LOOK AT THE EXTEMES!

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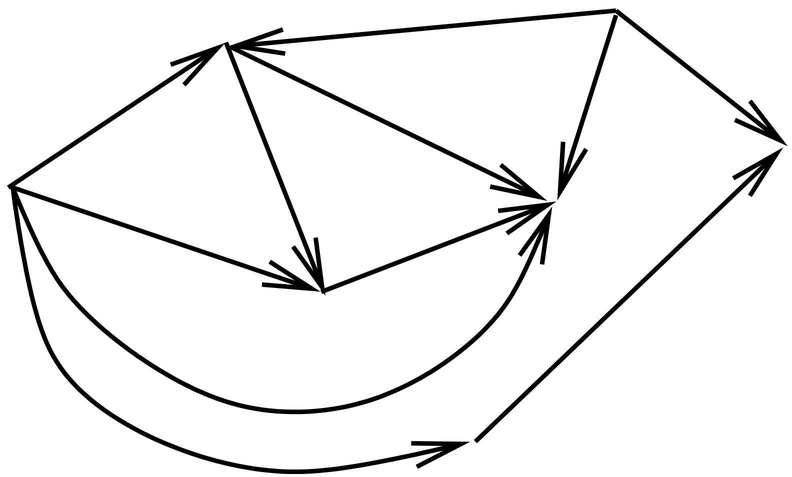
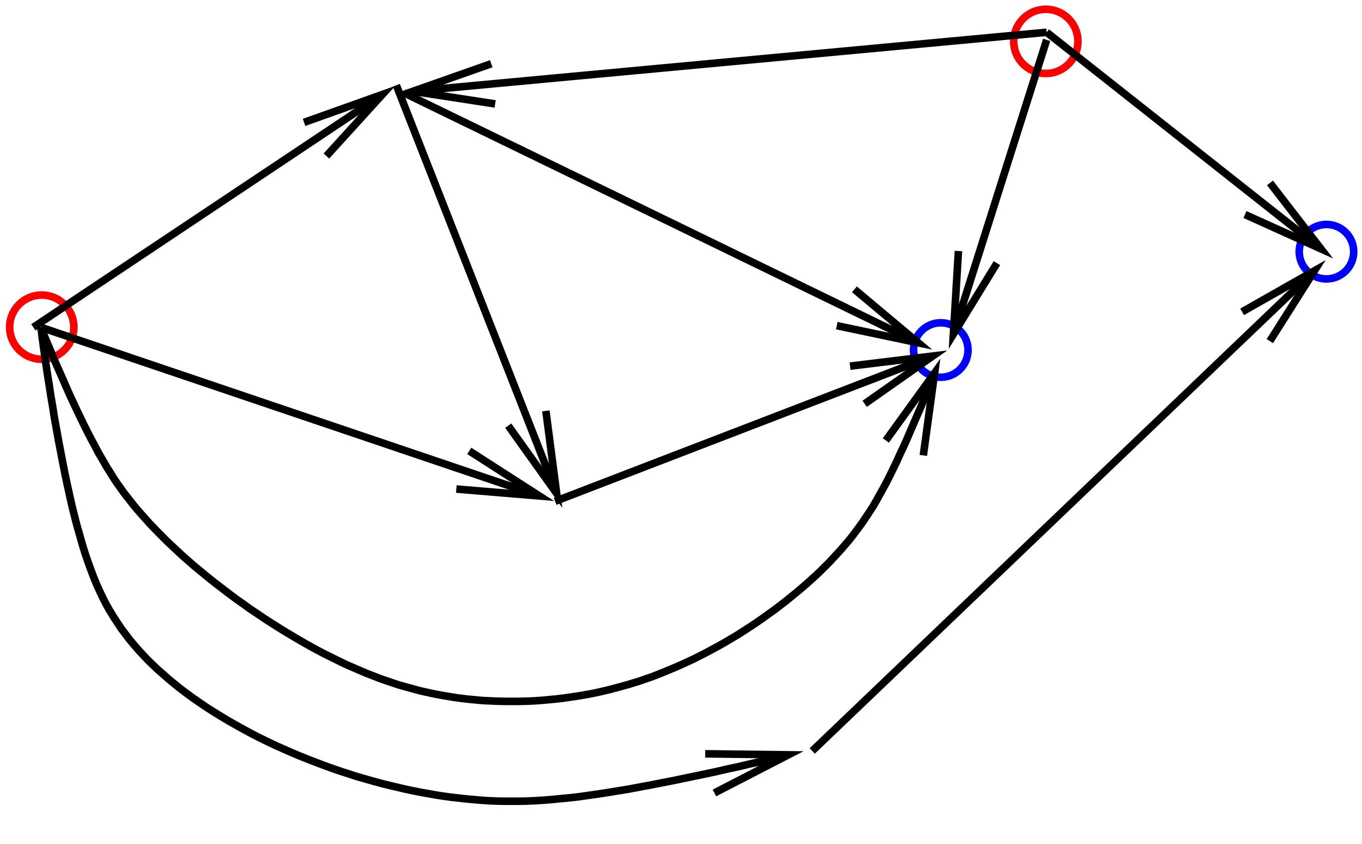
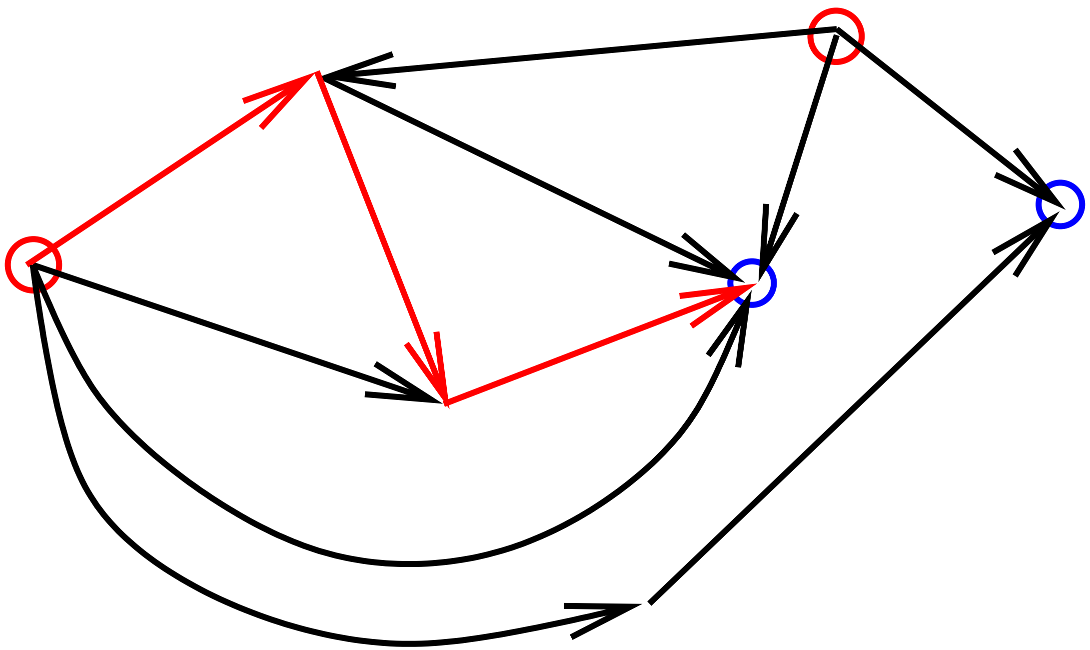
Problem 1: *In some country all roads between cities are one-way so that once you leave a city you cannot return to it anymore. Prove that there exists a city into which all roads enter and a city from which all roads exit.*

(Kvant)





Pick an itinerary that travels through a maximal number of cities (more than one such itinerary may exist). No roads enter the starting point of the itinerary, while no roads exit the endpoint.

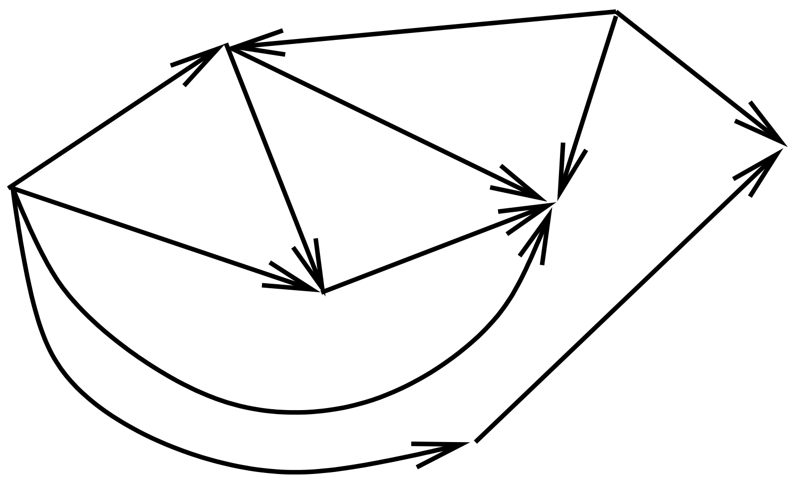


A more advanced solution: *Define a partial order of the cities, saying that $A \le B$ if one can reach B from A. It satisfies the following properties*

a. $A \le A$,

b. if $A \le B$ and $B \le A$ then $A = B$,

c. if $A \le B$ and $B \le C$ then $A \le C$.

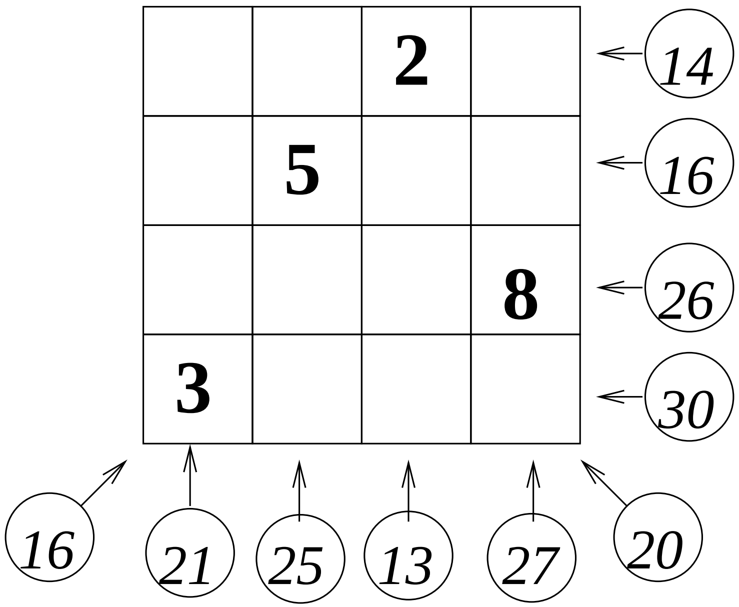


Theorem. *A partial order on a finite set has maximal and minimal elements.*

In a maximal city all roads enter, and from a minimal city all roads exit.

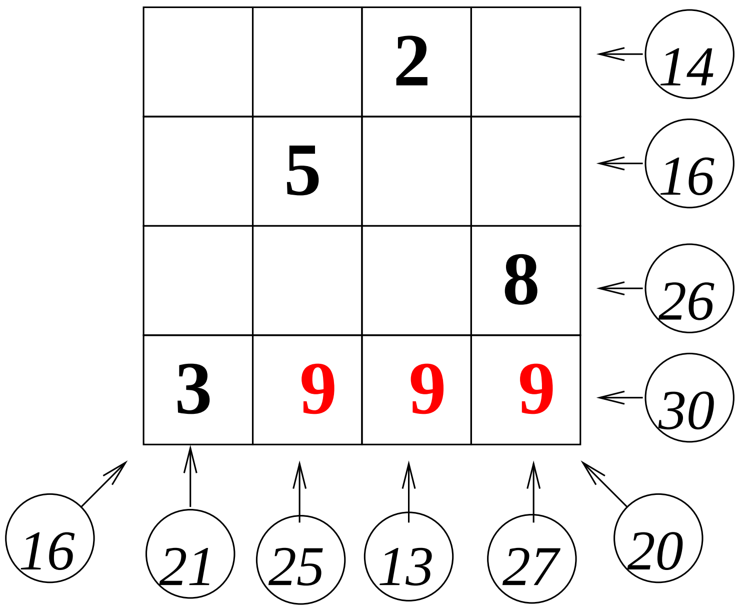
Problem 2. *Complete the following table with integers between* 1 *and* 9 *in such a way that the sum of the numbers on each row, column and diagonal is as indicated.*

(San Francisco Chronicle)



Solution: *Call the elements of the* 4×4 *tableau* a_{ij}, $i, j = 1, 2, 3, 4$, *according to their location. As such* $a_{13} = 2$, $a_{22} = 5$, $a_{34} = 8$ *and* $a_{41} = 3$.

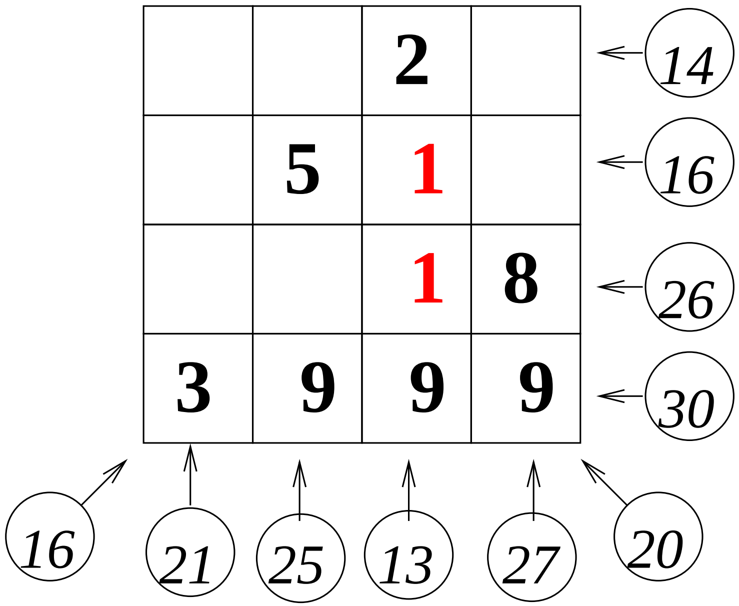
Look first at the row with the largest sum, namely the fourth. The unknown entries sum up to 27, *hence all three of them* a_{42}, a_{43} *and* a_{44} *must equal* 9.



Now we consider the column with smallest sum. It is the third, with

$$a_{13} + a_{23} + a_{33} + a_{43} = 2 + a_{23} + a_{33} + 9 = 13.$$

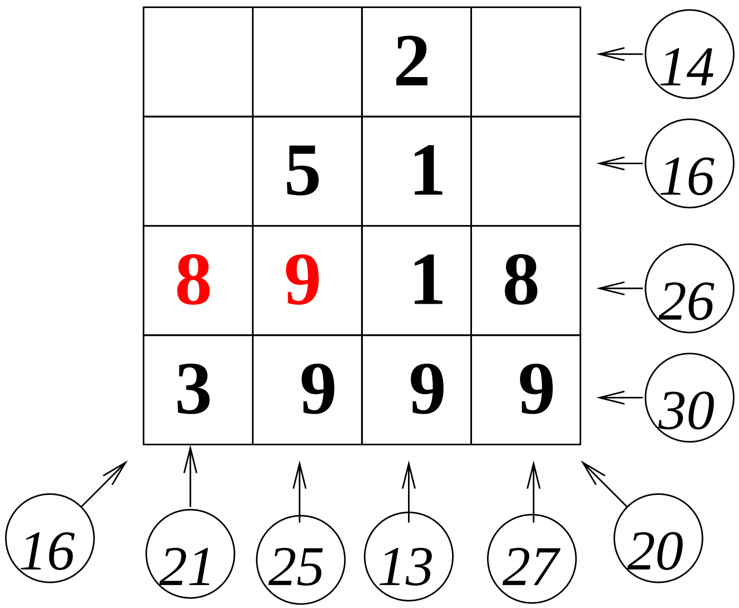
We see that $a_{23} + a_{33} = 2$, *therefore* $a_{23} = a_{33} = 1$.



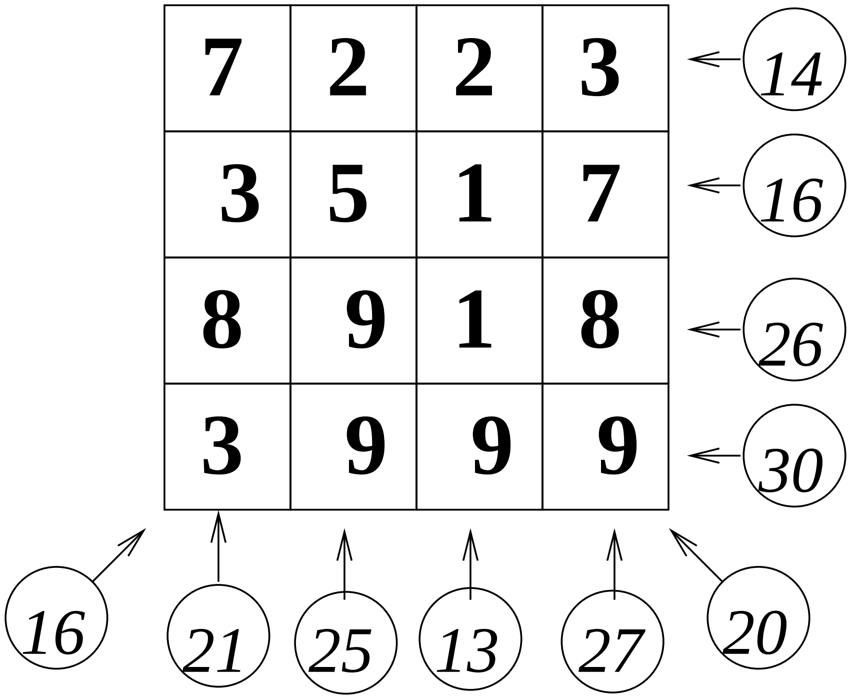
We than have

$$a_{31} + a_{32} + a_{33} + a_{34} = a_{31} + a_{32} + 1 + 8 = 26.$$

Therefore $a_{31} + a_{32} = 17$, *which can only happen if one of them is* 8 *and the other is* 9. *Checking separately the two cases we see that only* $a_{31} = 8$, $a_{32} = 9$ *yields a solution.*



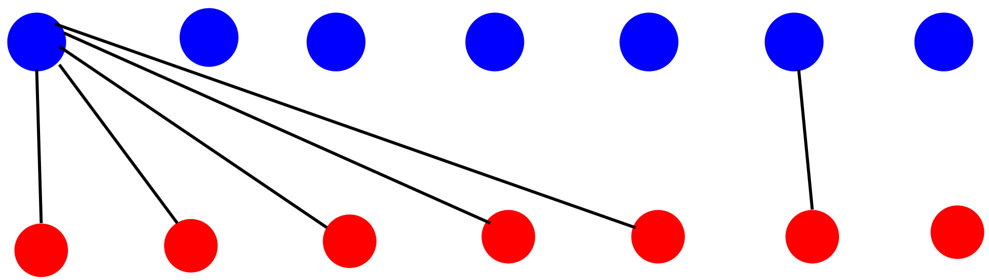
Here is the solution:



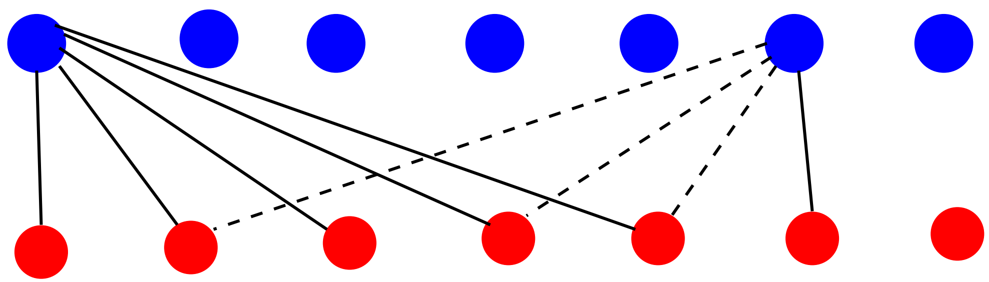
Problem 3. *At a party assume that no boy dances with all the girls, but each girl dances with at least one boy. Prove that there are two couples* gb *and* $g'b'$ *which dance, whereas* b *does not dance with* g'*, nor does* g *dance with* b'*.*

(W.L. Putnam Mathematical Competition)

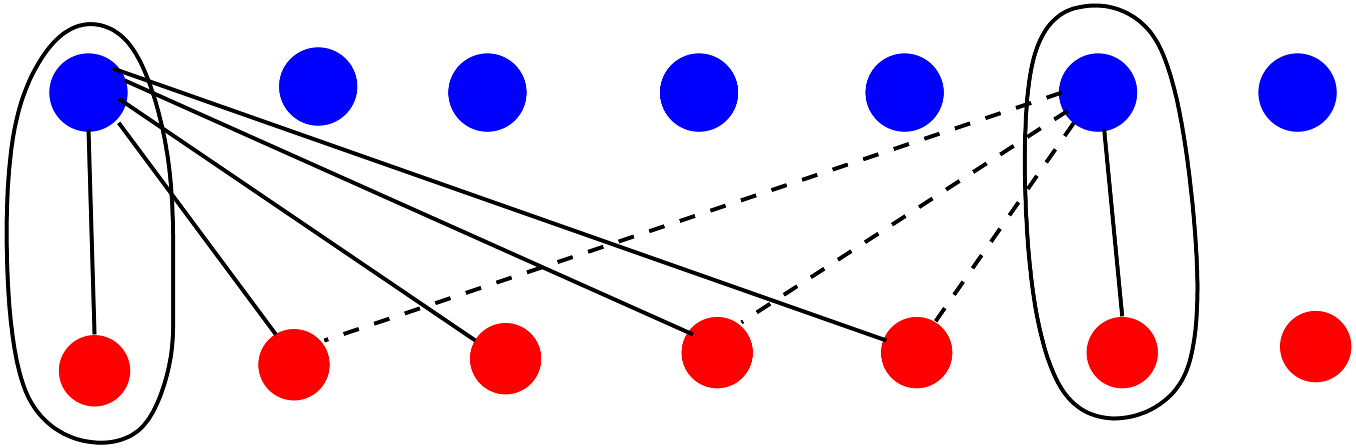
Solution: *Let b be a boy dancing with the largest number of girls. There is a girl g' he does not dance with. Choose as b' a boy who dances with g'.*



Let g be a girl who dances with b but not with b'. Such a girl exists because of the maximality of b, since b' already dances with a girl who does not dance with b.

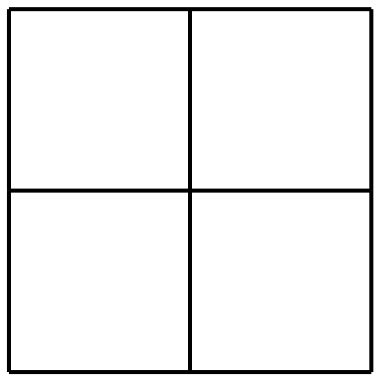


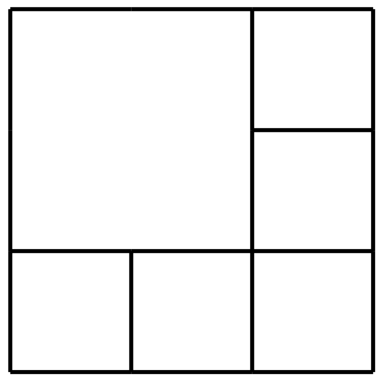
The pairs (b, g), (b', g') *satisfy the requirement of the problem.*



Problem 4. *Show that a square cannot be dissected into* 5 *squares.*

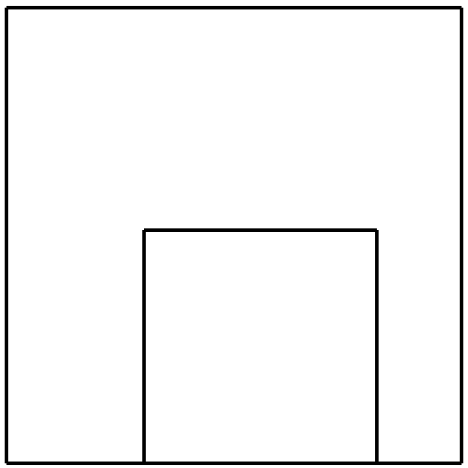
You can dissect it into 4 *or* 6 *squares!*



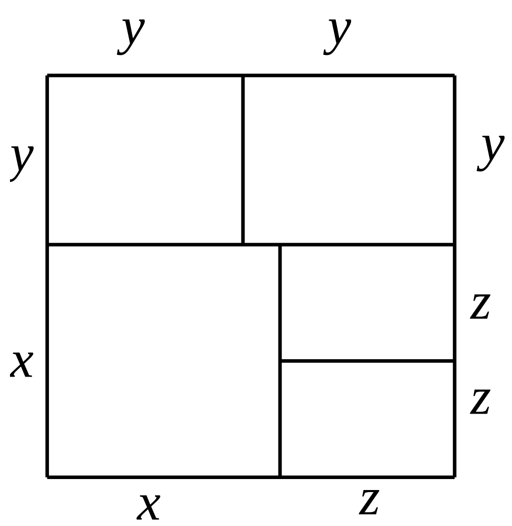


Solution: *Look at the largest square of the dissection!*

If it is not in one of the corners, then there are at least three squares that touch it. We end up with toooooo many squares.



If largest square is in the lower left corner, there are 2 more touching it, one above and one to the right. If both are smaller, then



This yields the system of equations

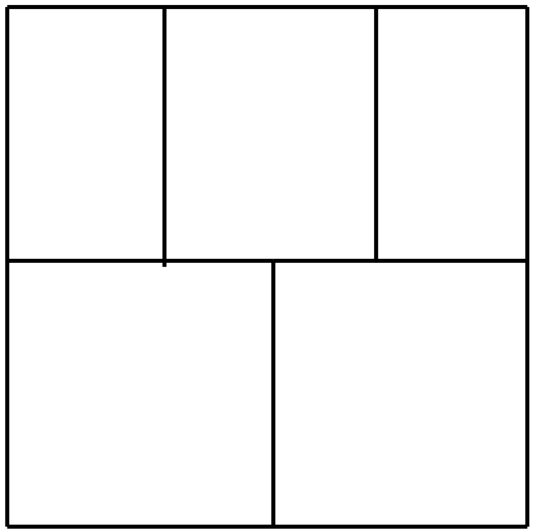
$$x + y = 2z + y$$

$$x + z = 2y$$

$$x + y = x + z$$

which has the unique solution $x = y = z = 0$.

If there are two big squares next to each other, it cannot happen that there is a third big square on top of the first. Then on top of them there should be at least three squares, to cover the entire length. The only possible configuration is



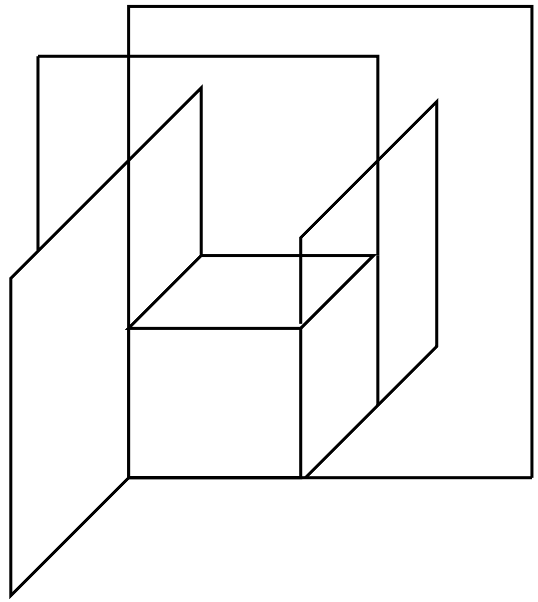
OOPS! This does not work. Can you see why?

Problem 5. *Prove that it is impossible to dissect a cube into finitely many cubes, no two of which having the same size.*

Solution: *For the solution, assume that a dissection exists, and look at the bottom face. It is cut into squares. Take the smallest of these squares, and look at the cube that lies right above it!*

This square does not touch the edges.

The smallest cube is surrounded, on all four sides, by bigger cubes, so its upper face must again be dissected into squares by the cubes that lie on top of it.

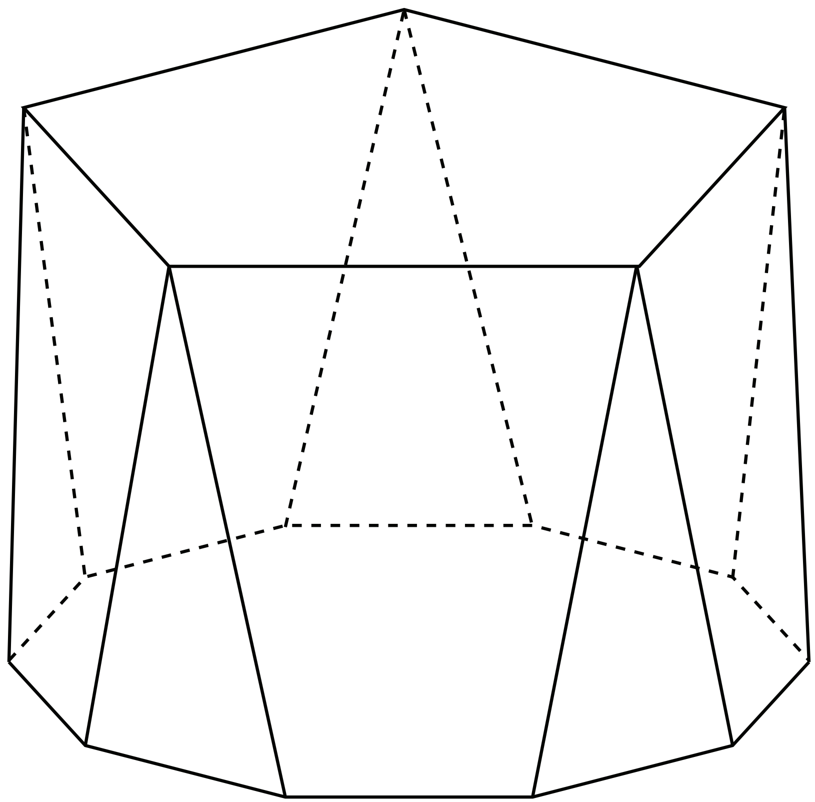


Take again the smallest of these cubes and repeat the argument.

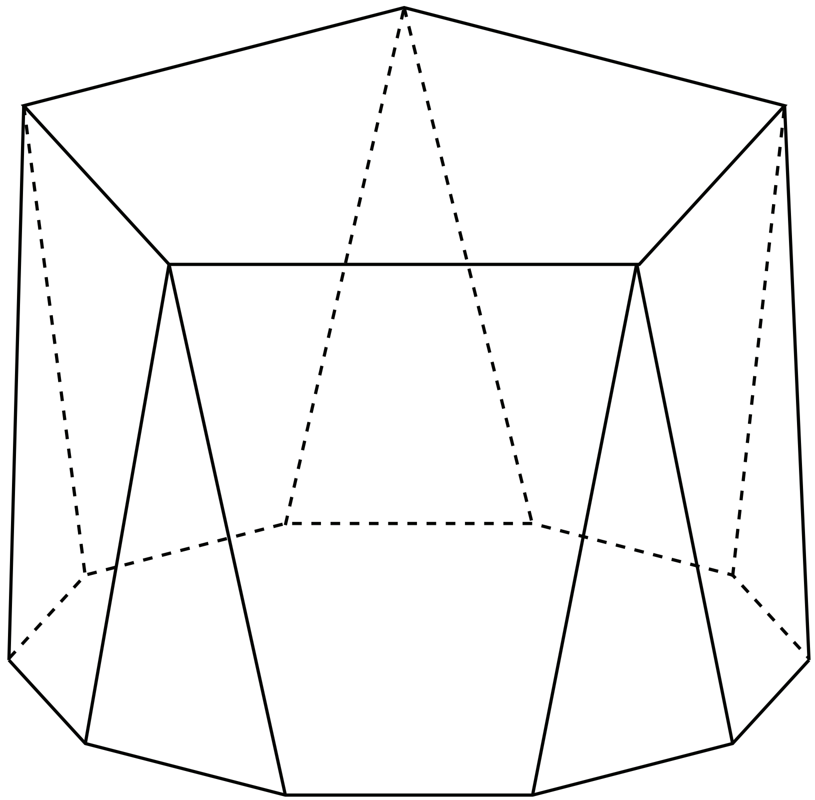
This process never stops, as the cubes that lie on top of one of these little cubes cannot end up touching simultaneously the upper face of the original cube. We contradicted therefore the finiteness of the decomposition. Hence the conclusion.

Problem 6. *Show that any convex polyhedron has two faces with the same number of edges.*

(Moscow Mathematical Olympiad)

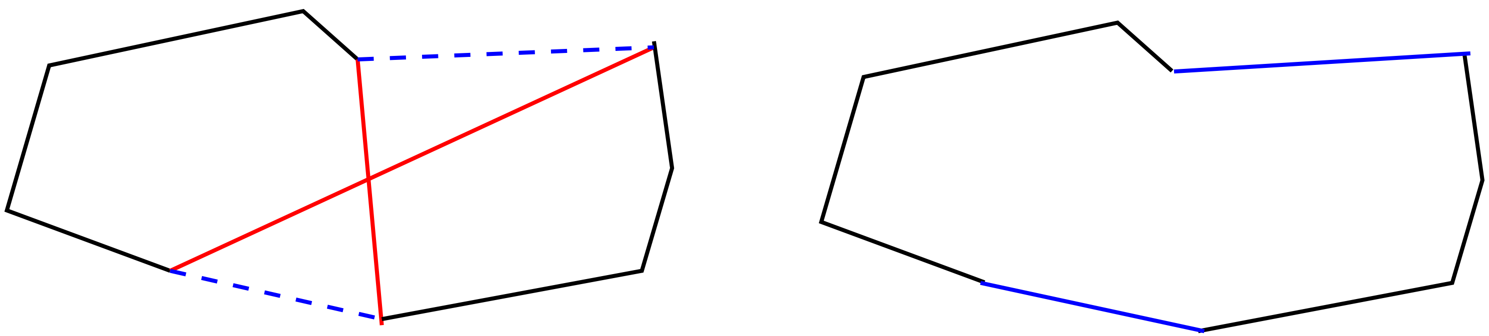


Solution: *Choose a face with maximal number of edges, and let n be this number. The number of edges of each of the n adjacent faces ranges between 3 and n, so, by the pigeonhole principle, two of these faces have the same number of edges.*



Problem 7. *Given n points in the plane, no three of which are collinear, show that there exists a closed polygonal line with no self-intersections having these points as vertices.*

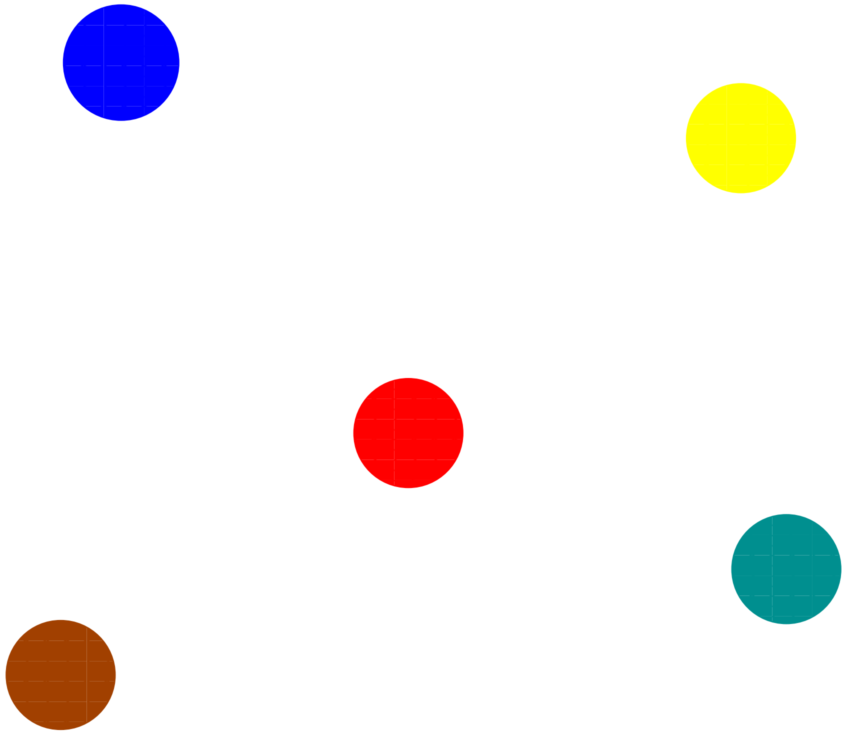
Solution: *There are only finitely many polygonal lines with these points as vertices. Choose the shortest. Because of the triangle inequality it cannot have a crossing.*



Theorem. In a triangle the sum of two sides is greater than the third.

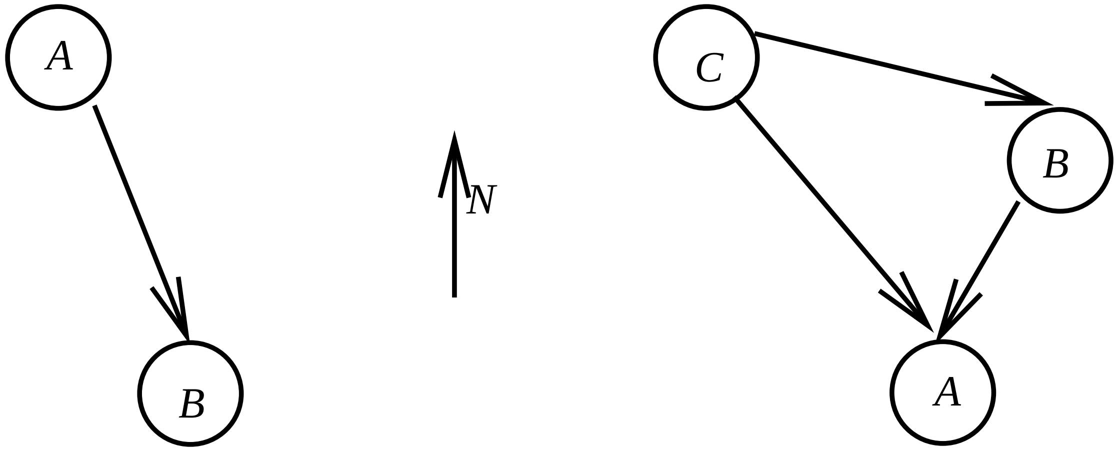
Problem 8. *Given is a finite set of spherical planets, all of the same radius and no two intersecting. On the surface of each planet consider the set of points not visible from other planets. Prove that the total area of these sets is equal to the surface area of one planet.*

(proposed for the IMO by Soviet Union)



Solution: *Choose a preferential direction in space, which defines the North Pole of each planet. Assume that the direction is not perpendicular to the axis of any two planets.*

Next, define an *order* *on the set of planets by saying that planet* $A \geq B$, *when removing all other planets from space, the North Pole of* B *is visible from* A.



1. $A \geq A$,

2. *if* $A \geq B$ *and* $B \geq A$ *then* $A = B$,

3. *if* $A \geq B$ *and* $B \geq C$ *then* $A \geq C$,

4. *either* $A \geq B$ *or* $B \geq A$.

This order has a unique maximal element M. This is the only planet whose North Pole is not visible from another.

Now consider a sphere of the same radius as the planets. Remove from it all North Poles defined by directions that are perpendicular to the axis of two of the planets. This is a set of area zero.

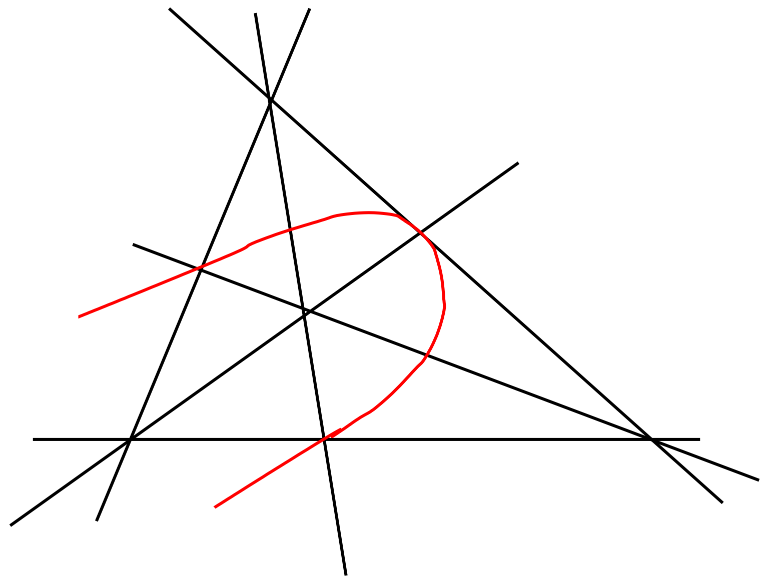
For every other point on this sphere, there exists a direction in space that makes it the North Pole, and for that direction, there exists a unique North Pole on one of the planets which is not visible from the others. Hence the total area of invisible points is equal to the area of this sphere, which in turn is the area of one of the planets.

Four famous problems!

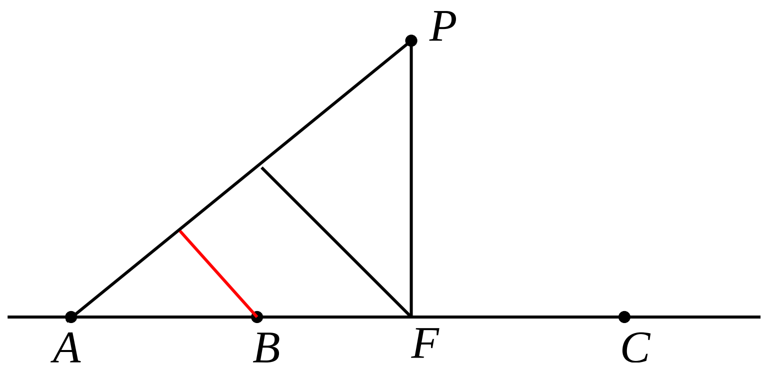
Problem 9. *A finite set of points in the plane has the property that any line through two of them passes through a third. Show that all the points lie on a line.*

(J.J. Sylvester)

It almost works:



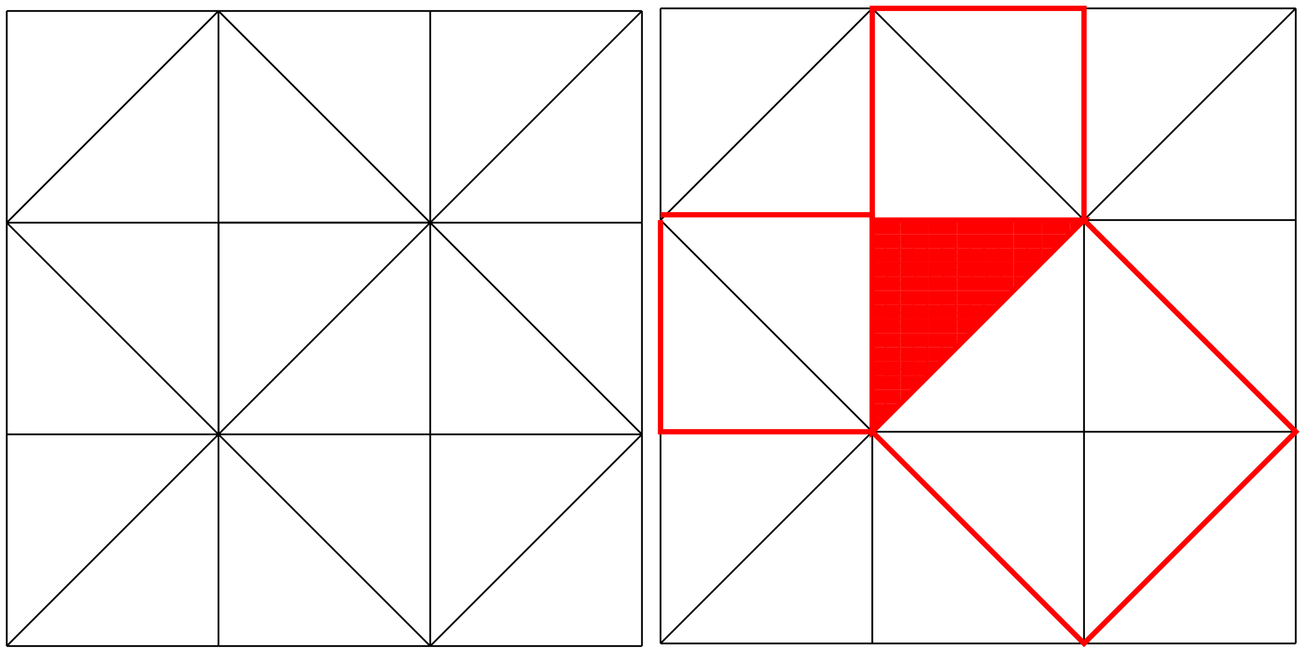
Solution: *(L.M.Kelly) Suppose the points are not collinear. Among the pairs* (P, L) *consisting of a line* L *and a point* P *not on that line, choose one which minimizes the distance* d *from* P *to* L*. Let* F *be the foot of the perpendicular from* P *to* L*.*

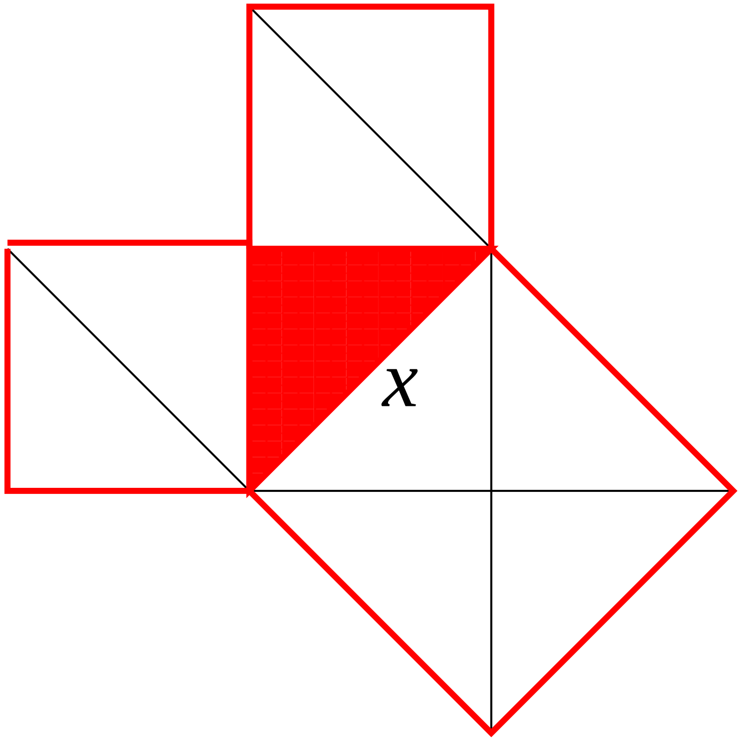


There are at least three points A, B, C *on* L*. Two of these, say* A *and* B*, are on the same side of* F*. Let* B *be nearer to* F *than* A*. Then the distance from* B *to the line* AP *is less than* d*. Contradiction! The conclusion follows.*

Problem 10. *Show that $\sqrt{2}$ is not the ratio of two positive integers.*

(Pythagoras)





$$x^2 = 1^2 + 1^2$$

Solution: *Assume that*

$$\sqrt{2} = \frac{m}{n}$$

where m *and* n *are two positive integers. Then* $n\sqrt{2} = m$ *is an integer. The set* S *of positive integers* n *for which* $n\sqrt{2}$ *is an integer has a smalest element* k.

Look at the number $(\sqrt{2} - 1)k$.

Since $(\sqrt{2} - 1)k = k\sqrt{2} - k$, *it follows that* $(\sqrt{2} - 1)k$ *is an integer. Is it in* S*?*

We compute

$$(\sqrt{2}-1)k\sqrt{2} = 2k - k\sqrt{2}.$$

This is an integer. Hence $(\sqrt{2}-1)k$ *is in* S*! But*

$$(\sqrt{2}-1)k \approx 0.41k < k,$$

contradicting the assumption that k *is the smallest element of* S*. This contradiction proves that our original assumption is false. Hence* $\sqrt{2}$ *is irrational.*

Problem 11. *Given a point inside a convex polyhedron, show that there exists a face of the polyhedron such that the projection of the point onto the plane of that face lies inside the face.*

Physical solution:

Construct the polyhedron out of an inhomogeneous material in such a way that the given point is its center of mass.

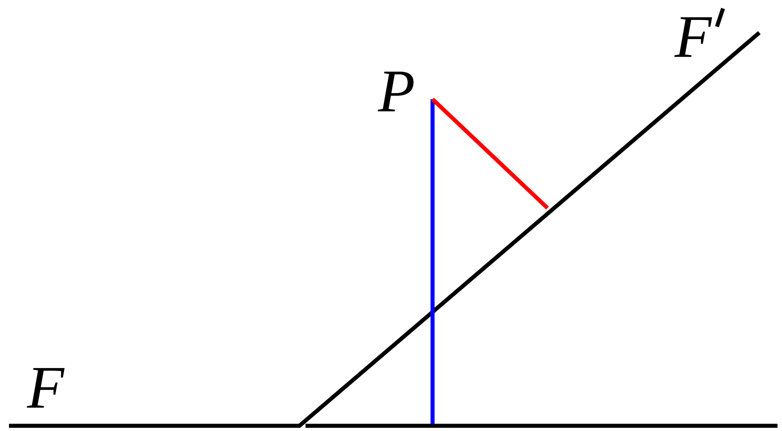
Assume that the property is not true.

Then, since the point always projects outside any face, if placed on a plane, the polyhedron will roll forever. Thus we have constructed a perpetuum mobile. This is physically impossible.

The movement clearly stops when the point reaches its lowest potential.

This suggests that the point projects inside the closest face.

The following figure explains it all



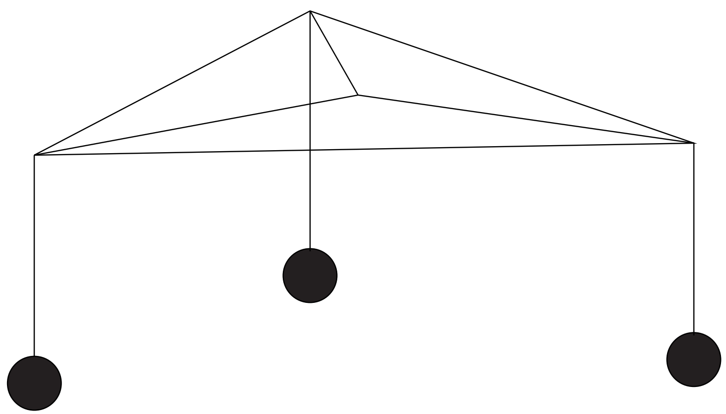
Problem 12. *Find the point in the plane of an acute triangle that mimimizes the sum of the distances to the vertices of the triangle.*

(P. Fermat)

The first to solve this problem was E. Torricelli.

Solution: *Physical solution (by G.W. Leibniz):*

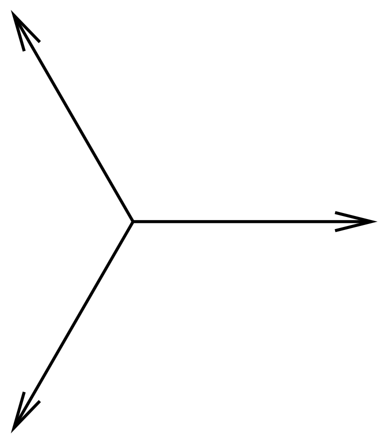
Place the triangle on a table, drill holes at each vertex, and suspend through the holes a ball equal weights, hanging on threads that are tied together.



The system reaches its equilibrium when the gravitational potential is minimal, hence when the sum of the lengths of the parts of the threads that are on the table is minimal.

The point P where the three threads are tied together is the one required by the problem.

The three equal forces that act at P, representing the weights of the balls, add up to zero, because there is equilibrium. These forces form equal angles.



Physical intuition helped us locate the point P.

Let us now prove rigorously that if

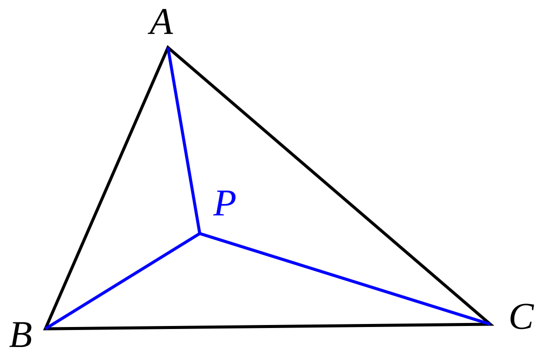
$$\angle APB = \angle BPC = \angle APC = 120^\circ,$$

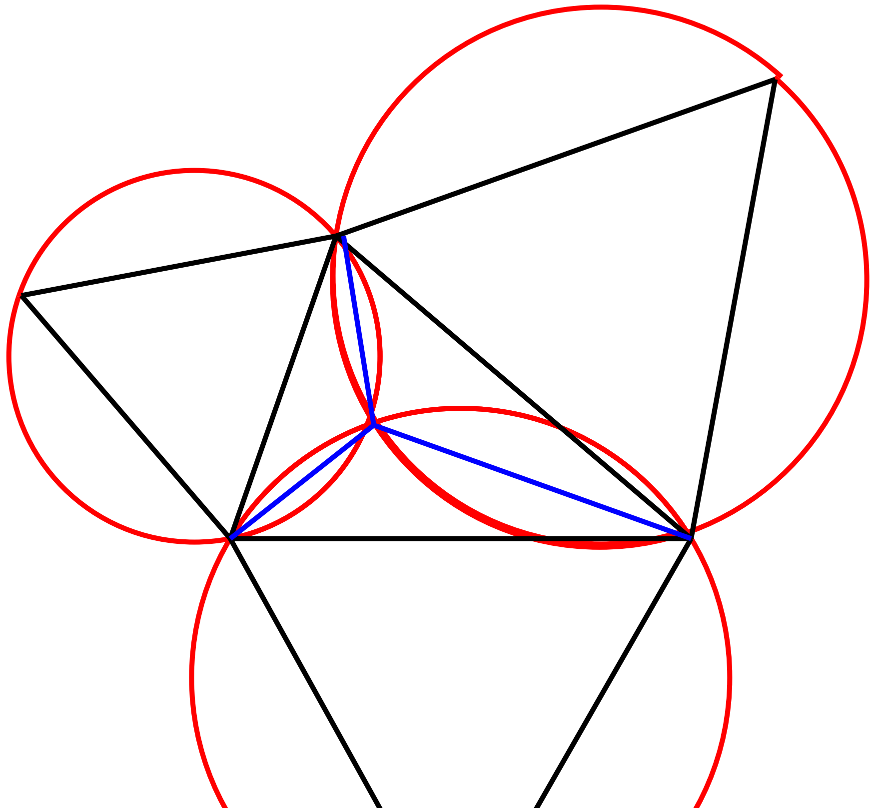
then

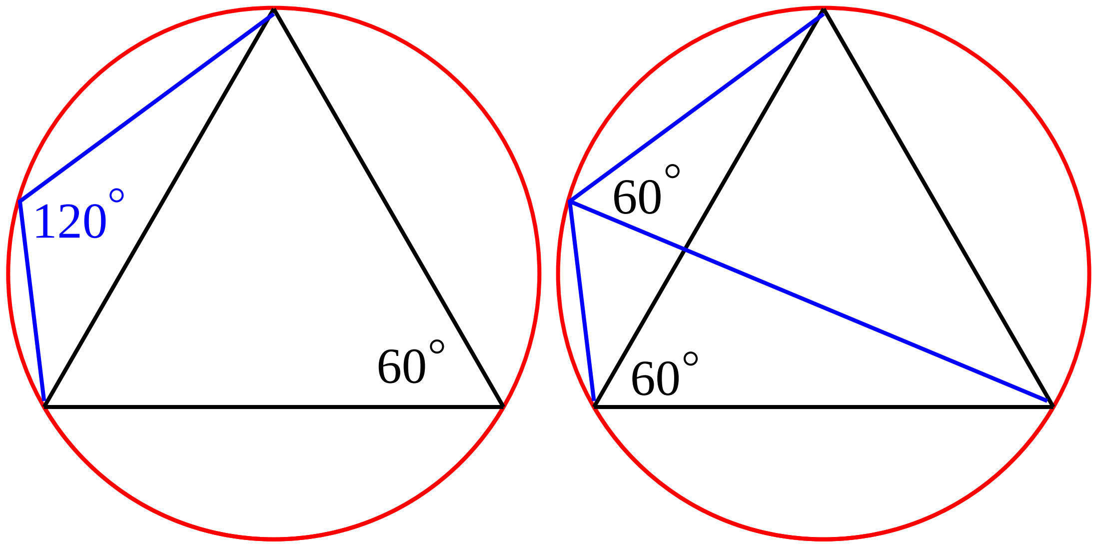
$$AP + BP + CP$$

is minimal.

First, how do you construct point P?



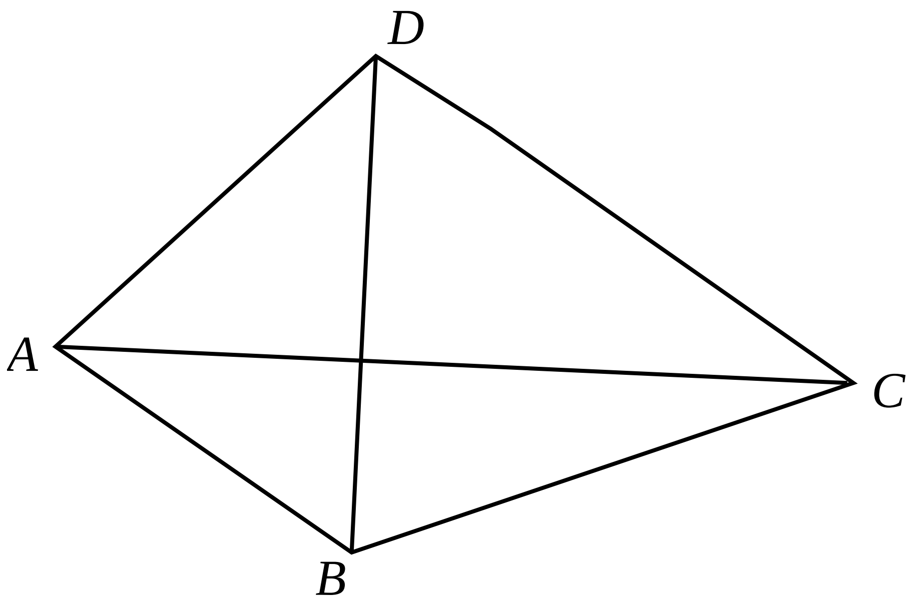


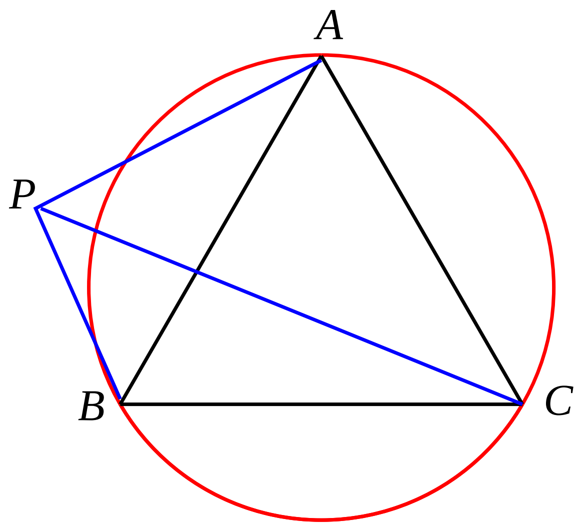

120°
60°
60°
60°

Ptolemy's theorem. In a quadrilateral $ABCD$,

$$AB \cdot CD + AD \cdot BC \geq AC \cdot BD$$

with equality if and only if $ABCD$ is cyclic.



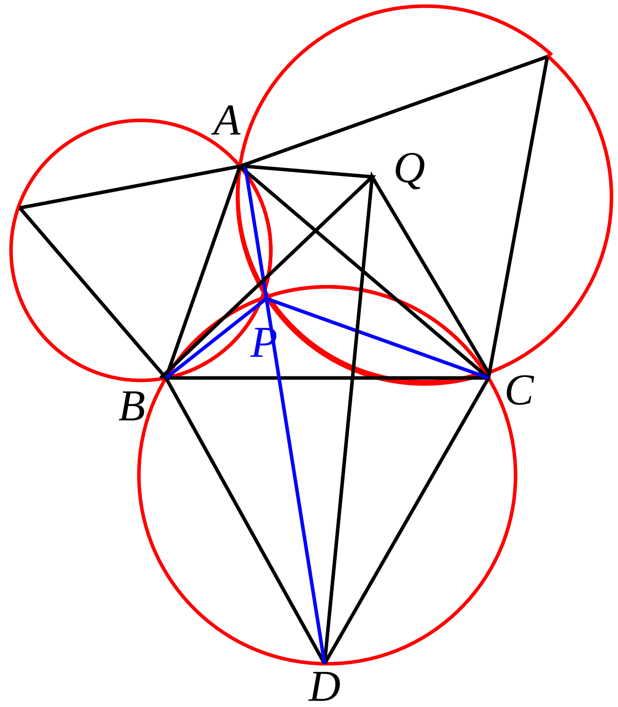


If $AB = AC = BC = a$ *then*

$$a \cdot PA + a \cdot PB \geq a \cdot PC$$

so

$$PA + PB \geq PC$$



In our case, if Q is another point, then

$$QA + QB + QC \geq QA + QD \geq AD.$$

Also

$$PA + PB + PC = PA + PD = AD.$$

So

$$PA + PB + PC \leq QA + QB + QC$$

quod erat demonstrandum.

As a by-product of this solution, we obtain

Pompeiu's theorem. Given an equilateral triangle ABC and a point P in its plane, with the segments PA, PB, PC one can form a triangle. This triangle is degenerate if and only if P is on the circumcircle of ABC.