(y^5 + x^2y^4 - y^4 - 4y^3 + x^2y^3 - 2x^2y^2 + 3y^2 + 3y - x^2y + x^2 - 1).$

A more general fact is true.

Lemma 7. *For any non-negative integer m, the minimal relation $R_m(-1)$ has the following decomposition:*

$$(y+2)\left(S_m(y) - S_{m-1}(y) + x^2\sum_{i=0}^{m-1} S_i(y)\right).$$

Moreover, the factor $S_m(y) - S_{m-1}(y) + x^2\sum_{i=0}^{m-1} S_i(y)$ is irreducible over $\mathbb{Q}$.

Proof. The first statement can be shown by using the properties of the Chebyshev polynomial S_m. The second statement can be proved by a result on the trace field shown by J. Hoste and P. Shanahan [HS]. Let us introduce the notation:

$$\widetilde{R}_m(x,y) := S_m(y) - S_{m-1}(y) + x^2\sum_{i=0}^{m-1} S_i(y).$$

In the case of $m = 0, 1$, (that is the case of the unknot and the right-handed trefoil which are non-hyperbolic knots), it was observed above. Next, consider the case

where $m \geq 2$. Then the twist knot K_m is hyperbolic. Hence there exists the discrete faithful representation
$$\rho_0 : \pi_1(E_{K_m}) \to SL(2, \mathbb{C})$$
of $\pi_1(E_{K_m})$. By Theorem 6, we can regard the skeins x and y as the functions $-t_x$ and $-t_y$, respectively. Here x is a meridional skein so we can assume that
$$x(\rho_0) = -t_x(\rho_0) = \pm 2, \ y(\rho_0) = -t_y(\rho_0) = -2 \text{ or } \alpha,$$
where α is a solution of $\widetilde{R}_m(\pm 2, y) = 0$ over $\mathbb{C}$.

Remark 1. *The diagram in Figure* 1 *is alternating and irreducible, so the minimal crossing number of K_m is exactly $m + 2$ (see* [K, M, T] *for details).*

Now, by Corollary 1 in [HS] and Remark 1, the extension field $\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m}))$ over $\mathbb{Q}$, called the trace field of K_m, has degree m. Namely,
$$[\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m})) : \mathbb{Q}] = m.$$
Here it can be shown that $\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m}))$ is simple extension, that is,
$$\mathbb{Q}(t_\gamma(\rho_0) : \gamma \in \pi_1(E_{K_m})) = \mathbb{Q}(\alpha).$$
(Refer to [CR, NR] for details). If $y = -2$, then $\mathbb{Q}(-2) = \mathbb{Q}$, a contradiction. Therefore y should be equal to α, which is a solution of $\widetilde{R}_m(\pm 2, y)$ with $\deg_y = m$. Hence $\widetilde{R}_m(\pm 2, y)$ must be irreducible over $\mathbb{Q}$. It is not so hard to see that if $\widetilde{R}_m(x, y)$ is reducible, then so is $\widetilde{R}_m(\pm 2, y)$. These two facts complete the proof. $\square$

It follows from the definition of the character variety that $X(M)$ for any compact orientable 3-manifold M is defined over $\mathbb{Q}$ (in fact over $\mathbb{Z}$). Hence by Lemma 7, we obtain:

Theorem 7. *Consider the character ring $\chi^{\mathbb{Q}}(\pi_1(E_{K_m}))$ and the KBSA $\mathcal{K}^{\mathbb{Q}}_{-1}(E_{K_m})$ whose coefficient fields are $\mathbb{Q}$. Then $\mathcal{K}^{\mathbb{Q}}_{-1}(E_{K_m})$ has trivial nilradical. Therefore the following holds:*
$$\chi^{\mathbb{Q}}(\pi_1(E_{K_m})) = \mathbb{Q}[x, y]/\langle R_m(-1)\rangle.$$

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