CHERN-SIMONS THEORY AND WEYL QUANTIZATION

Răzvan Gelca

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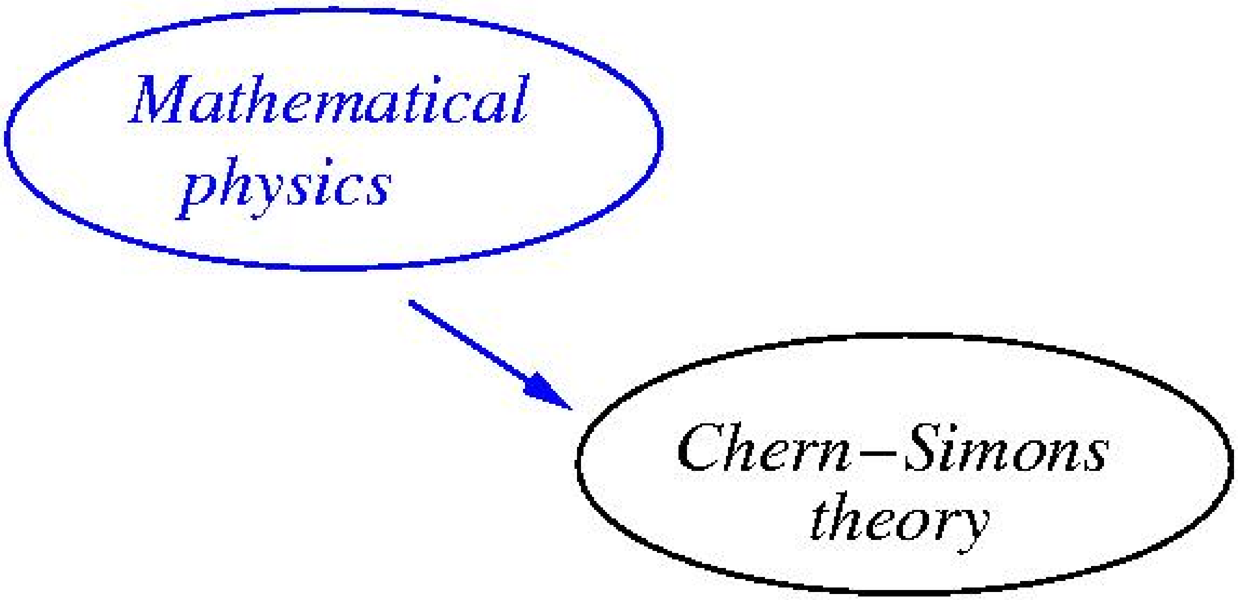
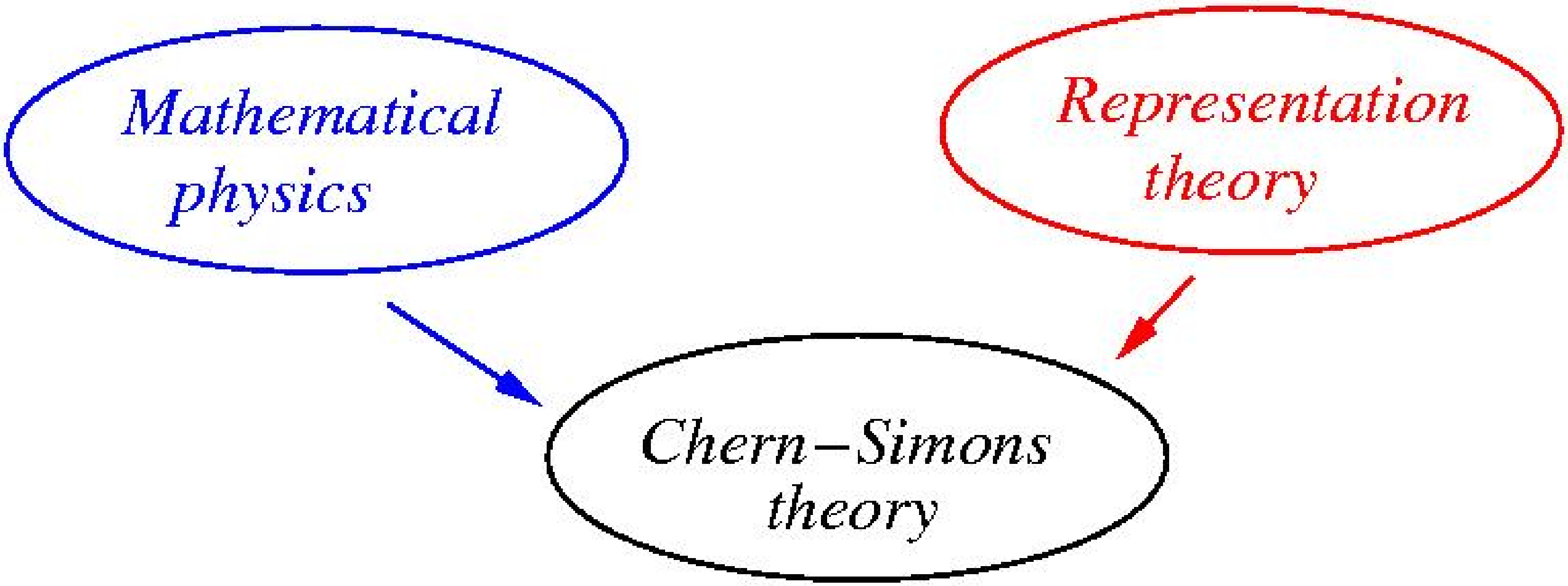
this talk is based on joint work and discussions with Alejandro Uribe, Alastair Hamilton, Charles Frohman, James Staff

Chern–Simons theory


Mathematical physics
Chern–Simons theory



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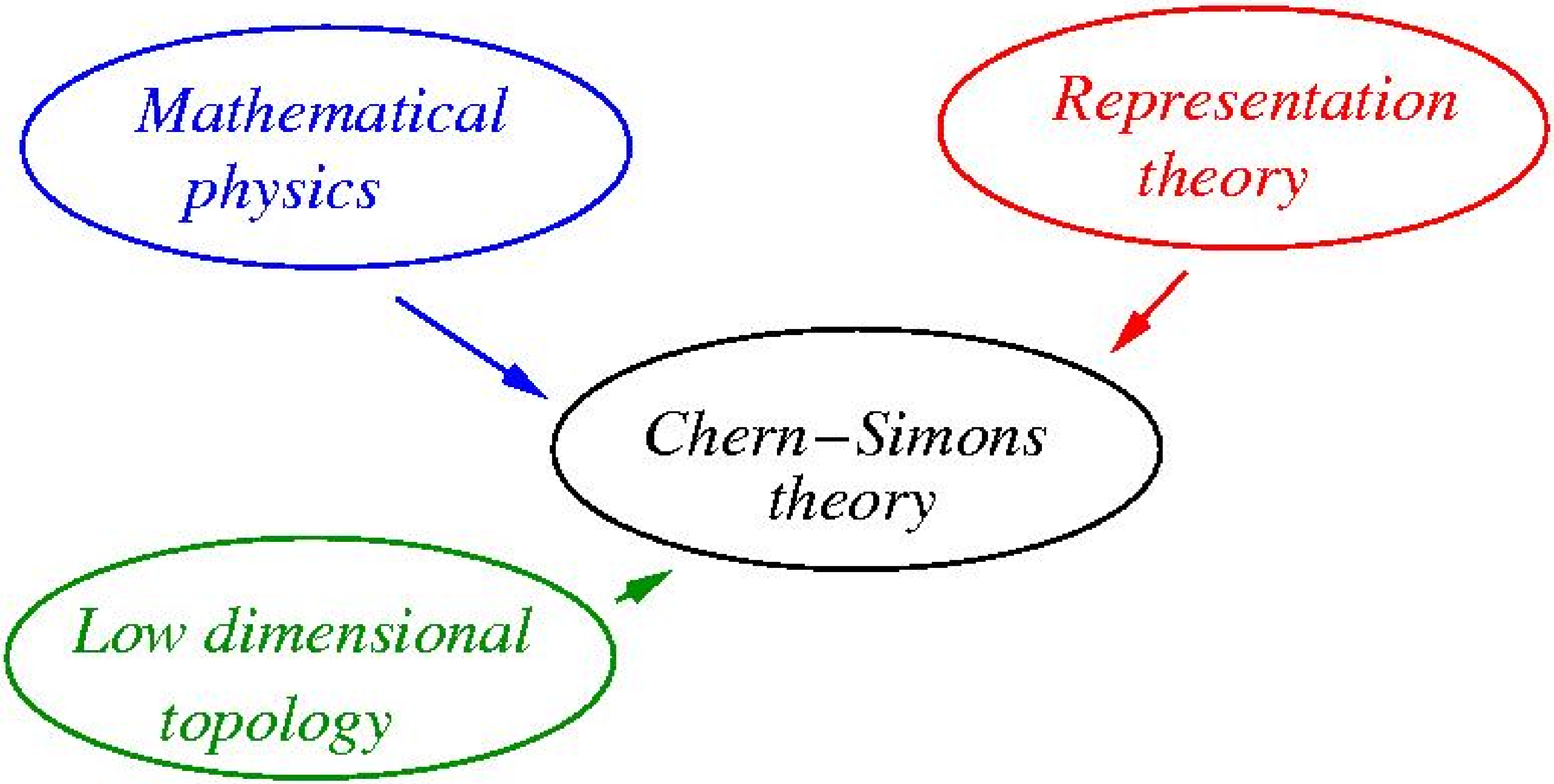
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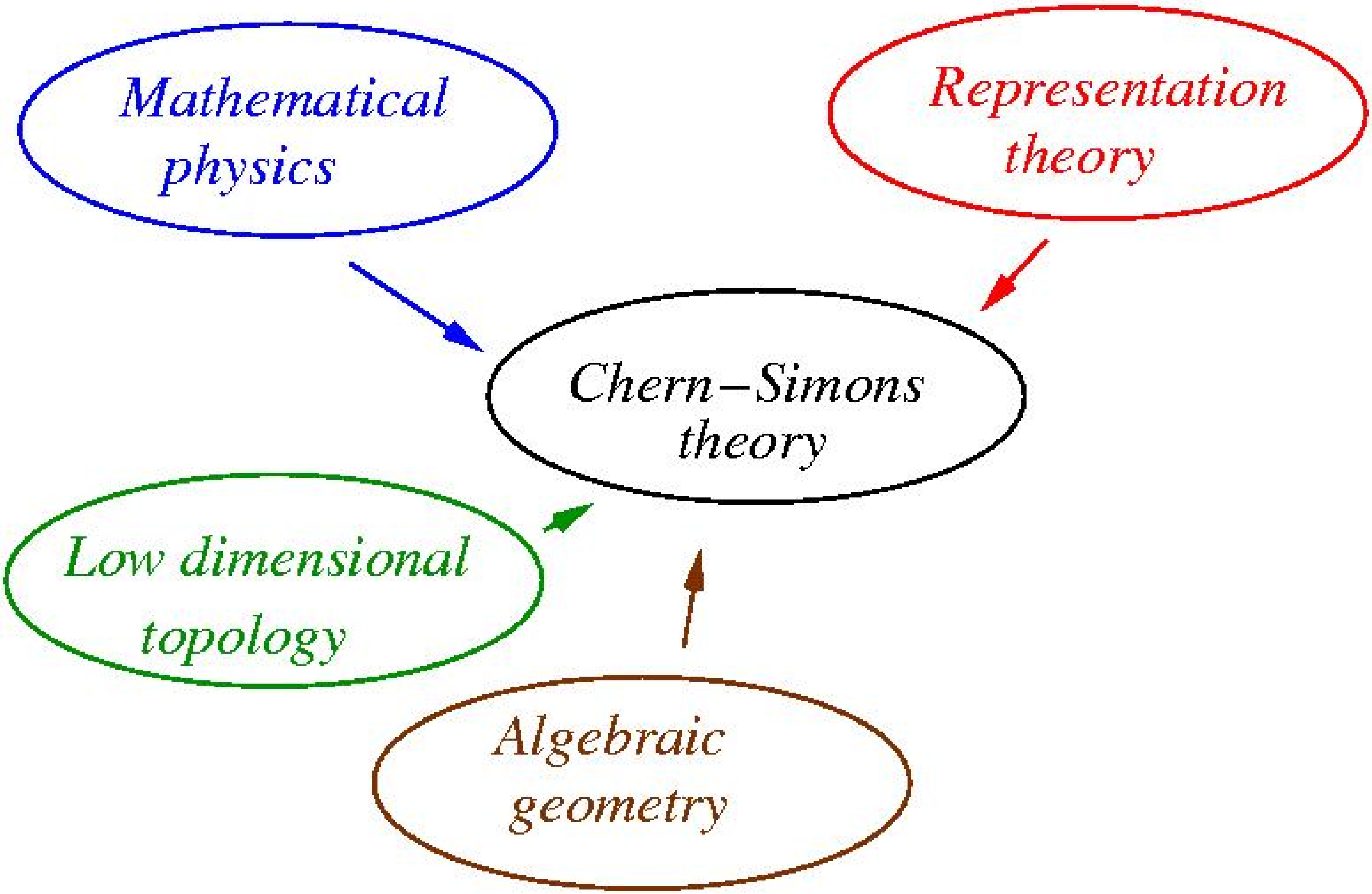
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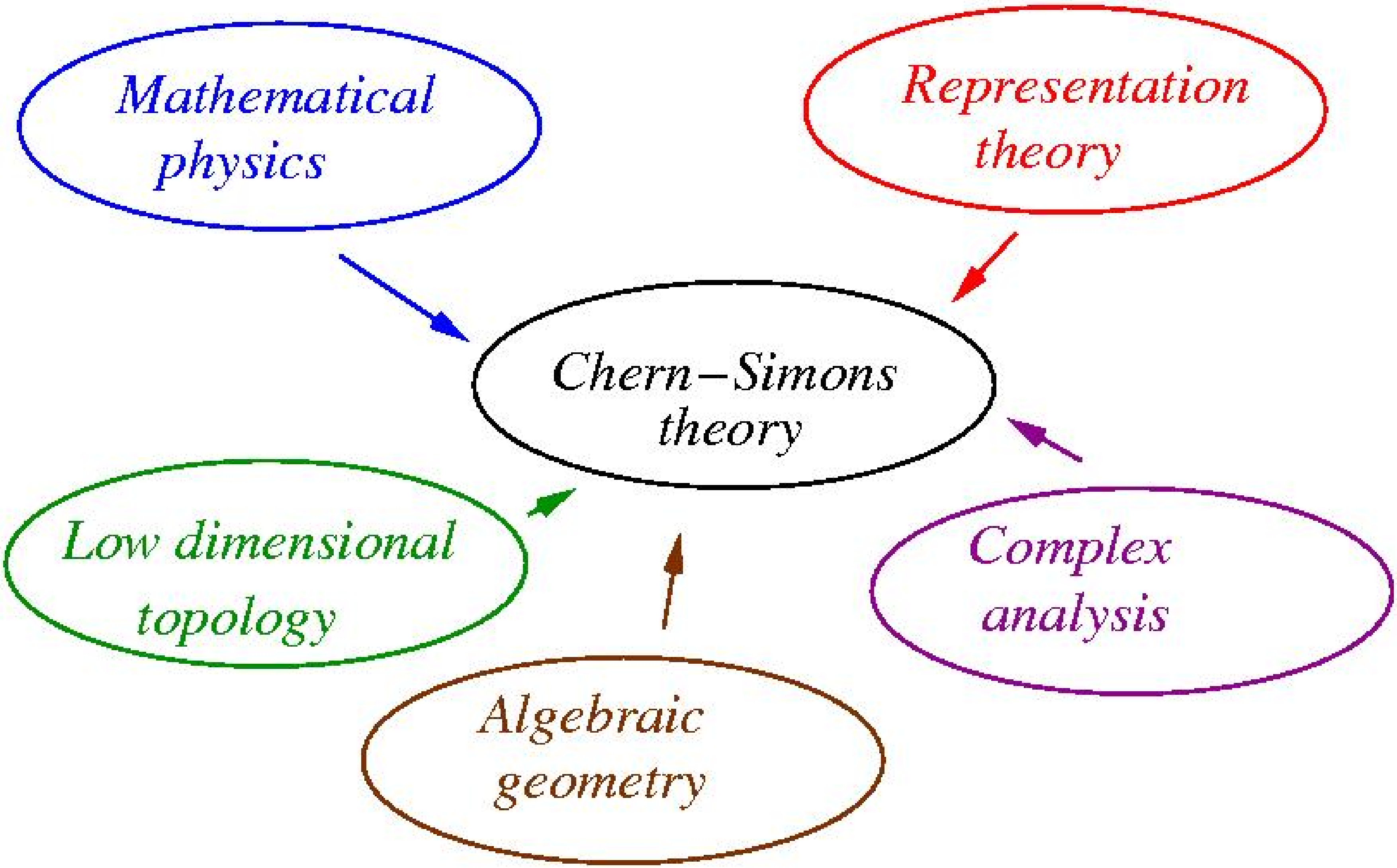
Mathematical physics
Representation theory
Chern–Simons theory
Low dimensional topology
Complex analysis
Algebraic geometry

Edward Witten's Chern-Simons theory

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Was constructed in order to give a geometric definition of the Jones polynomial of knots.

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- G *compact Lie group (the gauge group of the theory)*

Examples: $U(1)$, $SU(2)$, $SU(n)$, ...

- $\mathfrak{G}$ *its Lie algebra*

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Example:

$$SU(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\},$$

$$su(2) = \left\{ \begin{pmatrix} ir & z \\ -\bar{z} & -ir \end{pmatrix} \mid r \in \mathbb{R}, z \in \mathbb{C} \right\}.$$

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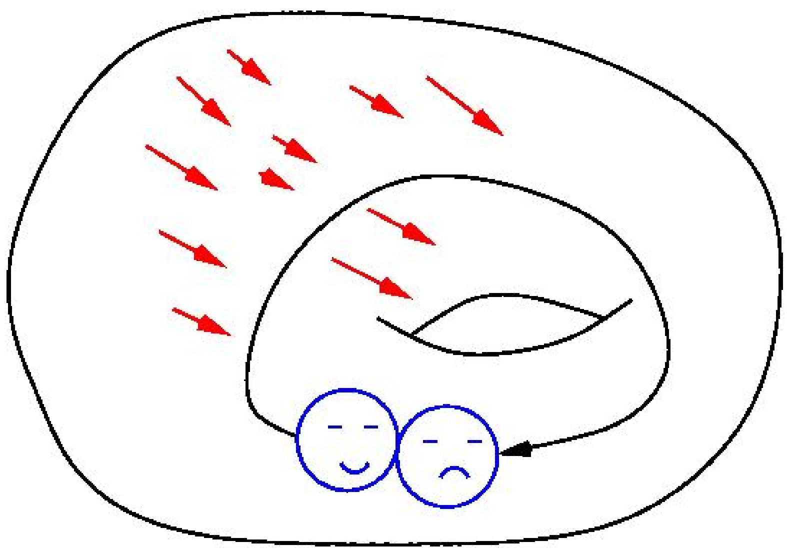

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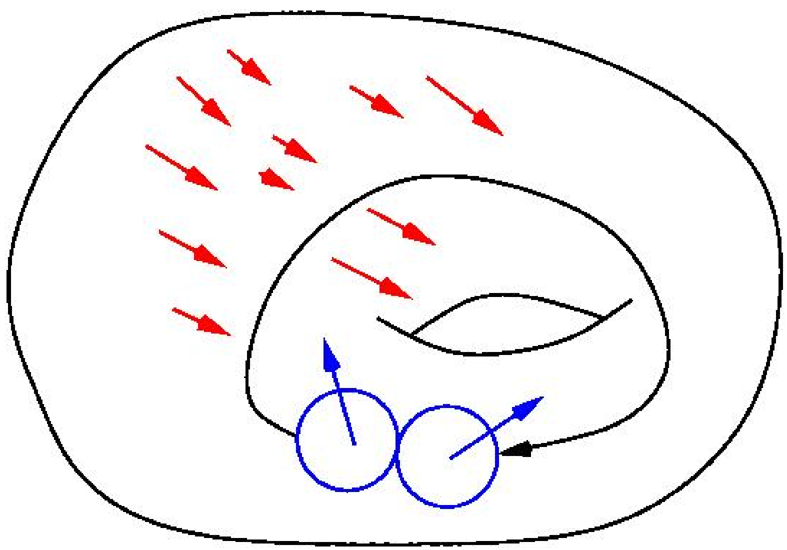

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- *Chern-Simons Lagrangian (functional)*

$$L(A) = \frac{1}{4\pi} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

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From the classical observable quantities of this physical theory we only care about Wilson lines:

$$W_{\gamma,V}(A) = \textit{trace}_V\, \textit{holonomy}_\gamma(A)$$

where V is a representation of G and γ is a curve (knot) in M.

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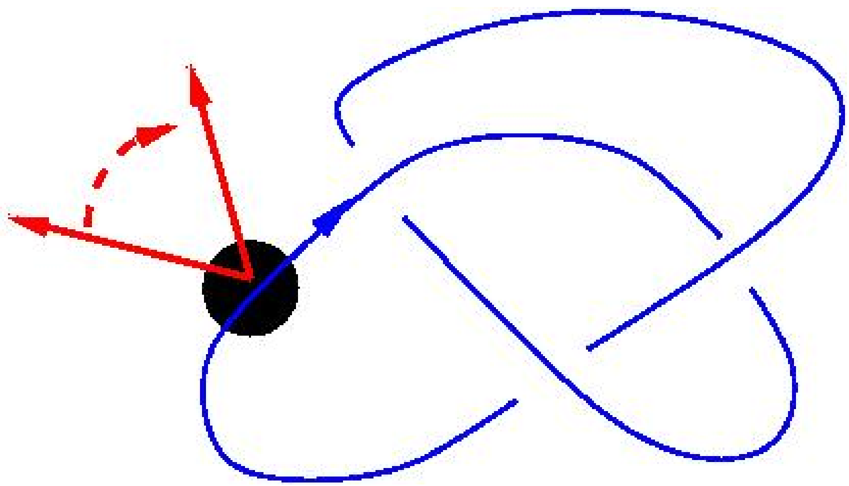

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$$\int W_{\gamma,V}(A) e^{ihL(A)} \mathcal{D}A.$$

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If $G = SU(2)$, $V = \mathbb{C}^2$, $M = S^3$, then this is the Jones polynomial of the trajectory evaluated at e^{ih}.

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The main objective of Witten's Chern-Simons theory is to study these quantized Wilson lines.

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Here you average the value of the Wilson line over the infinite-dimensional space of connections (fields) with oscillatory measure $e^{ihL(A)}\mathcal{D}A$ where h is Planck's constant.

Unfortunately this is a QUANTUM FIELD THEORY, and mathematics has made little progress in this area of physics.

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Fortunately Chern-Simons theory is a success story in quantum field theory, due to its many symmetries!

1. GAUGE SYMMETRIES

Let

$$\mathbf{g} : M \to G$$

be a smooth map, viewed as a change of coordinates.

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This gives rise to quantum mechanical models.

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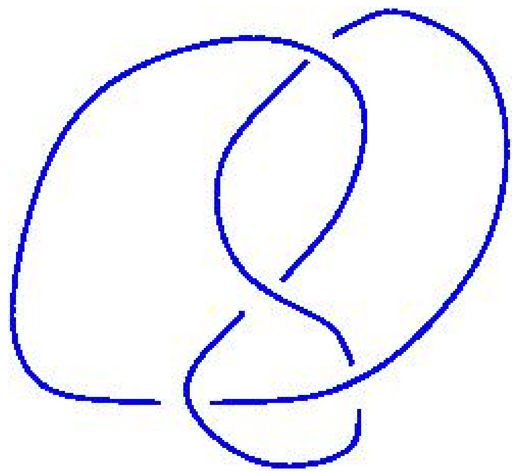

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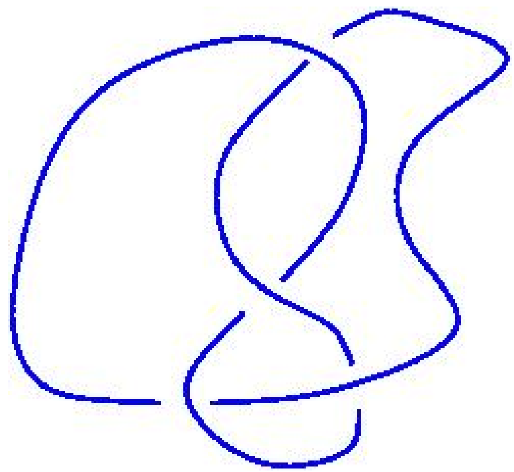

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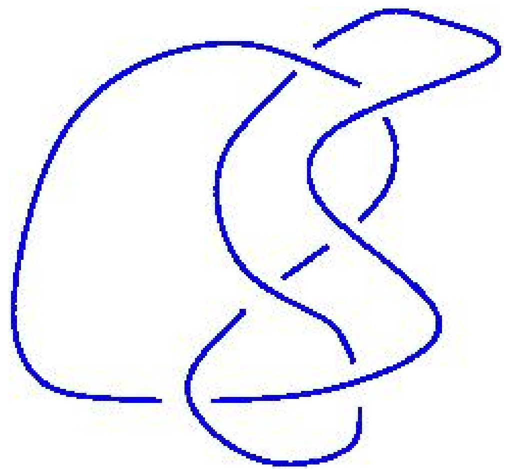

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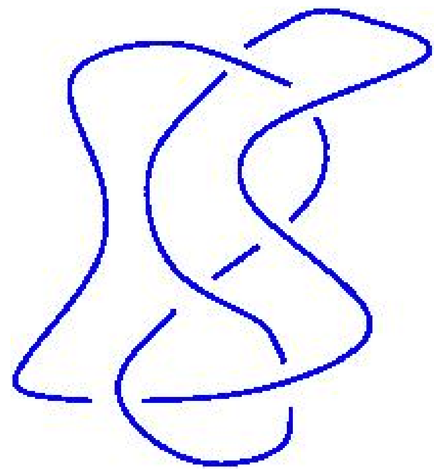

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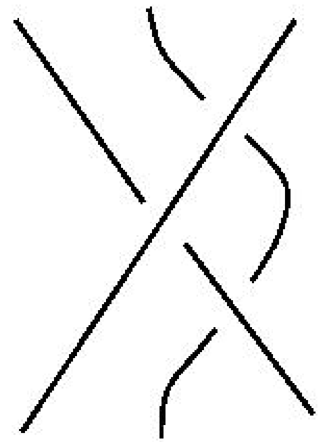
The quantized Wilson lines

$$\int W_{\gamma,V}(A)e^{ihL(A)}\mathcal{D}A$$

are invariant under isotopies. They are knot invariants.

Of the isotopies, the most important is the third Reidemeister move:



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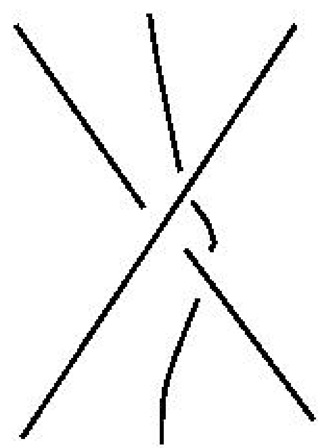

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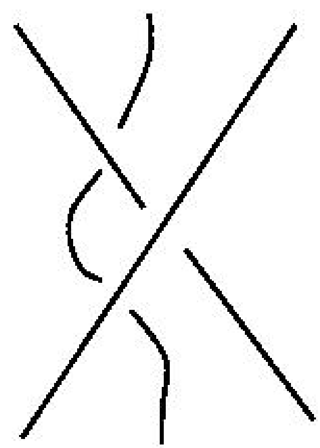

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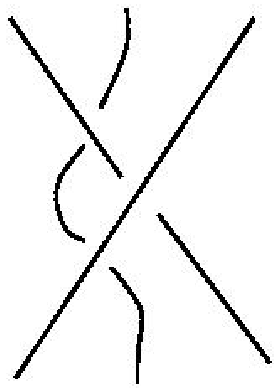

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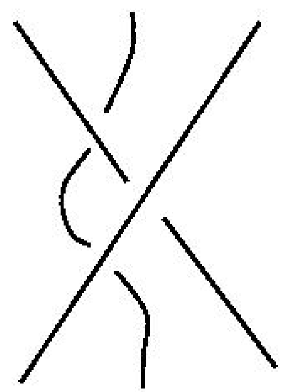

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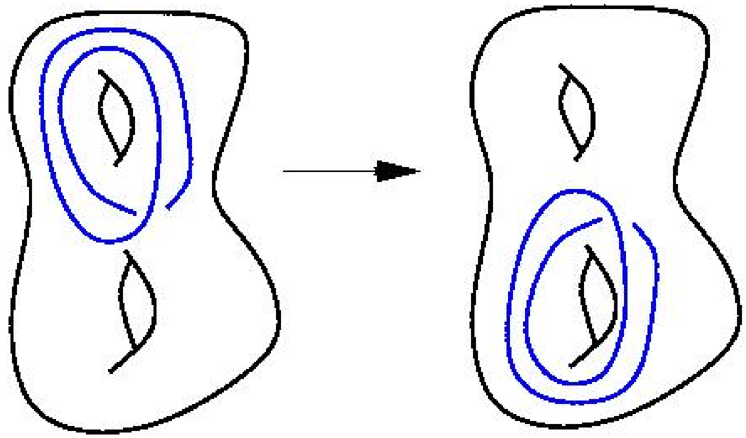
This gives rise to rigorous models (Reshetikhin-Turaev theory).

3. ORIENTATION PRESERVING DIFFEOMORPHISMS OF $\mathbb{M}$



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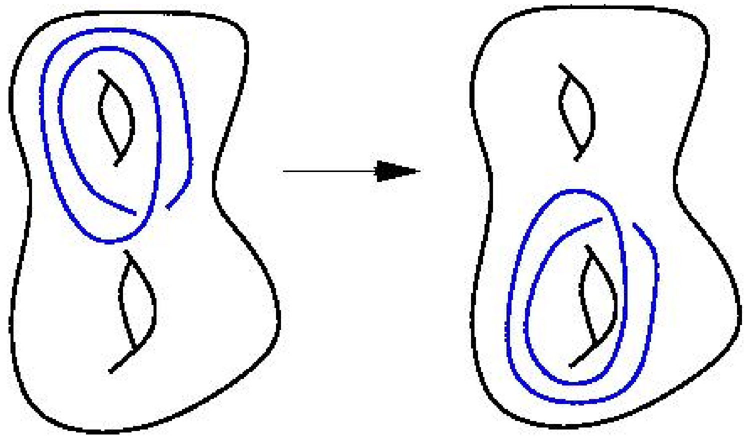

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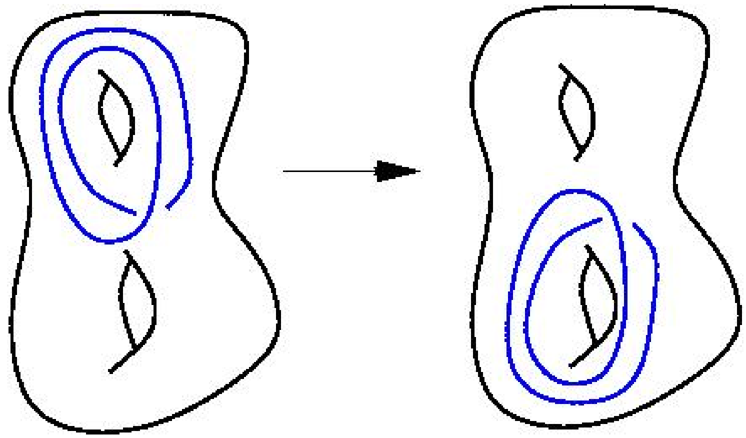

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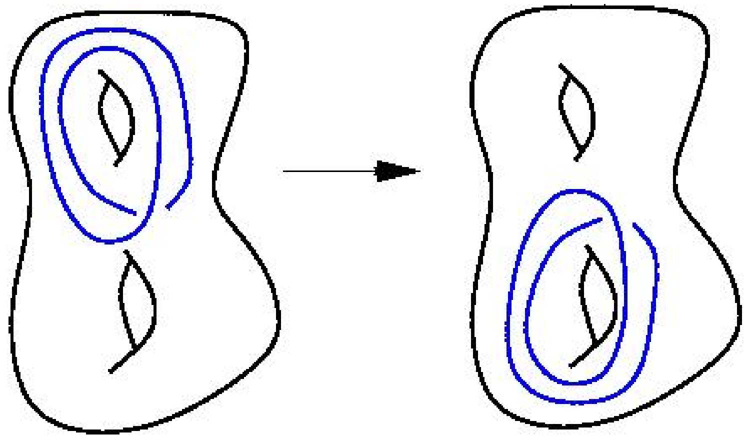

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This allows us to identify combinatorial models in Chern-Simons theory with analytical models.

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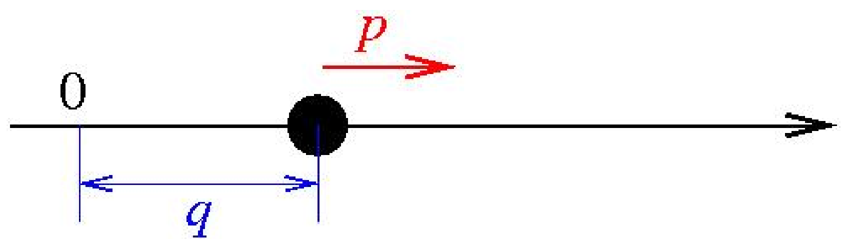
Consider a free particle.



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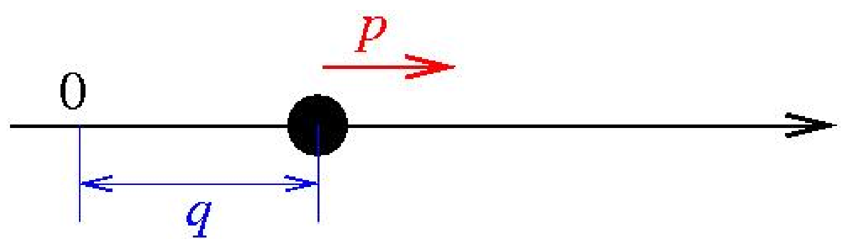

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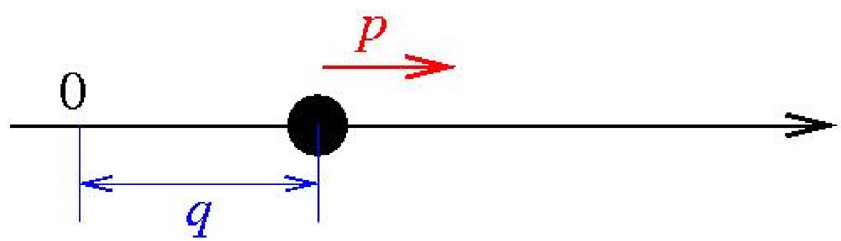

It has two coordinates: position q *and momentum* p*. They range freely in the phase space* $\mathbb{R}^2$*. Position and momentum are functions on* $\mathbb{R}^2$*:*

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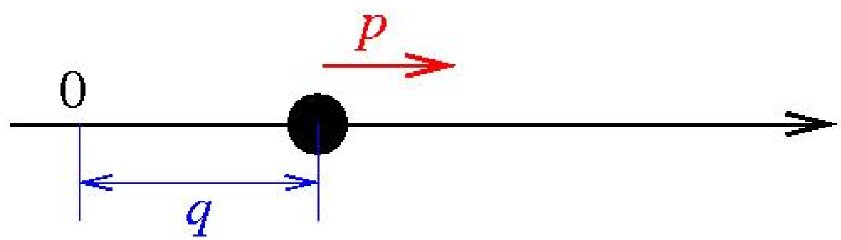

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$$(p, q) \mapsto q, \quad (p, q) \mapsto p.$$

Every other classical observable is a function of p and q. For example the total energy of the harmonic oscillator:

$$E(p, q) = \frac{1}{2m} p^2 + \frac{k}{2} q^2.$$

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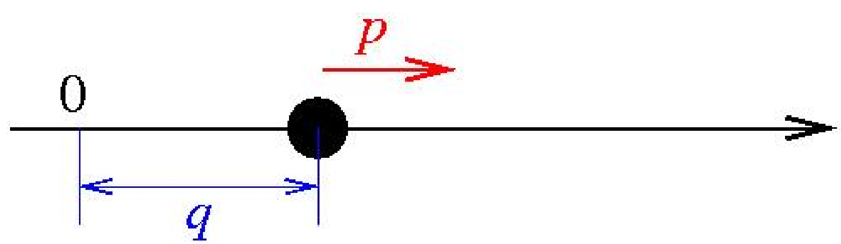

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The evolution of an observable is defined by Hamilton's equation

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q},$$

H: total energy.

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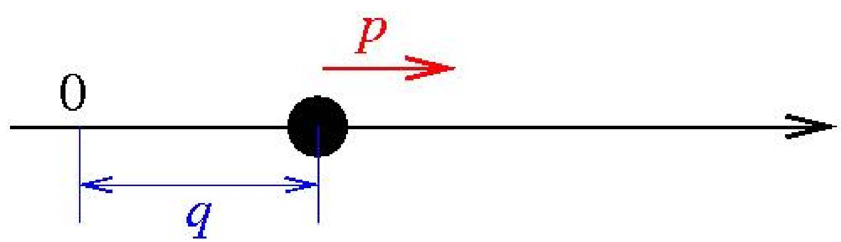

It has two coordinates: position q and momentum p. They range freely in the phase space $\mathbb{R}^2$.

According to W. Heisenberg we pass from classical to quantum mechanics by replacing

- phase space $\mapsto$ Hilbert space
- functions on the phase space $\mapsto$ linear operators on the Hilbert space

Hamilton's equation turns into Schroedinger's equation.

Postulate of quantum mechanics:

- $\mathbb{R}^2 \mapsto L^2(\mathbb{R})$
- $q \mapsto Q$, $Qf(q) = qf(q)$
  $p \mapsto P = -i\hbar\frac{\partial}{\partial q}$.

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$$\exp(iq) \mapsto \exp iQ$$
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Weyl quantization:

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then extend using the Fourier transform and the inverse Fourier transform

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are the only operators that satisfy the Heisenberg uncertainty principle:

$$PQ - QP = -i\hbar I.$$

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Corollary: If you change coordinates in classical mechanics and then quantize you get the same model.

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Corollary: If you change coordinates in classical mechanics and then quantize you get a unitary equivalent model.

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Let $h \in Sp(1)$. *The fact that after changing coordinates you obtain a unitary equivalent model means that there is a unitary map* $\rho(h) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ *such that*

$$\textit{op}(f \circ h) = \rho(h)\textit{op}(f)\rho(h)^{-1}.$$

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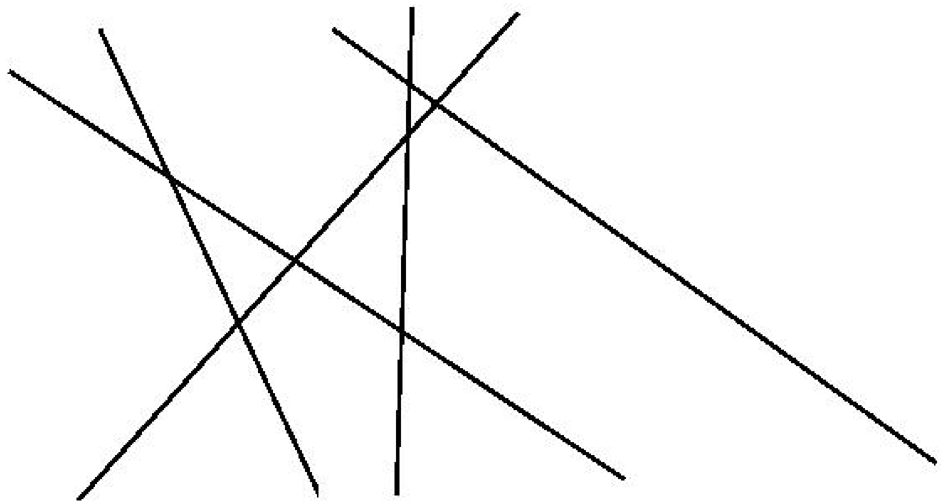
This is known as the Egorov condition satisfied exactly only for Weyl quantization. It is this symmetry of Weyl quantization that we related to the symmetry of Chern-Simons theory that comes from diffeomorphisms.

QUANTUM GROUPS

First appeared in the study, by the Russian school of mathematical physics, of exactly solvable models in statistical mechanics. The term was coined by V. Drinfel'd (see also the work of M. Jimbo).

QUANTUM GROUPS

A 2-dimensional statistical mechanics model:



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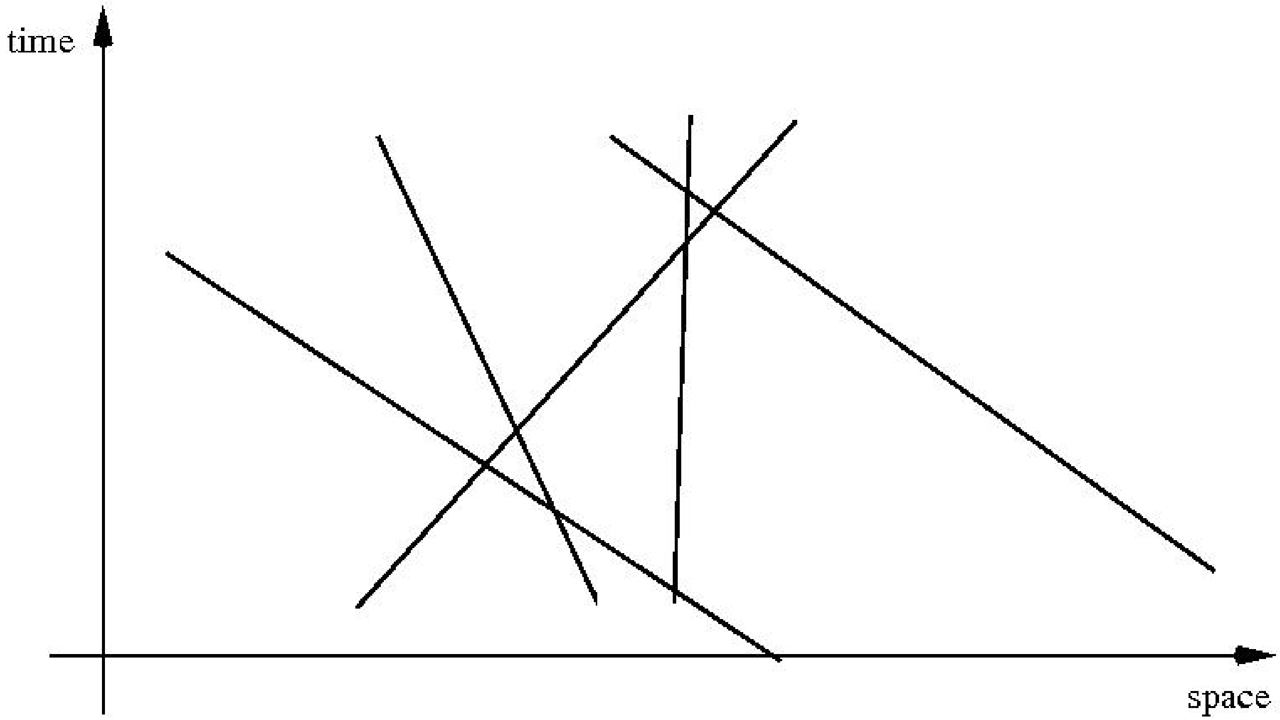
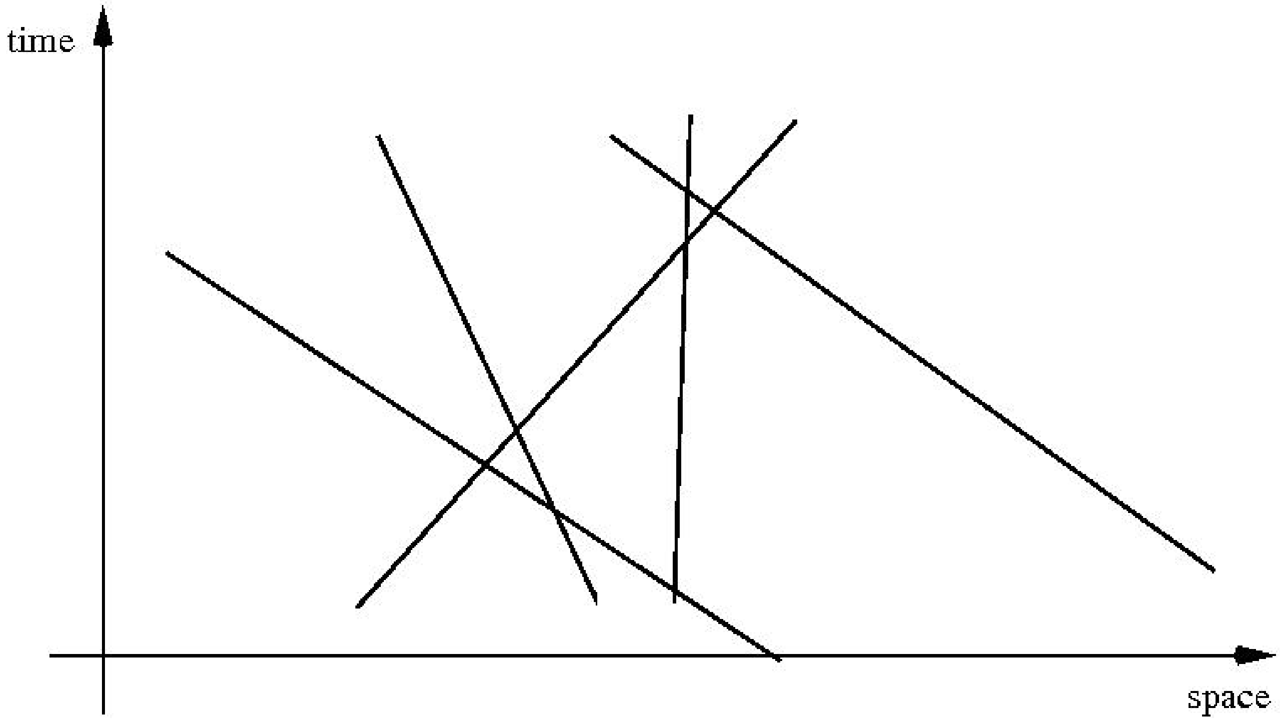


can be interpreted as a 1-dimensional quantum system with nodes being collisons (scattering of particles).

QUANTUM GROUPS

The Bethe Ansatz is a time symmetry that makes the system solvable



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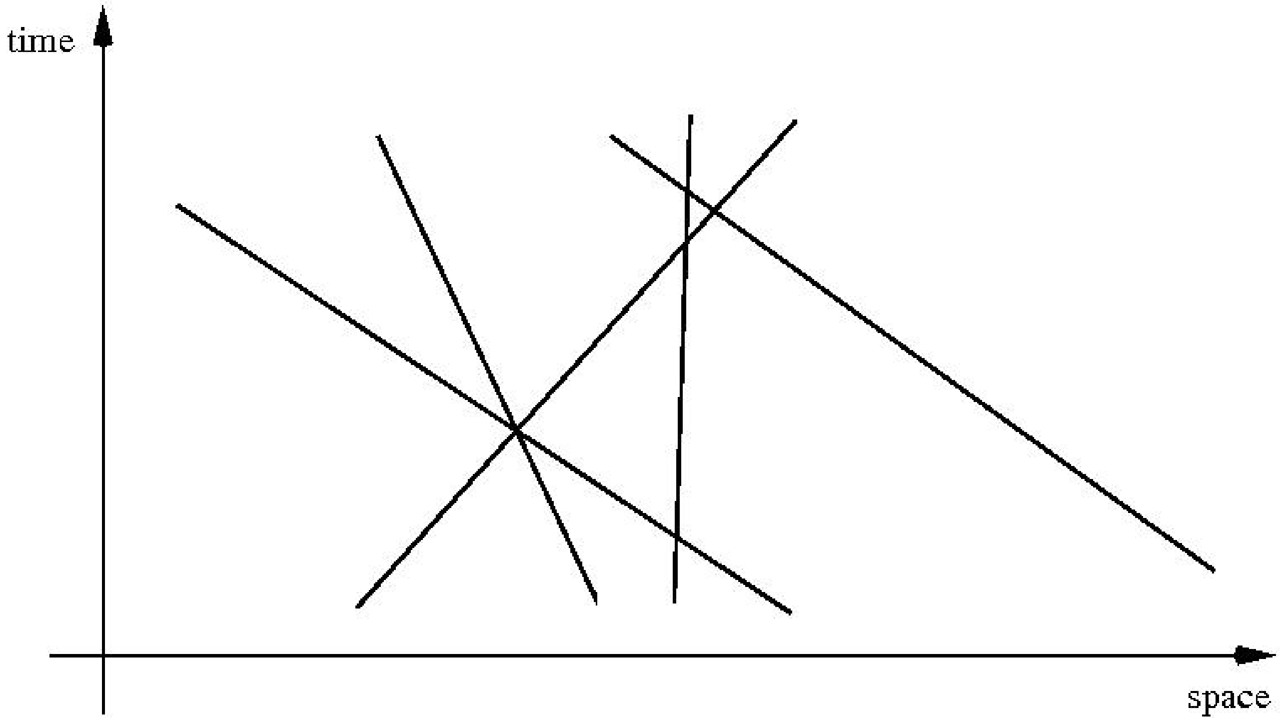

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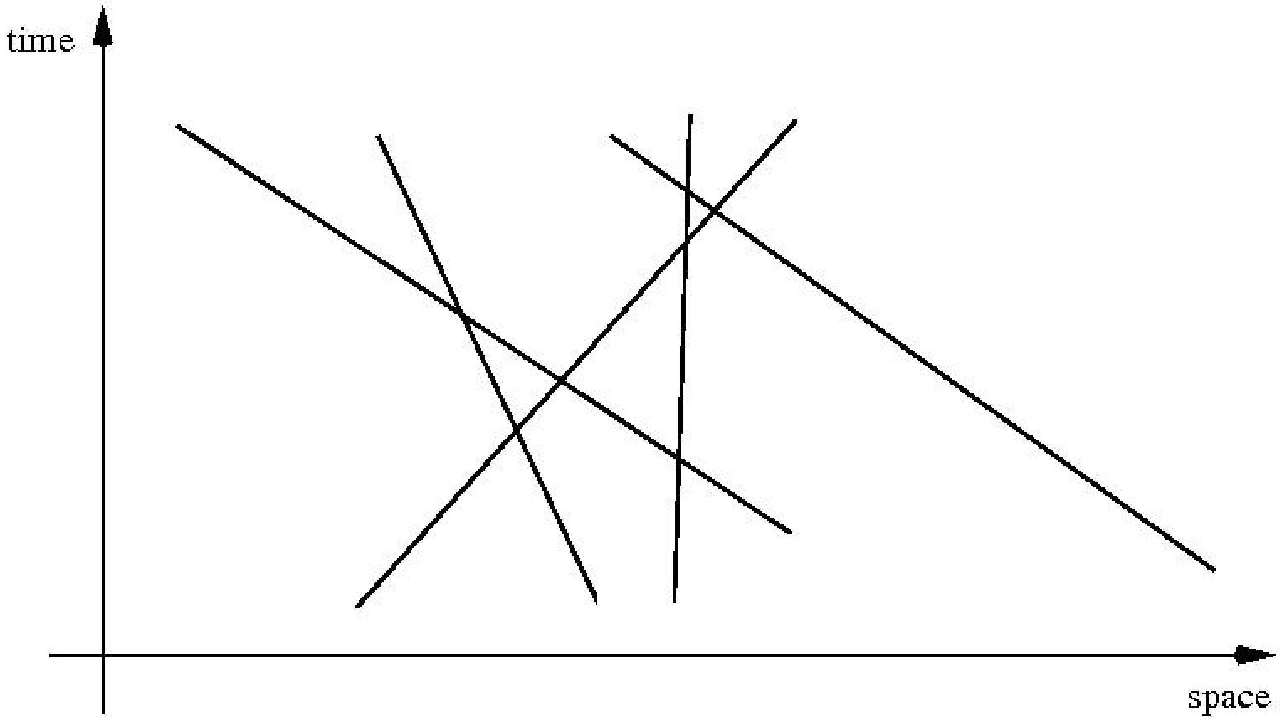

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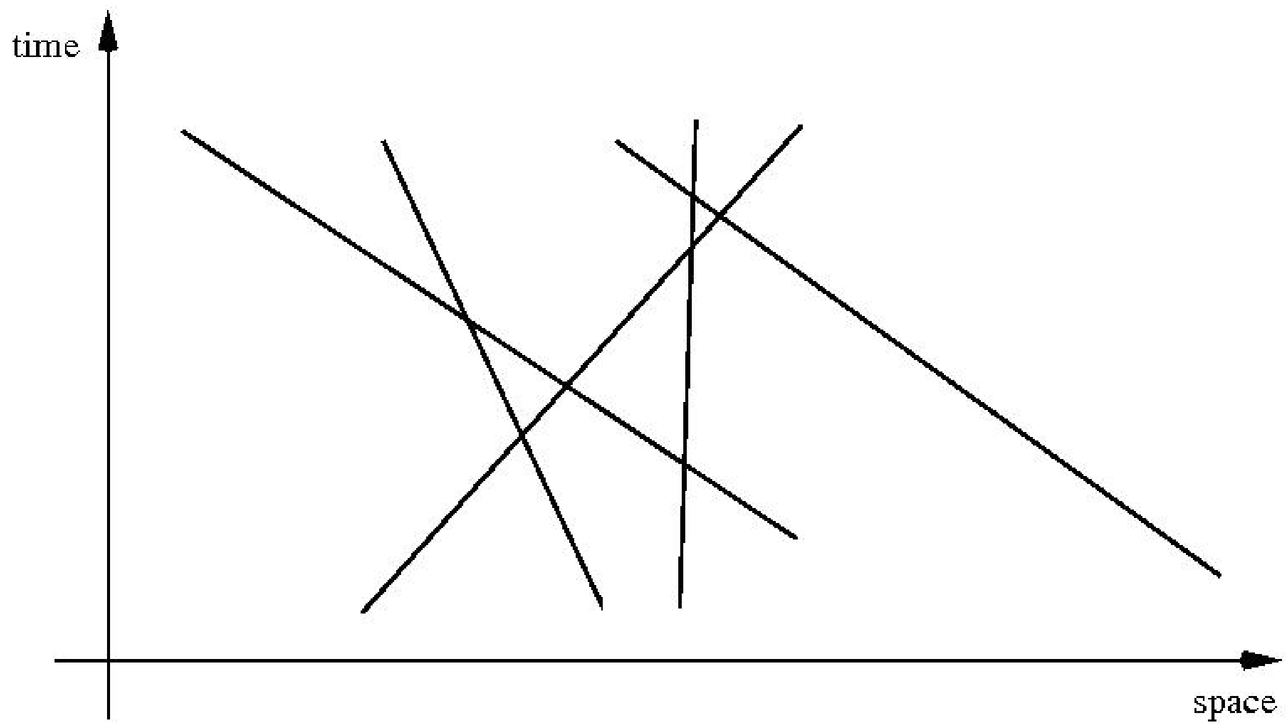

In statistical mechanics this symmetry is called the Yang-Baxter equation.

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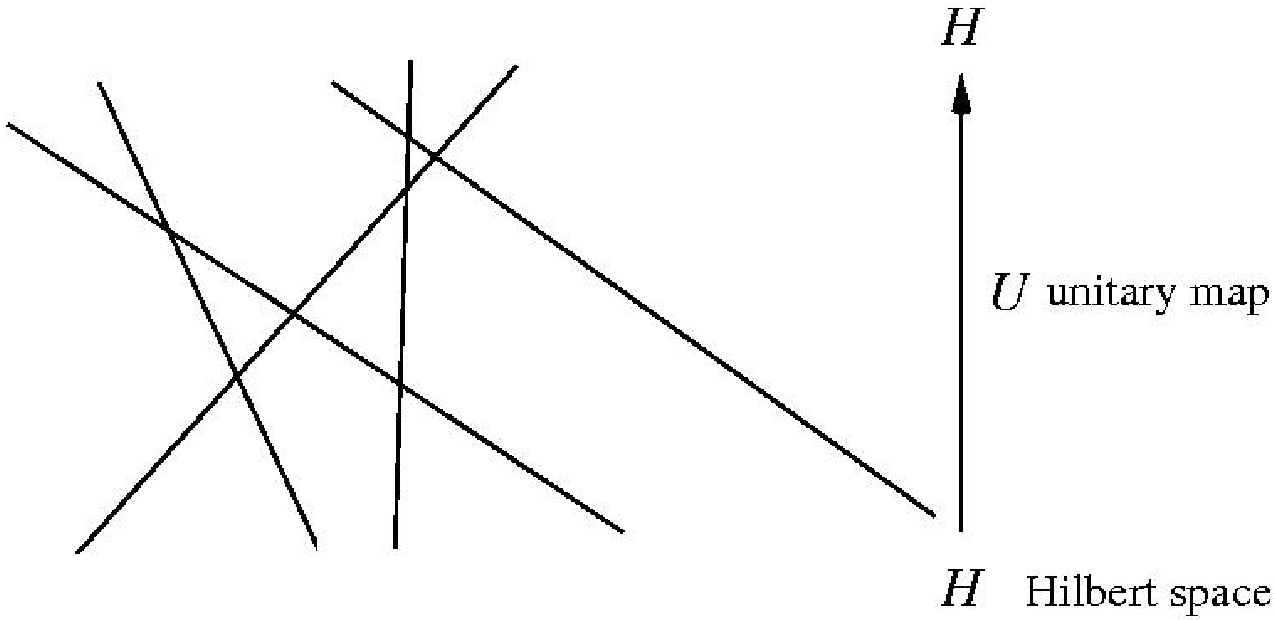

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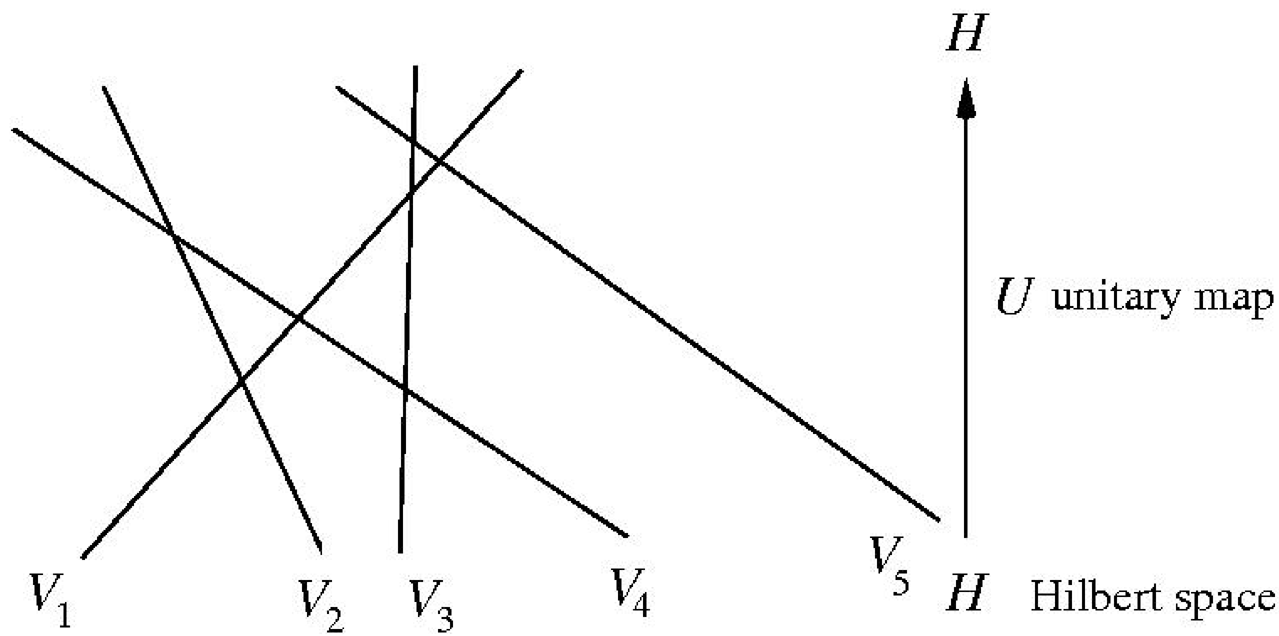

$$H = V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_5$$

V_j *are representations of the quantum group (which is a Hopf algebra).*

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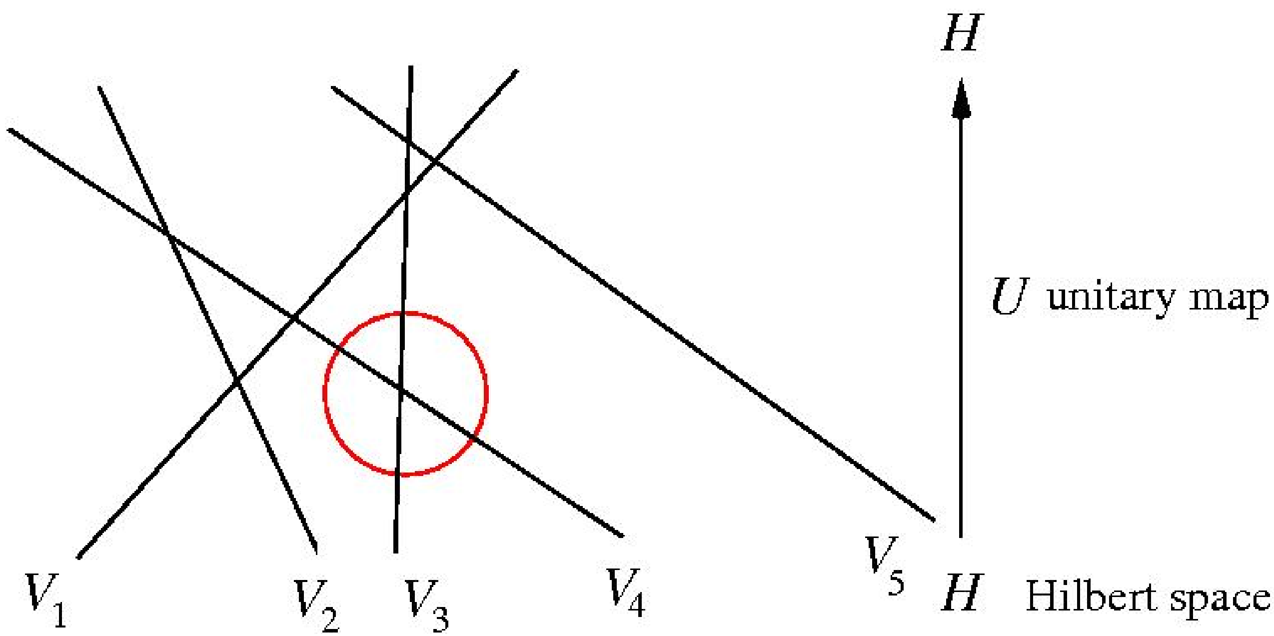

$$H = V_1 \otimes V_2 \otimes V_3 \otimes V_4 \otimes V_5$$

$$S : V_3 \otimes V_4 \to V_4 \otimes V_3.$$

The scattering matrix S *is a representation homomorphism.*

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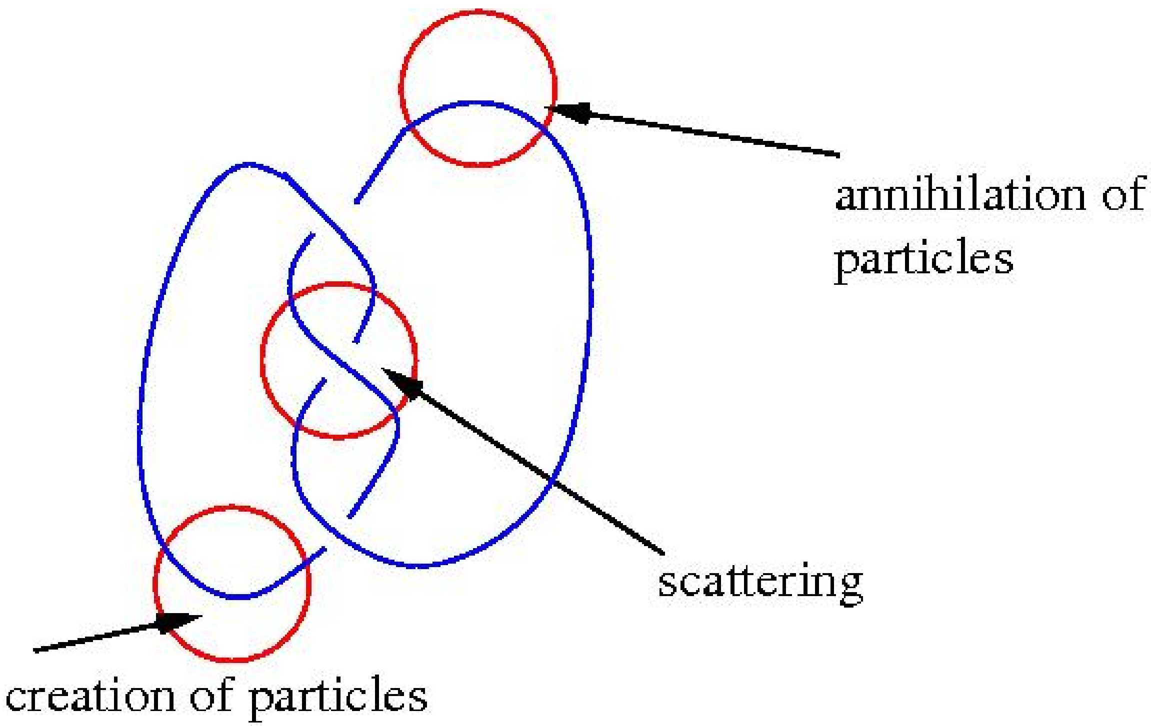

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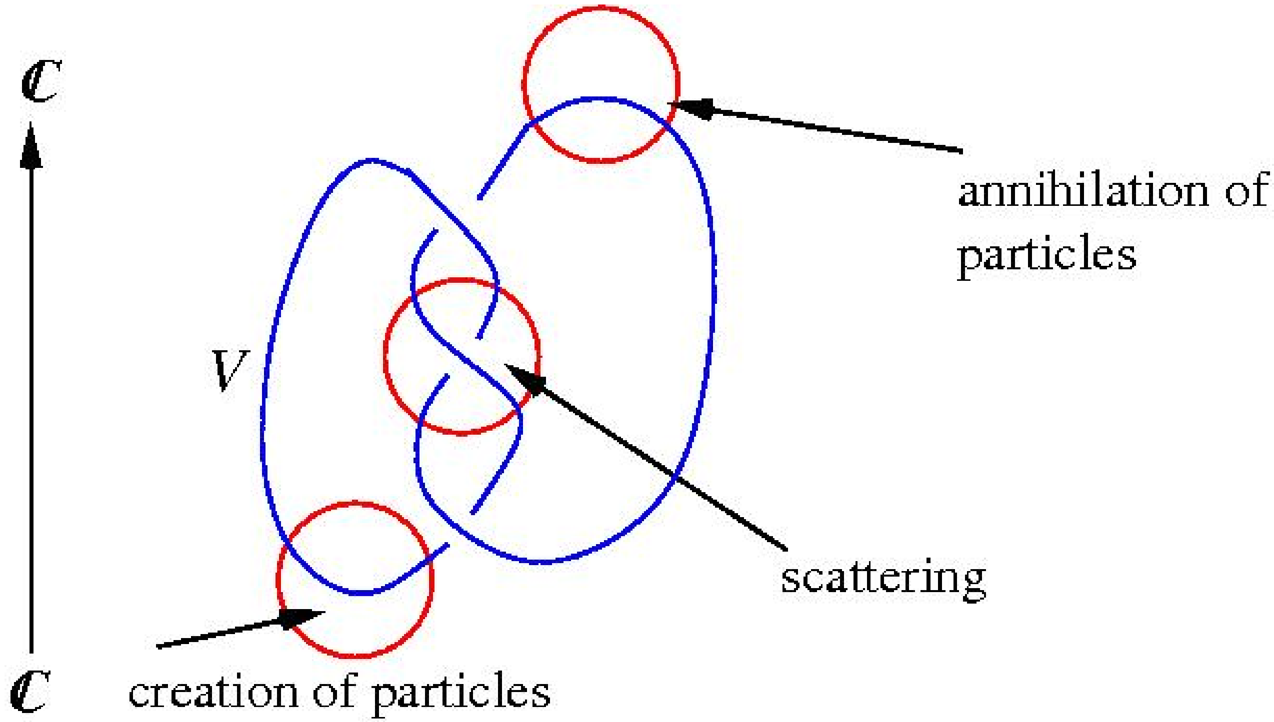
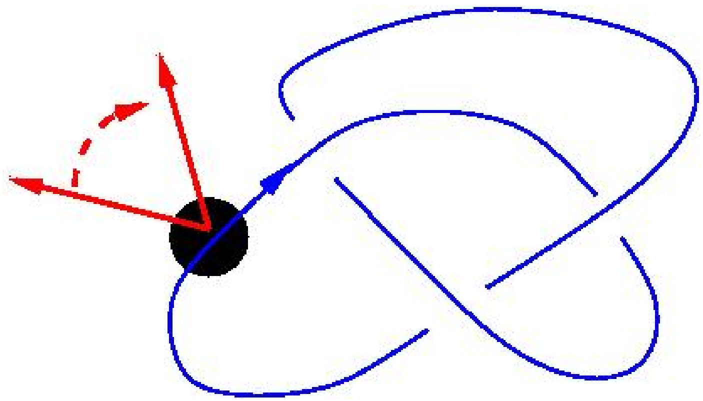


This is a 1-dimensional linear map, hence a number.

Reshetikhin's paradigm: This number is

$$\int W_{\gamma,V}(A)e^{ihL(A)}\mathcal{D}A$$

Remember our goal:



G *Lie group,* M *compact, orientable 3-manifold without boundary,* A $\mathfrak{G}$*-connection on* M*,*

$$L(A) = \frac{1}{4\pi}\int_M \mathit{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

$$W_{\gamma,V}(A) = \mathit{trace}_V\, \mathit{holonomy}_\gamma(A)$$

Understand: $\int W_{\gamma,V}(A) e^{ihL(A)} \mathcal{D}A$

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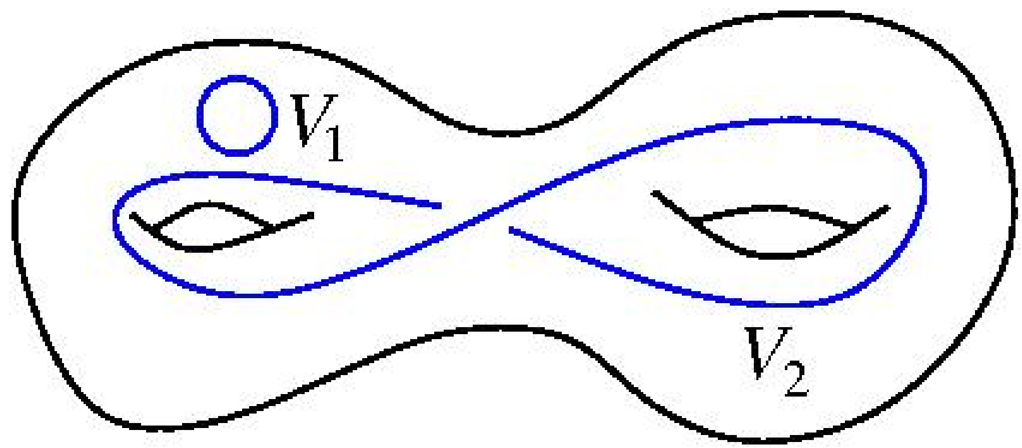

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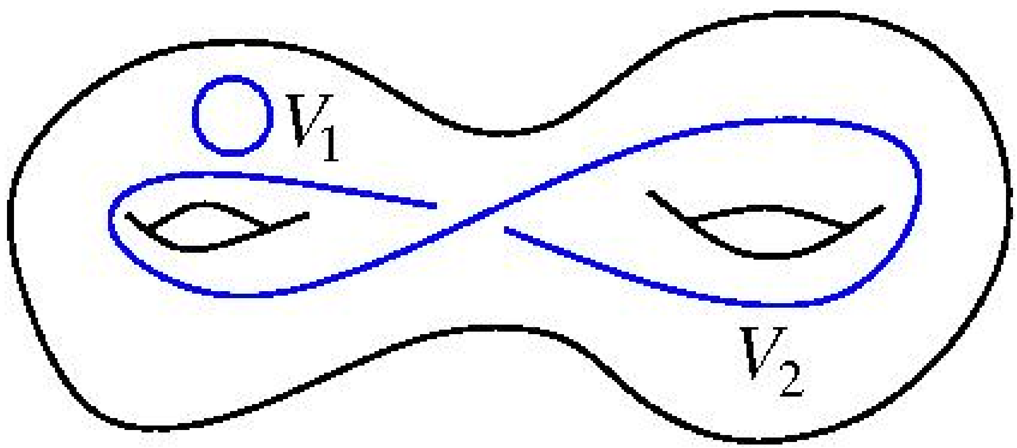

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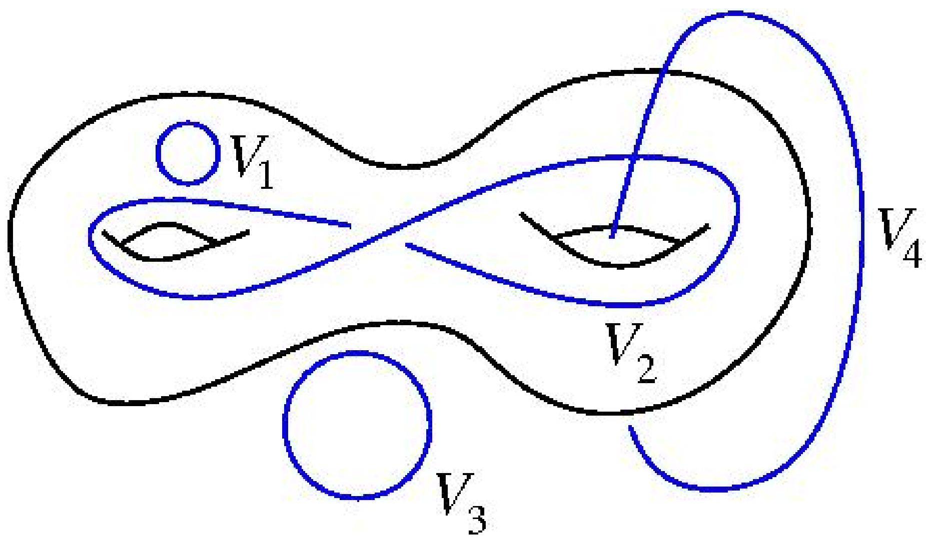

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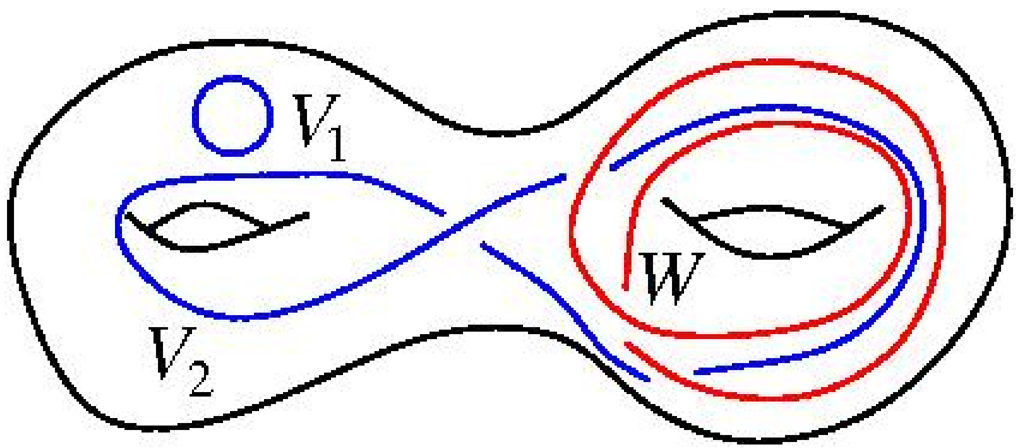

The linear operators are defined by the action of quantized Wilson lines on the boundary.

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A quantum physical model of WHAT???

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We obtain the quantum group quantization of the moduli space of flat $\mathfrak{G}$-connections on a surface.

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- *Atiyah and Bott (symplectic form)*
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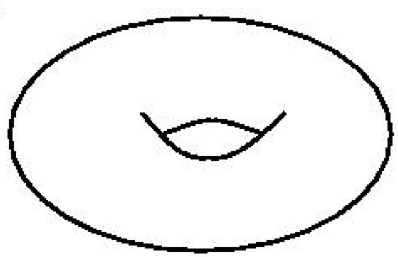
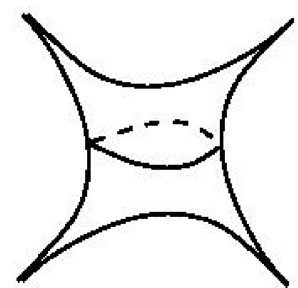
Examples for the torus:

G=U(1)



G=SU(2)



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Theorem: *Weyl quantization coincides with the quantum group quantization:*

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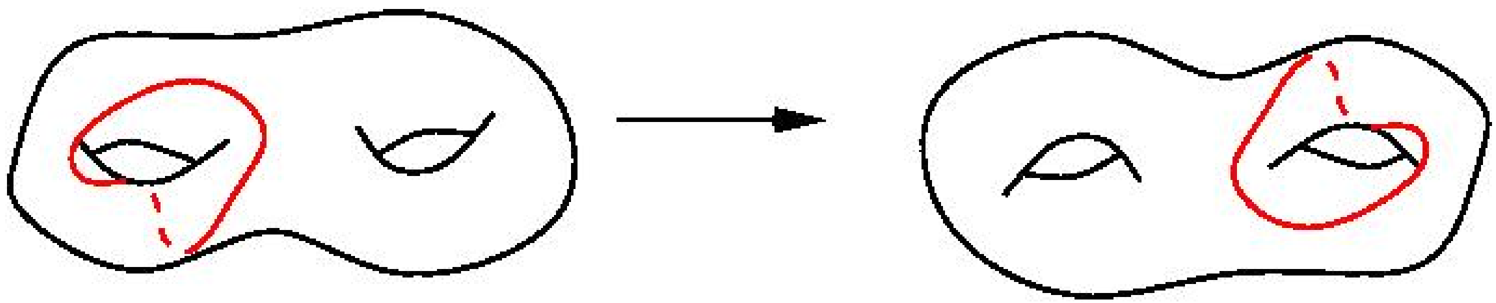

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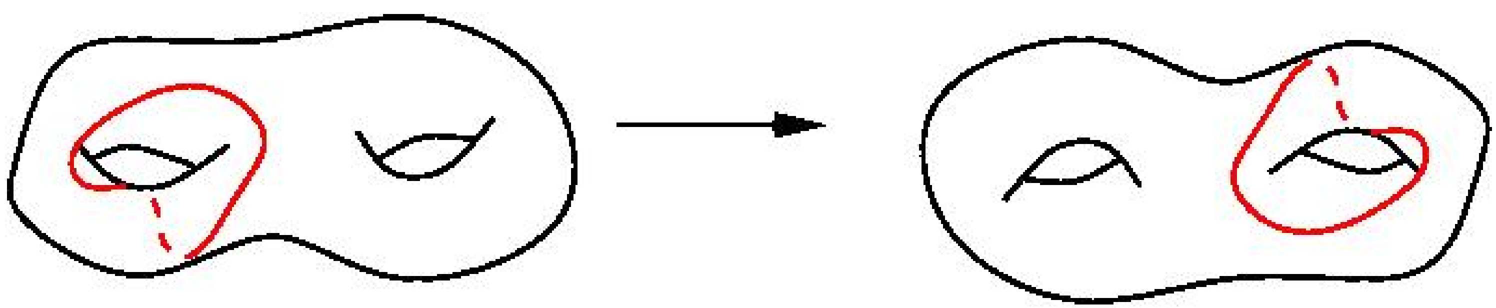

Exactly in the same way, Weyl quantization is well behaved with respect to the symmetries of $\mathbb{R}^2$*.*

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One should note that the quantum group quantization model is well behaved under the symmetries of the surface.

Every diffeomorphism of the surface induces a symplectomorphism of the moduli space.

Weyl quantization is well behaved with respect to the symplectomorphisms of the moduli space.

The exact Egorov identity is satisfied by Weyl quantization and the metaplectic representation:

$$\mathsf{op}(f \circ h) = \rho(h)\mathsf{op}(f)\rho(h)^{-1}.$$

A similar identity is satisfied by the quantum group quantization of the moduli space of flat $\mathfrak{G}$*-connections on a surface and the Reshetikhin-Turaev representation of the mapping class group of the surface.*

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Weyl quantization is one of the hardest quantization models to generalize! It is strange that it shows up in Chern-Simons theory.