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Chapter 1

Introduction

1.1 Why study complex analysis?

1.1.1 Polynomials

The equation

\[ x^2 + 3x + 2 = 0 \]

has the real roots \( x = 1 \) and \( x = 2 \), but the equation

\[ x^2 + 1 = 0 \]

has no real roots. For this latter equation we make up the number \( i \) so that \( i^2 = -1 \), and then this is a root for the equation. We make up complex numbers as \( a + bi \), \( a, b \in \mathbb{R} \), so that, according to Gauss and D’Alembert, every non-constant polynomial equation with constant coefficients has at least one complex root.

Operations with polynomials with complex coefficients and variables mirror those of polynomials with real coefficients and variables. In particular we can define the derivative of a polynomial by the same formula. But do we need it? In the case of real coefficients and variables, we know that between any real zeros of a polynomial lies a real zero of the derivative, and this is for example important to show that the zeros of certain special polynomials, such as the Legendre polynomials, are real. Does a such a result hold in complex? The answer is yes, according to Lucas’ theorem: the zeros of the derivative lie in the convex hull of the zeros of the polynomial.

Moreover, for a complex number \( z \) we can solve the equation \( x^2 = z \) and call the answer \( \sqrt{z} \) (actually one of the two answers). For example \( i = \sqrt{-1} \). Then

\[ -1 = \sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1. \]

What is going on? The answer lies in understanding the definition of the function \( z \mapsto \sqrt{z} \), and for that we need integrals.

Moreover, when you look at algebraic curves:

\[ P(x, y) = 0 \]

they don’t always contain points when you work over real numbers. But they do contain points, and are much nicer, when you work over complex numbers. For example, the group structure on an elliptic curve \( y^2 = x^2 + ax + b \), \( a, b \in \mathbb{R} \) looks quite mysterious when working over \( \mathbb{R} \), but when you work over complex numbers the elliptic curve is a torus and the group structure is the standard Lie group structure of the torus.
1.1.2 Differential equations

The solutions to the differential equation
\[ y'' - y = 0 \]
are \( y(x) = a_1 e^x + a_2 e^{-x} \), but the solutions to the differential equation
\[ y'' + y = 0 \]
are \( y(x) = a_1 \cos x + a_2 \sin x \). But they are also \( b_1 e^{ix} + b_2 e^{-ix} \), if we were to solve the characteristic equation and apply the standard formula. How are the two related? The answer is obtained by passing to complex variables. We can extend the definition of the exponential to complex numbers:
\[ z \mapsto e^z \]
and then define
\[ \cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i}. \]
In fact, these formulas explain the addition formulas in trigonometry:
\[ \cos(a + b) = \cos a \cos b - \sin a \sin b, \quad \sin(a + b) = \sin a \cos b + \cos a \sin b. \]
They are just consequences of \( e^{a+b} = e^a e^b \).

1.1.3 Elliptic integrals

If you integrate polynomials you get polynomials, when you integrate rational function you almost always get rational functions, and when you integrate irrational functions you sometimes get irrational functions.

We have
\[ \int \frac{1}{x} \, dx = \ln x, \]
and it is certainly more interesting to study the inverse function of this, which is \( e^x \). Similarly
\[ \int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x, \]
and it is more interesting to study the inverse function \( \sin x \). And we have seen that it is the complex setting where the two functions are related.

A similar situation is when we work with elliptic integrals, which are of the form
\[ \int R(x, y) \, dx \]
where \( R \) is a rational function and \( y = \sqrt{P(x)} \) with \( P \) a polynomial of degree 3 or 4 without multiple roots. This situation was considered by Lagrange, and studied intensively by Abel and Jacobi. They were the first to have the idea to pass to complex coordinates. Riemann perfected this idea, and considered arbitrary polynomials \( P(x, y) \) that defined \( y \) in terms of \( x \). For the polynomial equation \( P(z, w) = 0 \) Riemann introduced a complex surface (which we now call a Riemann surface) on which \( w(z) \) is univalent. Riemann’s programme was to study integrals of rational functions \( R(z, w) \) along paths in \( \Sigma \). Interesting enough, topology plays a major role in the computation, and the Cauchy theorem that we will study later is a good illustration of this phenomenon.
Chapter 2

Holomorphic functions

2.1 Polynomials and power series

2.1.1 Differentiation of Polynomials

We can of course define formally

\[ \frac{d}{dz} z^n = n z^{n-1}, \quad n = 0, 1, 2, 3, \ldots \]

And we can also define

\[ \frac{d}{dz} z^n = \lim_{h \to 0} \frac{(z + h)^n - z^n}{h}, \]

provided that we have a good definition of limits in the complex plane.

**Definition.** We say that for \( f : D \subset \mathbb{C} \to \mathbb{C} \), we have \( \lim_{w \to z} f(w) = L \) if for every \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( 0 < |w - z| < \delta \), then \( |f(w) - L| < \epsilon \).

So we have a well defined notion of differentiation in the complex variable \( z \) for a polynomial in \( z \). This differentiation satisfies all the nice rules (sum, product, quotient, chain) that differentiation with respect to a real variable satisfies.

How does this relate to the derivatives of two-variable functions? Let \( z = x + iy \). If \( P(z) \) is a polynomial in the complex variable \( z \) we can think of \( P \) as being a polynomial in the real variables \( x, y \) having complex coefficients.

**Example 1.** \( P(z) = z^2 + 3z + 1 \) can be thought of as \( P(x, y) = x^2 - y^2 + 2ixy + 3x + 3iy + 1 \).

How does \( \frac{d}{dz} \) relate to \( \frac{\partial}{\partial z} \) and \( \frac{\partial}{\partial y} \)? It turns out that

\[ \frac{d}{dz} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right). \]

**Example 2.** \( \frac{d}{dz} (z^2 + 3z + 1) = 2z + 3 \). And

\[ \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (x^2 - y^2 + 2ixy + 3x + 3iy + 1) = \frac{1}{2} (2x + 2iy + 3 + 2iy + 2x + 3) = 2x + 2iy + 3, \]

which is the same thing in the other system of coordinates.
In this setting, an interesting question arises. If I give you a two variable polynomial, say \( Q(x, y) = ix^3 + 3ixy^2 - 3x^2y + y^3 + (3 + 2i)x^2 - 3xy \), is this actually a polynomial in \( z \)? To answer this question, we introduce a second “variable” \( \bar{z} \). For this to be a true variable, we have to pass to the complexification of the real 2-dimensional plane, but let us not worry about this and just work formally.

Because \( z = x + iy \) and \( \bar{z} = x - iy \), you can also solve for \( x = \frac{z + \bar{z}}{2} \) and \( y = \frac{z - \bar{z}}{2i} \). So every polynomial in \( x \) and \( y \) can be written uniquely as a polynomial in \( z, \bar{z} \). The fact that the polynomial is in \( z \) only means that when you “differentiate” with respect to \( \bar{z} \) you get 0. We have

\[
\frac{d}{dz} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).
\]

So, the polynomial \( Q(x, y) \) is actually a polynomial in \( z \) if \( \frac{d}{d\bar{z}} Q(x, y) = 0 \).

How do we know that the two formulas for differentiation with respect to \( z \) and \( \bar{z} \) are correct? We can check easily that

\[
\begin{align*}
1 &= \frac{d}{dz} z = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (x + iy) = \frac{1}{2} (1 + 1) = 1, \\
0 &= \frac{d}{dz} \bar{z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (x - iy) = \frac{1}{2} (1 - 1) = 0, \\
1 &= \frac{d}{dz} z = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (x + iy) = \frac{1}{2} (1 - 1) = 0, \\
0 &= \frac{d}{dz} \bar{z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (x - iy) = \frac{1}{2} (1 + 1) = 1,
\end{align*}
\]

and for a general polynomial use the rule for the derivative of the sum and the derivative of the product.

So a 2-variable polynomial with complex coefficients \( P(x, y) \) is actually a polynomial in \( z \) if and only if \( \frac{d}{d\bar{z}} P(x, y) = 0 \). Separate the real and the complex parts of the polynomial, say \( P(x, y) = Q(x, y) + iR(x, y) \), where \( Q, R \) have real coefficients. Then

\[
\frac{d}{d\bar{z}} P(x, y) = \frac{1}{2} \left( \frac{\partial Q}{\partial x} + i \frac{\partial R}{\partial y} \right) (Q + iR)
\]

\[
= \frac{1}{2} \left( \frac{\partial Q}{\partial x} - \frac{\partial R}{\partial y} \right) + i \left( \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial x} \right).
\]

Setting the real and imaginary parts equal to zero we obtain that the necessary and sufficient condition that the 2-real variables polynomial complex coefficients \( P = Q + iR \) to be a polynomial in \( z \) is that

\[
\frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y} \quad \text{and} \quad \frac{\partial Q}{\partial y} = -\frac{\partial R}{\partial x}.
\]

### 2.1.2 Power series

One possible generalization of polynomials, dictated by the necessity to define \( e^z \), \( \sin z \), and \( \cos z \), is given by power series:

\[
\sum_{n=0}^{\infty} a_n z^n, \quad a_n \in \mathbb{C} \text{ for all } n.
\]
2.2. HOLOMORPHIC FUNCTIONS

There are questions to be addressed here. When does the series converge, and when is the resulting function in \( z \) differentiable? Can we differentiate term-by-term?

This story can be read in the book, Chapter III, section 1. I just want to emphasize the theorem about the radius of convergence:

**Theorem 1.** For the power series \( \sum_{n=0}^{\infty} a_n (z - a)^n \), let \( R = (\limsup |a_n|^{1/n})^{-1} \).

(a) If \( |z - a| < R \) the series converges absolutely,
(b) For \( r < R \) the series converges uniformly on the closed disk \( |z - a| \leq r \),
(c) If \( |z - a| > R \) the series diverges.

It is the uniform convergence that allows term-by-term differentiation. Note that the radii of convergence of a series and of the series obtained by term-by-term differentiation are the same, because \( \lim_{n \to \infty} \sqrt[n]{n} = 1 \).

**Example 3.** The series

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}, \quad \cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\]

converge and are differentiable everywhere, and we have

\[
\frac{d}{dz} e^z = e^z, \quad \frac{d}{dz} \sin z = \cos z, \quad \frac{d}{dz} \cos z = -\sin z.
\]

**Proposition 1.** Let \( \sum_{n=0}^{\infty} a_n (z-a)^n \) have the radius of convergence \( R \). If the limit \( \lim_{n \to \infty} |a_n/a_{n+1}| \) exists, then this limit equals \( R \).

**Proof.** This is a consequence of the discrete version of l’Hospital’s theorem:

**Theorem 2.** (Cesáro-Stolz) If \( (x_n)_n \) and \( (y_n)_n \) are two sequences of real numbers with \( (y_n)_n \) strictly positive, increasing, and unbounded, and if

\[
\lim_{n \to \infty} \frac{x_{n+1} - x_n}{y_{n+1} - y_n} = L
\]

then the limit

\[
\lim_{n \to \infty} \frac{x_n}{y_n}
\]

exists and is equal to \( L \).

Apply this theorem to \( x_n = \ln |a_n| \) and \( y_n = n \), and don’t forget to exponentiate.

\[ \square \]

### 2.2 Holomorphic functions

#### 2.2.1 The definition of holomorphic functions

I have a problem with the definitions from the textbook. So here is how I like to define things:

**Definition.** Let \( G \) be an open set in \( \mathbb{C} \). A function \( f : G \to \mathbb{C} \) is called (complex) differentiable at \( z \in \mathbb{C} \) if

\[
\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}
\]

exists and is finite. The limit is denoted by \( f'(z) \) and is called the derivative of \( f \) at \( z \).
**Definition.** Let $G$ be an open set in $\mathbb{C}$. A function $f : G \to \mathbb{C}$ is called holomorphic on $G$ if it is (complex) differentiable at every point in $G$.

**Definition.** Let $G$ be an open set in $\mathbb{C}$. A function $f : G \to \mathbb{C}$ is called analytic on $G$ if it is infinitely (complex) differentiable at every point in $G$ and in a neighborhood of every point it coincides with the Taylor series at that point.

The big result is that **every holomorphic function is analytic**. Surprisingly, the trickiest part to prove is that the derivative of a holomorphic functions is continuous. To avoid having to rephrase the statements later, let us for the moment add to the condition that a function is holomorphic the fact that the derivative is continuous, and later remove from the definition this redundant condition.

Note that as a direct consequence of this assumption we obtain that the real and the imaginary parts of $f$ are $C^1$ functions.

**Theorem 3.** Let $f = u + iv : G \subset \mathbb{R}^2 \to \mathbb{C}$ be such that $u$ and $v$ have continuous partial derivatives. Then $f$ is holomorphic if and only if $u$ and $v$ satisfy the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Using the operator $\frac{d}{dz}$ we have that $f$ is holomorphic if and only if

$$\frac{df}{dz} = 0.$$

The proof is in the book at pages 40-42.

Let $D$ be a connected open set, which is usually called a domain.

**Proposition 2.** If $f : D \to \mathbb{C}$ is holomorphic and $f'(z) = 0$ for all $z \in D$, then $f$ is constant.

**Proof.** Write $f = u + iv$. For every $w$ with $|w| = 1$, we have

$$f'(z) = \lim_{h \to 0} \frac{f(z + hw) - f(z)}{hw} = \lim_{h \to 0} \frac{u(z + hw) + iv(z + hw) - u(z) - iv(z)}{h} \cdot \frac{1}{w} = 0.$$

This means that the directional derivatives of $u$ and $v$ at any point are zero.

Let $z, w \in D$ such that the line segment $[z, w]$ is in $D$. We will show that $f(z) = f(w)$. Restrict $u$ to $[z, w]$. Then we have a one variable function whose derivative is identically equal to zero on an interval. It follows that $u(z) = u(w)$. Similarly $v(z) = v(w)$.

Finally, if we fix $z_0 \in D$ then the set $\{w \mid f(w) = f(z_0)\}$ is both open and closed in $D$. So it must be a connected component of $D$. It is therefore equal to $D$.

**Proposition 3.** The derivative satisfies:

1. $(f + g)' = f' + g'$,
2. $(fg)' = f'g + fg'$,
3. $(f/g)' = (f'g - fg')/g^2$,
4. $(f \circ g)' = (f' \circ g)g'$ (for a proof of this see page 34 in the book).

**Proposition 4.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$. Then
1. the two series have the same radius of convergence,
2. \( f' = g \).

**Proof.** Pages 35–37.

**Proposition 5.** If we define
\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},
\]
then
1. \( e^{z+w} = e^z e^w \) for all \( z, w \in \mathbb{C} \);
2. \( e^z = e^{\text{Re}z} (\cos \text{Im}z + i \sin \text{Im}z) \);
3. For every \( w \in \mathbb{C} \) there is \( z \in \mathbb{C} \) such that \( e^z = w \) if and only if \( w \neq 0 \);
4. \( e^{z+2k\pi i} = e^z \), for all \( z \in \mathbb{C}, k \in \mathbb{Z} \).

Define \( \cos z = \frac{(e^{iz} + e^{-iz})}{2}, \sin z = \frac{(e^{iz} - e^{-iz})}{2i} \).

**Proposition 6.** If \( f : G_1 \to G_2 \) is an invertible holomorphic between open sets \( G_1 \) and \( G_2 \), and if \( f'(z) \neq 0 \) for all \( z \), then \( f^{-1} \) is holomorphic as well, and
\[
(f^{-1})' = \frac{1}{f' \circ f^{-1}}.
\]

**Proof.** Page 40. (Note that our \( f \) is the \( g \) from the book, and the \( f \) from the book is our \( f^{-1} \)).

Let
\[
G_k = \mathbb{C} \setminus \{ z \mid \text{Im}z = 2k, \text{Re}z \leq 0 \}, \quad k \in \mathbb{Z}.
\]
For every \( k \),
\[
e^z : \{ z \mid (2k-1)\pi < \text{Im}z < (2k+1)\pi \},
\]
is invertible, so we can define a branch of the natural logarithm by
\[
\log : G_k \to \{ z \mid (2k-1)\pi < \text{Im}z < (2k+1)\pi \}, \quad \log(z) = \ln |z| + i(\arg z + 2k\pi).
\]
For \( k = 0 \) we have the principal branch of the logarithm.

We can define the set \( \Sigma \) by gluing the top part of the slit \( \{ \text{Re}z \leq 0 \} \) of \( G_k \) to the bottom part of the slit \( \{ \text{Re}z \leq 0 \} \) of \( G_{k+1} \). This set is a Riemann surface (we will return to this), and we obtain a one-to-one and onto holomorphic function
\[
\log : \Sigma \to \mathbb{C} \setminus \{0\}.
\]
For every \( z, w \in \mathbb{C} \), we define
\[
z^w = e^{w \log z}.
\]
2.2.2 Holomorphic maps as transformations, Möbius transformations

**Definition.** A $C^1$ path in $\mathbb{C}$ is a $C^1$ function $\gamma: [a, b] \subset \mathbb{R} \to \mathbb{C}$.

The angle between two $C^1$ paths $\gamma_1$ and $\gamma_2$ that intersect at $z_0 = \gamma_1(t_0) = \gamma_2(t_0)$ is

$$\arg \gamma_1'(t_0) - \arg \gamma_2'(t_0).$$

**Theorem 4.** Assume that $f: G \to \mathbb{C}$ is holomorphic and has continuous derivative, and that $f'(z_0) \neq 0$. Then $f$ preserves angles at $z_0$.

**Proof.** The chain rule

$$\frac{d}{dt} (f \circ \gamma)(t) = f'(\gamma(t)) \gamma'(t) = f'(z_0) \gamma'(t),$$

yields

$$\arg (f \circ \gamma)'(t_0) = \arg f'(z_0) + \arg \gamma'(t_0),$$

So $f$ rotates the tangent to every $C^1$ path through $z_0$ by the same angle $\arg f'(z_0)$.

A map is called conformal if it preserves angles at every point, and also at every point $a$, $\lim_{z \to a} \frac{|f(z) - f(a)|}{|z - a|}$ exists.

**Example 4.** Let $f: \mathbb{C} \to \mathbb{C}^2$, $f(z) = z^2$. Then $f(x, y) = x^2 - y^2 + 2i xy$. So $f$ maps the hyperbolas $x^2 - y^2 = c$, $2xy = d$ into the lines $x = c$, $y = d$. The hyperbolas $x^2 - y^2 = d$ and $xy = c$ intersect at $90^\circ$ angles. This is easy knowing the equation of the tangent at $(x_0, y_0)$ for the two hyperbolas:

$$xx_0 - yy_0 = d$$

$$\frac{1}{2} (xy_0 + yx_0) = c.$$

Now compute the slopes at $(x_0, y_0)$ to be $x_0/y_0$, respectively $-y_0/x_0$ (which are slopes of perpendicular lines). And the lines $x = c$ and $y = d$ are also perpendicular.

**Remark 1.** Holomorphic maps preserve orientation as long as they have nonzero derivative. This means that if we view a holomorphic map as a map from $\mathbb{R}^2 \to \mathbb{R}^2$, then it maps any pair of linearly independent vectors (the tangents to two trajectories that cross at a point) to two other vectors that have the same orientation. This is because both vectors are rotated by the same angle. This means that if a region $D$ whose boundary is a smooth curve $\Gamma$ that is defined by a closed path (loop) that, when traversed has the region on the left, and if $f$ is holomorphic on $D$ and extends continuously to $D \cup \Gamma$, and is such that it has nonzero derivative and is a one-to-one map onto the image, then $f(D)$ is to the left of $f \circ \Gamma$.

**Definition.** A mapping of the form $S(z) = \frac{az + b}{cz + d}$ with $ad - bc \neq 0$ is called a Möbius transformation.

Extend it to the one-point compactification of the plane, which is the Riemann sphere: $\mathbb{C} \cup \{\infty\}$.

$$S: \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}.$$
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The new function is differentiable at \( \infty \), meaning that if you replace \( z \) by \( 1/z \) in the expression of the function, then you get a function that is differentiable at 0. So we have a holomorphic function on the entire Riemann sphere.

Remark: \( c = 0, b = 0 \) defines a dilation, \( c = 0, a = d \) defines a translation, and \( a = 0, b = 1, c = 1, d = 0 \) is the inversion about the unit circle. This should be compared with geometric inversion which is the map \( z \mapsto \overline{1/z} \), and which should be interpreted as the reflection over the unit circle (in the Poincaré model of Lobachevskian geometry it is actually the reflection over a line). The dilation \( z \mapsto re^{i\theta}z \) is geometrically the composition of the dilation by ratio \( r \) and center 0, and the rotation about 0 by \( \theta \).

It is known that in geometry, for translations, dilations, rotations, and inversions:
\[
\{\text{lines, circles}\} \rightarrow \{\text{lines, circles}\}.
\]
The following proposition shows that the same is true for Möbius transformations.

**Proposition 7.** Every Möbius transformation is the composition of translations, inversion, and dilations.

**Proof.** The case \( c = 0 \) is easy. For \( c \neq 0 \), scale the \( a, b, c, d \) such that \( ad - bc = -c \). Now take the composition
\[
z \mapsto z + \alpha \mapsto \frac{1}{z + \alpha} \mapsto \frac{1}{\beta} \cdot \frac{1}{z + \alpha} \mapsto \frac{1}{\beta z + \alpha \beta} + \gamma.
\]
Then \( \beta = c, \alpha = \frac{d}{c}, \gamma = \frac{a}{c} \), and we are done. \( \square \)

A Möbius transformation can have at most 2 fixed points, so it is completely determined by the images of three points.

The map
\[
S_{z_2,z_3,z_4}(z) = \frac{z - z_3}{z - z_4} : \frac{z_2 - z_3}{z_2 - z_4}
\]
is the unique Möbius transformation that satisfies \( S_{z_2,z_3,z_4}(z_2) = 1, S_{z_2,z_3,z_4}(z_3) = 0, S_{z_2,z_3,z_4}(z_4) = \infty \).

The map \( S^{-1}_{w_2,w_3,w_4} \circ S_{z_2,z_3,z_4} \) is the unique Möbius transformation for which \( w_1 \mapsto z_1, w_2 \mapsto z_2, \) and \( w_3 \mapsto z_3 \).

**Corollary 1.** For any pair \( \{C,C'\} \in \{\text{circles, lines}\} \) there is a Möbius transformation that maps \( C \) to \( C' \).

Notation:
\[
(z, z_2, z_3, z_4) := S_{z_2,z_3,z_4}(z).
\]

**Remark 2.** \( z_1, z_2, z_3, z_4 \) lie on a circle or line if and only if \( (z_1, z_2, z_3, z_4) \in \mathbb{R} \), because 0, \( 1, \infty \) lie on a line, and Möbius transformations map lines and circles to lines and circles. If the four points lie on a circle and if \( (z_1, z_2, z_3, z_4) = -1 \) then the quadrilateral formed by the points is called harmonic.
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**Proposition 8.** For any Möbius transformation $T$,

$$(z_1, z_2, z_3, z_4) = (Tz_1, Tz_2, Tz_3, Tz_4).$$


Now we look at geometric inversion in more detail. This is the reflection over a line or a circle. The reflection over the $x$-axis is just $z \mapsto \bar{z}$. We define the reflection over an arbitrary circle passing through $z_2, z_3, z_4$ as $z \mapsto z^*$, where

$$(z^*, z_2, z_3, z_4) = \left(\frac{1}{z}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}\right).$$

We call $z^*$ the *symmetric* of $z$ with respect to the circle.

Geometrically, if the circle over which you are reflecting has radius $r$ and center $z_0$, then $z$ and $z^*$ lie on the same ray originating at $z_0$ and $|z - z_0| = R^2$. Here is a simpler proof than the one in the book (page 51). Because the cross-ratio is invariant under translation and dilation, and that geometric inversion is well behaved under translation and dilation, we may assume that the circle is the unit circle centered at the origin. Using Proposition 8 we have

$$(z^*, z_2, z_3, z_4) = \left(\frac{1}{z}, \frac{1}{z_2}, \frac{1}{z_3}, \frac{1}{z_4}\right).$$

And this is geometric inversion over the unit circle.

**Symmetry Principle.** A Möbius transformation maps a pair of points that are symmetric with respect to a circle, or a line, to a pair of points that are symmetric with respect to the image.

**Orientation Principle.** Möbius transformations preserve orientation.

All holomorphic maps preserve orientation as long as they have nonzero derivative.
Chapter 3

Complex integration

3.1 Cauchy’s theorem and integral formula

3.1.1 Line integrals and the Fundamental Theorem of Calculus

It is easy to integrate one variable functions or real variable, but what does it mean to integrate a function of complex variable? Should we integrate over a curve in the plane, or over a domain in the plane? Surprisingly, the answer is: both! The integration over curves in the plane grew out of the work of Abel, Jacobi, and Riemann on elliptic integrals, and this is the natural way to find antiderivatives.

If \( \gamma(t) = (x(t), y(t)) \), and we are given a function \( f = (u, v) : \mathbb{R}^2 \to \mathbb{R}^2 \), then we can define the line integral of \( f \) on \( \gamma \) by

\[
\int_{\gamma} (udx + vdy) = \int_{\gamma} f = \int_{a}^{b} f(\gamma(t))\gamma'(t)dt.
\]

Only the curve \( \gamma([a, b]) \) and the direction in which it is traveled matter, and not how it is parametrized in the same way as in one variable definite integrals we can change the variable without changing the value of the integral (and because of that).

If \( udx + vdy \) is the differential of a function \( F \), that is

\[
u = \frac{\partial F}{\partial x}, \quad v = \frac{\partial F}{\partial y},
\]

and if the path starts at \( p = \gamma(a) \) and ends at \( q = \gamma(b) \), then

\[
\int_{\gamma} (udx + vdy) = F(q) - F(p).
\]

This is known as the Fundamental theorem of calculus (Leibnitz-Newton). The Fundamental theorem of calculus is the natural way to find antiderivatives, as we will see below.

Now we can pass to complex variables and make the following definition:

**Definition.** Let \( \gamma \) be parametrized by \( z(t) \).

\[
\int_{\gamma} f(z)dz = \int_{\gamma} (udx - vdy) + i \int_{\gamma} (vdx + ugly).$$
Note that this is the same as
\[ \int_{\gamma} f = \int_{a}^{b} f(\gamma(t))\gamma'(t)dt, \]
where now we work with complex numbers.

To obtain this definition, you basically write \( f = u + iv \), and \( dz = dx + idy \) substitute in the formula on the left, and multiply out.

**Lemma 1.** Let \( \gamma \) be the unit circle, traveled counterclockwise. Then
\[ \int_{\gamma} \frac{1}{z}dz = 2\pi i. \]

**Proof.** Parametrize \( z = e^{i\theta} \). \( \square \)

**Corollary 2.** Let \( \gamma \) be a circle of center \( a \), traveled counterclockwise. Then
\[ \int_{\gamma} \frac{1}{z-a}dz = 2\pi i. \]

But if the curve \( \gamma \) does not cross the ray \( \{z \in \mathbb{C} | \text{Im } z \leq 0\} \), then because
\[ \frac{d}{dz} \log z = \frac{1}{z}, \]
we have
\[ \int_{\gamma} \frac{1}{z}dz = \log \gamma(b) - \log \gamma(a), \]
and this is zero if the curve is closed. Can you prove Lemma 1 using this? Indeed, we can. We have the following result.

**Lemma 2.** Assume that \( \gamma : [a,b] \to \mathbb{C} \) is a rectifiable curve in the domain of the holomorphic function \( f \) (whose first derivative is continuous). Then
\[ \int_{\gamma} \frac{df}{dz}dz = f(\gamma(b)) - f(\gamma(a)). \]

**Proof.** Write \( f(z) = u(x,y) + iv(x,y) \). Then
\[
\int_{\gamma} \frac{df}{dz}dz = \int_{\gamma} \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (u + iv)(dx + idy)
= \frac{1}{2} \int_{\gamma} \frac{\partial u}{\partial x}dx + i \frac{\partial u}{\partial x}dy + i \frac{\partial v}{\partial x}dx - \frac{\partial v}{\partial x}dy - \frac{i}{\partial y}dx + \frac{\partial u}{\partial y}dy + \frac{\partial v}{\partial y}dx + \frac{i}{\partial y}dy
\]
Using the Cauchy-Riemann equations
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \]
we obtain that this is equal to
\[ \int_{\gamma} \frac{\partial}{\partial x}(u + iv)dx + \frac{\partial}{\partial y}(u + iv)dy = (u + iv)(\gamma(b)) - (u + iv)(\gamma(a)), \]
where for the last step we used the fundamental theorem of calculus. \( \square \)
3.1. CAUCHY’S THEOREM AND INTEGRAL FORMULA

3.1.2 Green’s Formula

We will look at this fact in a different perspective, but for that we need integrals over 2-dimensional domains. For that we need

**Theorem 5.** (Stokes’ Theorem)

\[ \int_{\partial D} \omega = \int_D d\omega. \]

If \( D \subset \mathbb{C} \) is a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and if \( u \, dx + v \, dy \) is such that \( u, v \) are differentiable with continuous partial derivatives, then

\[ \int_{\partial D} u \, dx + v \, dy = \iint_D \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx \wedge dy, \]

where for the last step we used \( dx \wedge dx = dy \wedge dy = 0 \) and \( dx \wedge dy = -dy \wedge dx \). This is the well known Green’s formula.

Let us see what Green’s formula becomes when switching to complex integration.

\[ \int_{\partial D} f(z) \, dz = \int_{\partial D} (u + iv) \, (dx + idy) = \iint_{\partial D} (udx - vdy) + i \iint_{\partial D} (udy + vdx) \]

\[ = \iint_D \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \, dx \wedge dy + i \iint_D \left( \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \right) \, dx \wedge dy \]

\[ = i \iint_D \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) (u + iv) \, dx \wedge dy = 2i \iint_D \frac{df}{dz} \, dx \wedge dy. \]

Now we can also do the computation

\[ dz \wedge d\bar{z} = (dx + idy) \wedge (dx - idy) = dx \wedge dx - idx \wedge dy + idy \wedge dx - dy \wedge dy = -2idx \wedge dy. \]

We obtain the complex form of Green’s formula:

**Theorem 6.** (Green’s formula) Let \( D \subset \mathbb{C} \) be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let \( \bar{D} = D \cup \partial D \). Let \( f : \bar{D} \to \mathbb{C} \) be a function that is continuous on \( \bar{D} \) and has continuous partial derivatives in \( D \). Then

\[ \int_{\partial D} f(z) \, dz = -\iint_{\bar{D}} \frac{df}{dz} \, dz \wedge d\bar{z}. \]

An example of such an open set is shown in Figure 3.1.

3.1.3 Cauchy’s theorem and Cauchy’s formula

Here are several corollaries to Green’s formula, which are probably the most important results in this course.

**Theorem 7.** (Cauchy’s Theorem) Let \( D \subset \mathbb{C} \) be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let \( \bar{D} = D \cup \partial D \). Let \( f : \bar{D} \to \mathbb{C} \) be a function that is continuous on \( \bar{D} \) and holomorphic (with continuous derivatives) in \( D \). Then

\[ \int_{\partial D} f(z) \, dz = 0. \]
Theorem 8. (The Cauchy-Pompeiu Formula) Let $D \subset \mathbb{C}$ be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let $\overline{D} = D \cup \partial D$. Let $f : \overline{D} \rightarrow \mathbb{C}$ be a function that is continuous on $\overline{D}$ and has continuous partial derivatives in $D$. Let $a \in D$. Then

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-a} \, dz + \frac{1}{2\pi i} \iint_{D} \frac{df}{dz} \, dz \wedge d\bar{z}.$$

Proof. Let $\epsilon > 0$ be small, and take out from $D$ the open disk $B(a, \epsilon)$, as shown in Figure 3.2. The boundary of this is the circle $|z-a| = \epsilon$, which we can parametrize clockwise (so that $D \setminus B(0, \epsilon)$ is to the left) by $a + \epsilon e^{-it}$, $0 \leq t \leq \epsilon$. We write Green’s formula for

$$g(z) = \frac{f(z)}{z-a},$$

and notice that, using the product rule,

$$\frac{dg}{d\bar{z}} = \frac{df}{dz} \frac{1}{z-a}.$$

We have

$$\int_{\partial D} \frac{f(z)}{z-a} \, dz + \int_{0}^{2\pi} \frac{f(a + \epsilon e^{-it})}{a + \epsilon e^{-it} - a} (-i\epsilon e^{-it}) \, dt = -\iint_{D} \frac{df}{dz} \, dz \wedge d\bar{z}.$$

This can be rewritten as

$$\int_{0}^{2\pi} \frac{f(a + \epsilon e^{-it})}{e^{-it}i\epsilon e^{-it}} \, dt = \int_{\partial D} \frac{f(z)}{z-a} \, dz + \iint_{D} \frac{df}{dz} \, dz \wedge d\bar{z},$$

Figure 3.1: Open set in $\mathbb{C}$.

Figure 3.2: Open set in $\mathbb{C}$ with a disk removed.
3.1. CAUCHY’S THEOREM AND INTEGRAL FORMULA

or

\[ i \int_0^{2\pi} f(a + \epsilon e^{-it}) dt = \int_{\partial D} \frac{f(z)}{z - a} \, dz + \iint_D \frac{df}{dz} \, dz \wedge d\bar{z}. \]

Because \( f \) is continuous, when \( \epsilon \to 0 \), \( f(a + \epsilon e^{-it}) \to f(a) \), and so \( \int_0^{2\pi} f(a + \epsilon e^{-it}) dt \to 2\pi f(a) \).

Therefore

\[ f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - a} \, dz + \frac{1}{2\pi i} \iint_D \frac{df}{dz} \, dz \wedge d\bar{z}. \]

Here at the last step we have used a result which is good to emphasize.

**Lemma 3.** Let \( \gamma \) be a rectifiable path, and assume that the continuous functions \( F_n \) converge uniformly to \( F \) on \( \gamma \). Then

\[ \int_{\gamma} F(z) \, dz = \lim_{n \to \infty} \int_{\gamma} F_n(z) \, dz. \]

**Proof.** This follows from

\[ \left| \int_{\gamma} F(z) \, dz - \int_{\gamma} F_n(z) \, dz \right| = \left| \int_{\gamma} (F(z) - F_n(z)) \, dz \right| \leq \int_{\gamma} |F(z) - F_n(z)| \, |dz| \]

\[ \leq \sup_{\gamma} |F(z) - F_n(z)| \int_{\gamma} |dz| = \sup_{\gamma} |F(z) - F_n(z)| \cdot |\text{length}(\gamma)|, \]

where \( |dz| \) is the measure given by the arclength.

**Theorem 9.** (Cauchy’s Formula) Let \( D \subset \mathbb{C} \) be a domain whose boundary is a finite union of rectifiable curves that are oriented so that the domain is to the left, and let \( \bar{D} = D \cup \partial D \). Let \( f : \bar{D} \to \mathbb{C} \) be a function that is continuous on \( \bar{D} \) and holomorphic (with continuous derivatives) in \( D \). Then

\[ f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - a} \, dz. \]

**Proof.** This follows from the previous result, because for holomorphic functions the double integral is zero.

This theorem has a shocking consequence.

**Theorem 10.** Let \( f : D \to \mathbb{C} \) be a holomorphic function (with continuous derivative) in the open set \( D \), and assume that the open disk \( B(a, R) \) lies inside \( D \). Then on \( B(a, R) \), \( f \) coincides with a power series \( \sum_{n=0}^{\infty} a_n z^n \), whose radius of convergence is at least \( R \). Consequently, \( f \) is analytic.

**Proof.** Let \( 0 < r < R \), so that \( B(a, r) \in D \). Set \( \gamma(t) = a + re^{it} \), \( 0 \leq t \leq 2\pi \). Using the Cauchy formula we can write

\[ f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w - z} \, dw, \quad \text{for } z \in B(a, r). \]
CHAPTER 3. COMPLEX INTEGRATION

The Taylor series expansion of \( g_w(z) = f(w)/(w - z) \) around \( a \) is

\[
f(w) = f(w) / (w - a) - (z - a) = \sum_{n=0}^{\infty} f(w)(w - a)^{n+1}(z - a)^n.
\]

Note that

\[
|f(w)| / |w - a|^{n+1} |z - a|^n \leq \sup_{w \in \gamma} |f(w)| (|z - a| / r)^n,
\]

so by the Weierstrass \( M \)-test, the series converges uniformly for \( w \in \gamma \). So the integral commutes with the series expansion

\[
\int_\gamma \sum_{n=0}^{\infty} f(w)(w - a)^{n+1}(z - a)^n \, dw = \sum_{n=0}^{\infty} \left( \int_\gamma f(w) / (w - a)^{n+1} \, dw \right) (z - a)^n,
\]

Adding a factor of \( 1 / 2\pi i \) you obtain that

\[
f(z) = \sum_{n=0}^{\infty} \left( \frac{1}{2\pi i} \int_\gamma f(w) / (w - a)^{n+1} \, dw \right) (z - a)^n,
\]

and the latter is just a power series expansion about \( a \). Write this as \( f(z) = \sum_n a_n z^n \).

For the moment it seems that \( a_n \) depends on \( r \) (that is, it depends on \( \gamma \)). But, since the power series converges uniformly, the summation commutes with differentiation, and consequently

\[
a_n = \frac{f^{(n)}(a)}{n!},
\]

and the latter is clearly independent of \( \gamma \).

**Theorem 11.** (Cauchy’s estimate) Let \( f \) be holomorphic (with continuous derivative) in \( B(a, R) \), and suppose \( |f(z)| \leq M \) in \( B(a, R) \). Then

\[
|f^{(n)}(a)| \leq \frac{n! M}{R^n}.
\]

**Proof.** For \( r < R \),

\[
|f^{(n)}(a)| = n! \left| \frac{1}{2\pi i} \int_\gamma f(w) / (w - a)^{n+1} \, dw \right| \leq \frac{n!}{2\pi} \int_\gamma |f(w)| / |w - a|^{n+1} \, dw \leq \frac{n!}{2\pi} \cdot \frac{M}{r^{n+1}} \cdot 2\pi r = \frac{n!}{r^n}.
\]

Letting \( r \to R \), we obtain the conclusion. \( \square \)

### 3.2 Every complex differentiable function is analytic

Now let us assume that we are given a holomorphic function \( f \) (with continuous derivative) on an open set \( G \) that is homeomorphic to a disk (meaning that it has no holes, which is phrased mathematically as saying that it is simply connected). Then we can find an antiderivative of \( f \) as
3.2. EVERY COMPLEX DIFFERENTIABLE FUNCTION IS ANALYTIC

follows. First, fix a point \( a \in G \). Then for every point \( z \in G \), connect \( a \) to \( z \) by a polygonal line \( \gamma \). Define the antiderivative of \( f \) as

\[
F(z) = \int_{\gamma} f(w)dw.
\]

Of course we can make this construction for any continuous function \( f \), but it only yields an antiderivative when \( f \) is holomorphic (with continuous derivative). This is because of Cauchy’s theorem. Otherwise different paths give different derivatives, and because of that the entire definition falls apart. In fact, from Cauchy’s theorem we only need one condition: that

\[
\int_T f(w)dw = 0
\]

for all triangles \( T \) included in \( G \).

**Theorem 12.** (Morera’s Theorem) Let \( G \) be an open set and let \( f : G \to \mathbb{C} \) be a continuous function such that \( \int_T f(w)dw = 0 \) for every triangular path \( T \) in \( G \). Then \( f \) is analytic in \( G \).

**Proof.** The condition of \( f \) to be analytic is local, so we can just prove it is analytic in the neighborhood of every point. Then we can work inside a disk. Let \( a \) be the center of the disk, and define the antiderivative of \( f \) as above. Note that the antiderivative does not depend on the path. Indeed, if \( \gamma_1 \) and \( \gamma_2 \) are two such paths, then

\[
\int_{\gamma_1 \cup \bar{\gamma}_2} f(w)dw = 0,
\]

(where \( \bar{\gamma}_2 \) is \( \gamma_2 \) traced backwards, because \( \gamma_1 \cup \bar{\gamma}_2 \) can be decomposed into nonskew polygons, and these can be decomposed into triangles, and on the boundary of each triangle is zero, so on each polygon it is zero. Now, for two points \( z \) and \( z_0 \), we have that

\[
F(z) - F(z_0) = \int_{[z_0, z]} f(w)dw,
\]

because to define \( F(z) \) we can use a path whose last segment is \([z_0, z]\). Then

\[
\left| \frac{F(z) - F(z_0)}{z - z_0} - f(z_0) \right| \leq \sup_{w \in [z, z_0]} |f(w) - f(z_0)|,
\]

and the latter goes to 0 as \( z \to z_0 \) since \( f \) is continuous. Therefore \( F \) is holomorphic with continuous derivative (which is \( f \)), and so it is analytic. Its derivative, \( f \), is also analytic.

**Theorem 13.** (Goursat’s Theorem) Let \( G \) be an open set and let \( f : G \to \mathbb{C} \) be a complex differentiable function. Then \( f \) is analytic in \( G \).

**Proof.** All we have to do is show that it satisfies the hypothesis of Morera’s theorem. Again, it suffices to check analyticity on a disk. We argue by contradiction, assuming there is \( f \) that is complex differentiable, but does not satisfy the hypothesis of Morera’s theorem. We can assume that there is a function \( f \) such that there is a triangle of perimeter 1 (just rescale the complex plane to make the perimeter 1) on which the integral of \( f \) is 1 (if not, just multiply \( f \) by the appropriate variable). Divide the triangle by midlines into 4 equal triangles. On one of them the integral has the absolute value at least \( 1/4 \), because the integral on the big triangle is the sum of the integrals
on the small triangles. Repeat. At the nth step there is a triangle of perimeter $1/2^n$ on which the integral has the absolute value at least $1/4^n$. Repeating the process we get a decreasing sequence of closed triangular regions with diameters going to 0. By Cantor’s theorem, the intersection consists of one point, call it $z_0$.

Now for every $\epsilon > 0$, there is $\delta > 0$ such that if $|z - z_0| < \delta$,

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon|z - z_0|.$$ 

Note that $f(z_0) + f'(z_0)(z - z_0)$ is analytic, so its integral on any triangle is zero. Then for a triangle that lies within the disk of radius $\delta$ around $z_0$,

$$\left| \int_T f(z)dz \right| = \left| \int_T (f(z) - f(z_0) - f'(z_0)(z - z_0))dz \right| \leq \epsilon \int_T |z - z_0||dz| \leq \frac{\epsilon}{4^n+1}.$$ 

But for a triangle from the decreasing sequence that lies in the disk of radius $\delta$, the first integral is greater than $1/4^n$. And this is absurd. Hence the conclusion. 

\[ \square \]

### 3.3 The winding number and the generalization of Cauchy’s Formula

**Definition.** The winding number (also known as the index) of a rectifiable curve with respect to a point is $(\theta(1) - \theta(0))/2\pi$, where the point has coordinates $(x_0, y_0)$ and the curve is parametrized in polar coordinates centered at $(x_0, y_0)$ as $(x_0 + r(t)\cos\theta(t), y_0 + r(t)\sin\theta(t))$.

We denote the winding number of $\gamma$ with respect to $a$ by $n(\gamma; a)$.

**Proposition 9.**

$$n(\gamma; a) = \frac{1}{2\pi i} \int_{\gamma} (z - a)^{-1}dz.$$ 

As a corollary of Theorem 9 we obtain

**Theorem 14.** (General form of Cauchy’s formula) Let $G$ be an open subset of the plane and let $f : G \to \mathbb{C}$ be an analytic function. If $\gamma_1, \gamma_2, \ldots, \gamma_n$ are closed rectifiable curves in $G$ such that $n(\gamma_1; w) + n(\gamma_2; w) + \cdots + n(\gamma_n; w) = 0$ for all $w \in \mathbb{C}\setminus G$, then for $a \in G - \bigcup_{k=1}^m \gamma_k$,

$$f(a) \sum_{k=1}^m n(\gamma_k; a) = \sum_{k=1}^m \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(z)}{z - a}dz.$$ 

Let $S_1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$.

**Definition.** Two loops $\gamma_0, \gamma_1 : S^1 \to G$ are homotopic if there is a continuous map $H : S^1 \times [0, 1] \to G$ such that $H|S^1 \times \{j\} = \gamma_j$, $j = 0, 1$.

**Definition.** Two paths $\gamma_0, \gamma_1 : [0, 1] \to G$ are homotopic relative to the endpoints if there is a continuous map $H : [0, 1] \times [0, 1] \to G$ such that $H|[0, 1] \times \{j\} = \gamma_j$, $j = 0, 1$, and $H|[\{0\} \times [0, 1]$ and $H|[\{1\} \times [0, 1]$ are constant.

**Theorem 15.** (General form of Cauchy’s formula) If two loops are homotopic or two paths are homotopic relative to the endpoints, then the integrals of a holomorphic function on them are equal.
For people who know algebraic topology, we have the following reformulations, which are more appropriate generalizations, given the relationship to Green’s theorem and, implicitly to de Rham cohomology. One should note that the homological formulation generalizes to Riemann surfaces, and for compact surfaces it relates to Hodge theory and the construction of Jacobian varieties.

**Theorem 16.** Let $G$ be an open subset of the plane and let $f : G \to \mathbb{C}$ be an analytic function. Let $\gamma$ be a collection of finitely many rectifiable curves in $G$ whose homology class in $H_1(G, \mathbb{Z})$ is zero. Let $a \in G$ and assume that the homology class of $\gamma$ in $H_1(\{\mathbb{C}\}\{a\}, \mathbb{Z})$ is $n$ under the isomorphism $H_1(\{\mathbb{C}\}\{a\}, \mathbb{Z}) \cong \mathbb{Z}$ that maps a circle centered at $z$ and oriented counterclockwise to 1. Then

$$nf(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - a} dz.$$

**Theorem 17.** Let $G$ be an open subset of the plane and let $f : G \to \mathbb{C}$ be an analytic function. Let $\gamma$ be a collection of finitely many rectifiable curves in $G$ whose homology class in $H_1(G, \mathbb{Z})$ is zero. Then $\int_\gamma f(z) dz = 0$.

**Example 5.** Let $\gamma$ consist of the circles $\{z \in \mathbb{C} \mid |z - 1 - i| = 2\}$, $\{z \in \mathbb{C} \mid |z - 2| = 3\}$, and $\{z \in \mathbb{C} \mid |z - 3| = 2\}$, the first two oriented counterclockwise, the third oriented clockwise. Let us find

$$\int_\gamma \frac{1}{z^2 + 1} dz.$$

We have

$$\int_\gamma \frac{1}{z^2 + 1} dz = \frac{1}{2i} \left( \int_\gamma \frac{dz}{z - i} + \int_\gamma \frac{dz}{z + i} \right) = \frac{1}{2i} (2\pi i + 4\pi i) = 3\pi.$$
Chapter 4

Zeros and poles; classification of singularities

4.1 Zeros of a holomorphic function; the fundamental theorem of algebra

**Definition.** A number \( a \in \mathbb{C} \) is called a zero of a holomorphic function \( f \) if \( f(a) = 0 \).

**Theorem 18.** (Liouville’s theorem) If \( f : \mathbb{C} \to \mathbb{C} \) is a bounded holomorphic function, then \( f \) is constant.

**Proof.** Page 77

**Theorem 19.** (The Gauss-d’Alembert fundamental theorem of algebra) Every nonconstant polynomial with complex coefficients has at least one complex zero.

**Proof.** page 77

**Definition.** If \( f : G \to \mathbb{C} \) is holomorphic and \( a \in G \) is so that \( f(a) = 0 \), we say that \( a \) is a zero of multiplicity \( m \) (where \( m \) is a positive integer) if there is a holomorphic function \( g : G \to \mathbb{C} \) such that \( g(a) \neq 0 \) and \( f(z) = (z - a)^mg(z) \).

Because of the power series expansion about every point, every zero of a non-identically zero holomorphic function has some multiplicity. Indeed, we have the following result:

**Theorem 20.** Let \( G \) be a connected open set and let \( f : G \to \mathbb{C} \) be a holomorphic function. The following are equivalent

(a) \( f \) is identically equal to zero;

(b) there is a point \( a \) in \( G \) at which all derivatives of \( f \) are zero;

(c) the set of zeros of \( f \) has a limit point in \( G \).

**Proof.** page 78

**Example 6.** For \( f(z) = \sin z \), 0 is a zero of multiplicity 1. Because of periodicity, every zero of \( \sin z \) is of the first order.
Example 7. The situation should be contrasted with the real case. The function \( f(x) = x^{3/2} \) does not have a zero of integer multiplicity at zero. Note that at zero it is not analytic, while it is analytic at every other point. You see that \( f(z) = z^{3/2} \) is not holomorphic everywhere, you have to remove \( \{ z \mid \text{Re}z \leq 0 \} \) from its domain.

Note that holomorphic functions have only isolated zeros, and consequently, if two holomorphic functions coincide on a set that has an accumulation point, then they are equal.

4.2 Poles; Meromorphic functions

The concept of zero of a holomorphic function comes naturally associated with that of a pole. In short, \( a \) is a zero of \( f \) if it is a pole of \( 1/f \) and it is a pole of \( f \) if it is a zero of \( 1/f \). For example 0 is a zero of second order of \( f(z) = z^2 \), and so it is a pole of second order of \( g(z) = z^{-2} \). It is not just the elegance of formulation that brings these two notions together, they also appear together in various situations, for example every meromorphic function on a compact Riemann surface has as many zeros as poles (multiplicity counted). We will see this later.

There is a problem with identifying poles, in that if \( a \) is a pole of \( f \) then naturally \( f(a) \) is not defined. So then \( 1/f(a) \) is not a priori defined, so then how can \( a \) be a zero of \( f(a) \). The fact is that \( 1/f \) can be extended to \( a \) as well. For this we need the notion of removable singularity. For that we first need the notion of isolated singularity.

**Definition.** A function \( f \) has an isolated singularity at \( a \) if its domain is an open set in \( \mathbb{C} \setminus \{a\} \) that contains a set of the form \( \{ z \mid 0 < |z-a| < R \} \).

**Definition.** A function \( f \) has a removable singularity at \( a \) if \( a \) is an isolated singularity of \( f \) and there is a holomorphic function \( g \) on some \( B(a, R) \), such that \( f(z) = g(z) \) for \( 0 < |z-a| < R \).

**Theorem 21.** An isolated singularity \( a \) if \( f \) is removable if and only if \( \lim_{z \to a} (z-a)f(z) = 0 \).

**Proof.** pages 103-104

In particular, if \( \lim_{z \to a} f(z) \) exists, the singularity is removable. This is not an obvious fact, and not that in the real setting, 0 is not a “removable singularity” of \( f(x) = |x| \), despite the fact that \( f \) is analytic everywhere but at zero and \( \lim_{z \to 0} f(x) = 0 \).

**Example 8.** For \( f(z) = \frac{\sin z}{z} \), 0 is a removable singularity.

Based on this, we can define

**Definition.** An isolated singularity \( a \) is a pole for \( f \) if \( \lim_{z \to a} |f(z)| = \infty \).

An alternative definition is that \( a \) is a removable singularity for \( 1/f \) and \( (1/f)(a) = 0 \). A corollary of the above discussion is the following.

**Proposition 10.** If \( f : G \setminus \{a\} \to \mathbb{C} \) is holomorphic (where \( G \) is open) with a pole at \( z = a \), then there is a holomorphic function \( g : G \to \mathbb{C} \) such that \( g(a) \neq 0 \) and

\[
f(z) = \frac{g(z)}{(z-a)^m}, \quad \text{for all } z \in G.
\]

**Proof.** Find a disk \( B(a, R) \) on which \( f \) has no zeros. Then \( 1/f \) is analytic on \( B(a, R) \setminus \{a\} \), and has a removable singularity at \( a \). Write \( (1/f) = (z-a)^m h(z) \) with \( h(a) \neq 0 \). Let \( g = 1/h \). Then \( f(z) = \frac{g(z)}{(z-a)^m} \) on \( B \setminus \{a\} \), so \( g(z) = (z-a)^m f(z) \) on this set. But this formula allows us to extend \( g \) to the entire \( G \). We are done.
Definition. If $f$ has a pole at $a$ and $m$ is the smallest integer so that $f(z)(z-a)^m$ has a removable singularity at $a$, then $m$ is called the order of the pole.

Proposition 11. If $f$ is defined on $B(a,R)\{a\}$ and has a pole of order $m$ at $a$, then on $B(a,R)\{a\}$

$$f(z) = \sum_{k=-m}^{\infty} a_k(z-a)^k.$$  

The part $\sum_{k=-m}^{-1} a_k(z-a)^k$ is called the singular part of $f$.

I have said before that zeros and poles should be studied together, and we have seen before that poles are defined using zeros. But there is a more profound reason why poles and zeros should be studied together. For that we should add the point at infinity. First, a definition.

Definition. A meromorphic function on $G$ is a function that is holomorphic on $G$ except of some poles.

Now let us add the point at infinity to the range. We say that $f : G \to \mathbb{C} \cup \{\infty\}$ is holomorphic if $f$ is holomorphic in some neighborhood of every point where $f$ takes a finite value and $1/f$ is holomorphic in some neighborhood of every point where $f$ takes the value $\infty$.

Then a meromorphic function is just a holomorphic function $f : G \to \{\infty\}$. Now let us consider $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. We say that $f$ is holomorphic at $\infty$ if $f(1/z)$ is holomorphic at $0$. Now we have included the concept of a meromorphic function in that of holomorphic functions on Riemann surfaces. This is yet another reason why to study Riemann surfaces.

Example 9. Let $p(z)$ be a polynomial of $n$th degree; then it has $n$ zeros multiplicities counted. Let us look at the order of the pole at $\infty$. We write

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0.$$  

Then after changing $z \to 1/z$, the order of the pole of $p(z)$ at $\infty$ is the same as the order of the pole of

$$p(1/z) = \frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + a_0.$$  

at $0$. And this is $n$. As a corollary, the number of the zeros on the Riemann sphere is equal to the number of the poles.

Let me explain the big picture. If we have two Riemann surfaces $X$ and $Y$, and $f : X \to Y$ holomorphic, then for $w \in Y$ the equation $f(z) = w$ has the same number of solutions regardless of $w$, with multiplicities counted. The number of solutions just counts the number of times $f$ "wraps" $X$ around $Y$. A particular case is $X = Y = \mathbb{C} \cup \{\infty\}$, $w = 0, \infty$. Then this just says that for a meromorphic function, the number of zeros equals the number of poles, multiplicities counted. We will return to this when we talk about Riemann surfaces. For the moment let us prove a particular case. For it we need the logarithmic derivative:

$$\frac{d}{dz} \log f = \frac{f'}{f}.$$  

Note that $\frac{d}{dz} \log(fg) = \frac{d}{dz} \log f + \frac{d}{dz} \log g$ and $\frac{d}{dz} \log(f/g) = \frac{d}{dz} \log f - \frac{d}{dz} \log g$. 


Theorem 22. (The Argument Principle) Let \( f : G \to \mathbb{C} \) be meromorphic with zeros and poles counted with multiplicity. If \( \gamma \) is a closed, rectifiable, curve (consisting of maybe several closed curves) that lies in \( G \) and bounds an open subset \( D \) in \( G \) to its left and does not pass through the zeros and poles of \( f \), then

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \text{number of zeros in } D - \text{number of poles in } D.
\]

Proof. Let \( \gamma \) bound the surface \( D \). Let the zeros be \( z_1, z_2, \ldots, z_m \) and poles \( p_1, p_2, \ldots, p_n \). Let \( B(z_j, \epsilon_j), j = 1, 2, \ldots, m \), and \( B(p_k, \delta_k), k = 1, 2, \ldots, n \) be disjoint disks in \( D \) that don’t contain other zeros or poles inside or on the boundary, let their boundaries be \( \gamma_j, j = 1, 2, \ldots, m \), \( \gamma_k \), \( k = 1, 2, \ldots, n \) oriented counterclockwise. Then \( D \setminus (\bigcup_{j} B(z_j, \epsilon_j) \cup \bigcup_{k} B(p_k, \delta_k)) \) is bounded by \( \gamma \cup (\bigcup_{j} \gamma_j^\prime) \cup (\bigcup_{k} \gamma_k^\prime) \). In this set \( f'/f \) is holomorphic, and is continuous on its closure, so by Cauchy’s theorem its integral is zero. Hence

\[
\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_j \frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z)} \, dz + \sum_k \frac{1}{2\pi i} \int_{\gamma_k} \frac{f'(z)}{f(z)} \, dz.
\]

Note that if \( f \) has a zero of order \( m_j \) at \( z_j \), then there is a holomorphic function on \( B(z_j, \epsilon) \) such that \( f(z) = (z - z_j)^{m_j} g(z) \) where \( g \) is holomorphic and nonzero in \( B(z_j, \epsilon) \). Then

\[
\frac{f'(z)}{f(z)} = \frac{m_j}{z - z_j} + \frac{g'(z)}{g(z)}.
\]

Consequently,

\[
\frac{1}{2\pi i} \int_{\gamma_j} \frac{f'(z)}{f(z)} \, dz = \frac{1}{2\pi i} \int_{\gamma_j} \frac{m_j}{z - z_j} \, dz + \frac{1}{2\pi i} \int_{\gamma_j} \frac{g'(z)}{g(z)} \, dz = m_j + 0 = m_j.
\]

Similarly if \( f \) has a pole of order \( m_k \) at \( p_k \), then there is a holomorphic function on \( B(p_k, \epsilon) \) such that \( f(z) = (z - z_k)^{-m_k} g(z) \) where \( g \) is holomorphic and nonzero in \( B(p_k, \epsilon) \). Then

\[
\frac{f'(z)}{f(z)} = -\frac{m_k}{z - z_j} + \frac{g'(z)}{g(z)}.
\]

Consequently,

\[
\frac{1}{2\pi i} \int_{\gamma_k} \frac{f'(z)}{f(z)} \, dz = -\frac{1}{2\pi i} \int_{\gamma_k} \frac{m_k}{z - p_k} \, dz + \frac{1}{2\pi i} \int_{\gamma_k} \frac{g'(z)}{g(z)} \, dz = -m_k + 0 = -m_k.
\]

Adding we obtain the formula. For a slightly different proof, see page 123 in the book. \( \square \)

Theorem 23. (Rouché’s Theorem) Suppose \( f \) and \( g \) are meromorphic in a neighborhood of \( \bar{B}(a, R) \) with no zeros or poles on the circle \( \gamma = \{ z \mid |z - a| = R \} \). If \( Z_f, Z_g, P_f, P_g \) are the number of zeros, respectively poles of \( f \) and \( g \), inside \( \gamma \) counted with multiplicity, and if

\[
|f(z) + g(z)| < |f(z)| + |g(z)| \text{ for all } z \in \gamma,
\]

then

\[
Z_f - P_f = Z_g - P_g.
\]
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Proof. Page 125.

The Rouche’s theorem has a beautiful consequence.

**Theorem 24.** Let \( a \) be a zero of multiplicity \( m \) of \( f \) such that \( f \) has no other zeros or poles in \( B(a, \epsilon) \). Then there is \( \delta > 0 \) such that for every \( \alpha \in B(a, \delta) \), \( f(z) = \alpha \) has \( m \) zeros (multiplicities counted) in \( B(a, \epsilon) \).

**Proof.** Let \( R < \epsilon \). Since \( f \) is continuous, \( t \mapsto |f(a + Rei^{t})| \) has a minimum \( \delta \) on \( a + Rei^{t}, 0 \leq t \leq 2\pi \). For \( |\alpha| < \delta \), apply Rouche’s theorem to \( f \) and \( g = f - \alpha \).

As a corollary we obtain:

**Theorem 25.** (Open Mapping Theorem) A holomorphic function maps open sets to open sets.

**Proof.** Let \( w \in f(G) \). Then by the previous theorem there is \( \delta > 0 \) such that \( B(w, \delta) \in f(G) \). Done.

We conclude this section with a result about meromorphic functions on the Riemann sphere.

**Theorem 26.** Let \( f : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\} \) be a holomorphic function (in other words \( f : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \) is meromorphic). Then there exist polynomials \( P(z) \) and \( Q(z) \) such that

\[
f(z) = \frac{P(z)}{Q(z)} \text{ for all } z.
\]

**Proof.** Around every finite pole \( \alpha_k, k = 1, 2, \ldots, p \),

\[
f(z) = \sum_{n \geq -m_k} a_{n,k}(z - \alpha_k)^n = R_k(z) + \sum_{n \geq 0} a_{n,k}(z - \alpha_k)^n = R_k(z) + g_k(z),
\]

where \( R_k \) is rational, and \( g_k(z) \) is in a neighborhood of \( \alpha_k \). Also, at infinity

\[
f(z) = \sum_{n = -\infty}^{m_\infty} a_{n,\infty}z^n = R_\infty(z) + \sum_{n \leq 0} z^n = R_\infty(z) + g_\infty(z),
\]

where \( R_\infty(z) \) is a polynomial and \( g_k \) is holomorphic in a neighborhood of \( \infty \). Then

\[
f(z) - \sum_{k=1}^{p} R_k(z) = R_\infty(z) \]

is holomorphic on \( \mathbb{C} \cup \{\infty\} \) and takes values in \( \mathbb{C} \), so by Liouville’s theorem, it is a constant function. Let \( c \) be its value. Then

\[
f(z) = \sum_{k=1}^{p} R_k(z) + R_\infty(z) + c,
\]

which is a rational function.
4.3 Essential singularities

**Definition.** An isolated singularity is essential if it is neither a zero nor a pole.

**Example 10.** $0$ is an essential singularity of $f(z) = e^{1/z}$. Indeed,

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{1}{n!} z^n.$$

This example prompts us to look at Laurent series. And as we will see, Laurent series cover all singularities.

**Definition.** A Laurent series is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n z^n.$$

**Theorem 27.** (Laurent series development) Let $f$ be analytic in the annulus $R_1 < |z-a| < R_2$ ($0 \leq R_1 < R_2 \leq \infty$). Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n,$$

where the convergence is absolute and uniform in any annulus $r_1 \leq |z-a| < r_2$, and the coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz, \quad \gamma(t) = a + re^{it}, \quad 0 \leq t \leq 2\pi, R_1 < r < R_2.$$

**Proof.** Denote $A = \{z \mid R_1 < |z-a| < R_2\}$. Choose $\rho_1, \rho_2$ such that $R_1 < \rho_1 < \rho_2 < R_2$, and define $\gamma_1(t) = a + \rho_1 e^{it}$, $\gamma_2 = a + \rho_2 e^{it}, 0 \leq t \leq 2\pi$. Let $\gamma = \gamma_1 \cup \gamma_2$ (where the bar means that we reverse orientation of the path). Then Cauchy’s formula yields

$$f(z) = \int_{\gamma} \frac{f(w)}{w-z} dw, \text{ for } \rho_1 < |z-a| < \rho_2.$$

On $r_1 < |z-a| < r_2$, define

$$f_1(z) = \int_{\gamma_1} \frac{f(w)}{w-z} dz, \quad f_1(z) = \int_{\gamma_2} \frac{f(w)}{w-z} dz.$$

Note that $f_1$ and $f_2$ are holomorphic in $\rho_1 < |z-a| < \rho_2$ and $f = -f_1 + f_2$. Note that for $w \in \gamma_1$, $|w-a| < |z-a|$, while for $w \in \gamma_2, |w-a| > |z-a|$. Thus for $w \in \gamma_1$,

$$\frac{1}{w-z} = -\frac{1}{(z-a)-(w-a)} = -\sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}} = \sum_{n=-\infty}^{-1} \frac{(z-a)^n}{(w-a)^{n+1}},$$

and for $w \in \gamma_2$,

$$\frac{1}{w-z} = \frac{1}{(w-a)-(z-a)} = -\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}.$$
The two series converge uniformly and absolutely for $\rho_1 < r_1 \leq |z - a| \leq r_2 < \rho_2$, and using Fubini (summation commutes with integration) we have

$$f(z) = -\frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \frac{f(w)}{(w-a)^{n+1}} dw (z-a)^n + \sum_{n=0}^{\infty} \left( \frac{f(w)}{(w-a)^{n+1}} dw \right) (z-a)^n,$$

as desired. (Note that in the first sum the two minus signs in the first sum, one from $-f_1$ and one from the series expansion, cancel.)

So a singularity is removable if for all negative indices $n$, $a_n = 0$, is a pole if for all but finitely many negative indices $n$, $a_n = 0$, and essential if for infinitely many negative indices $n$, $a_n \neq 0$.

**Theorem 28.** (Casorati-Weierstrass theorem) If the holomorphic function $f$ has an essential singularity at $a$ then for every $\delta > 0$, $f(\{z \mid 0 < |z - a| < \delta\})$ is dense in $\mathbb{C}$.

**Proof.** Assume that for some holomorphic function $f$ and some annulus $A = \{z \mid 0 < |z - a| < \delta\}$ this is not true. Then there is a number $c \in \mathbb{C}$ and some $\epsilon > 0$ such that $|f(z) - c| > \epsilon$ for all $z \in A$. Then $1/(f - c)$ is bounded in $A$, so $a$ is a removable singularity for it. This means that either $a$ is a removable singularity for $f - c$ or $a$ is a pole of $f - c$ (if when you remove the singularity $1/(f - c)$ is extended to have value 0 at $a$). Consequently $a$ is either a removable singularity or a pole of $f$ itself, a contradiction. The conclusion follows.

### 4.4 Residues

#### 4.4.1 How to find residues

If $f$ has an isolated singularity at $a$, let its Laurent series expansion around $a$ be

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n.$$

The residue of $f$ is $\text{Res}(f; a) = a_{-1}$.

**Theorem 29.** (Residue Theorem) Let $f$ be analytic in the open set $G$ except for the isolated singularities $\alpha_1, \alpha_2, \ldots, \alpha_m$. If $\gamma$ is a closed rectifiable curve in $G$ which does not pass through any of the singularities of $f$, and if $\gamma$ bounds an open set in $G$ to its left then

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{k=1}^{n} n \text{Res}(f; \alpha_k).$$

More generally, if $\gamma = 0 \in H_1(G, \mathbb{R})$, then

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{k=1}^{n} n(\gamma; \alpha_k) \text{Res}(f; \alpha_k).$$

**Proof.** page 112, 118
• If \( f \) has a pole of order \( m \) at \( a \). Let \( g(z) = (z - a)^m f(z) \). Then
\[
\text{Res}(f; a) = \frac{1}{(m-1)!} g^{(m-1)}(a).
\]

• If \( f \) has a simple pole at \( a \),
\[
\text{Res}(f; a) = \lim_{z \to a} (z - a) f(z).
\]

• If \( f \) has a double pole at \( a \), then
\[
a_2 = \lim_{z \to a} (z - a)^2 f(z)
\]
and
\[
\text{Res}(f; a) = \lim_{z \to a} (z - a) \left( f(z) - \frac{a_2}{(z - a)^2} \right)
\]
Alternatively, \( \text{Res}(f; a) \) is the value at \( a \) of the function
\[
\frac{d}{dz} [(z - a)^2 f(z)].
\]

• If \( f, g \) are analytic, and \( g \) has a simple zero at \( a \), then
\[
\text{Res}(f; g; a) = f(a) g'(a).
\]

4.4.2 Computations of integrals using residues

Example 11. Let us compute
\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.
\]
Let \( R > 0 \), and consider the contour \( \gamma = \gamma_1 \cup \gamma_2 \), where \( \gamma_1(t) = t, -R \leq t \leq R \), and \( \gamma_2(t) = Re^{it}, 0 \leq t \leq \pi \). Note that \( \gamma \) bounds a semidisk, and inside \( \gamma \) there is only one isolated singularity \( i \), which is a pole of order 1. Thus
\[
2\pi ia_{-1} = \int_{\gamma_1} \frac{dz}{1 + z^2} = \int_{\gamma_1} \frac{dz}{1 + z^2} + \int_{\gamma_2} \frac{dz}{1 + z^2}.
\]
Set \( f(z) = 1 \), and \( g(z) = 1 + z^2 \). Then
\[
\text{Res} \left( \frac{f}{g}; i \right) = \frac{f(i)}{g'(i)} = \frac{1}{2i}
\]
Also, note that
\[
\lim_{R \to \infty} \int_{\gamma_2} \frac{dz}{1 + z^2} = \lim_{R \to \infty} \int_{0}^{\pi} \frac{Re^{it}}{1 + Re^{2it}} dt = \lim_{R \to \infty} \frac{1}{R} \int_{0}^{\pi} \frac{e^{it}}{1 + e^{2it}} dt = 0.
\]
Hence
\[
\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{\gamma} \frac{dz}{1 + z^2} = 2\pi i \frac{1}{2i} = \pi.
\]
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Example 12. Evaluate

\[ \int_0^{2\pi} \frac{1}{a + \cos \theta} \, d\theta, \quad a > 1. \]

Set \( z = e^{i\theta} \), then \( \cos \theta = \frac{z + \frac{1}{z}}{2} \), and \( d\theta = \frac{1}{iz} \, dz \). The integral becomes

\[ \frac{2}{i} \int_\gamma \frac{1}{z^2 + 2az + 1} \, dz, \]

where \( \gamma \) is the unit circle. The only singularity inside the unit circle is \( -a + \sqrt{a^2 - 1} \), which is a simple pole whose residue is obtained by setting \( f = 1, \quad g = z^2 + az + 1 \), and then

\[ \text{Res} \left( \frac{f}{g} ; -a + \sqrt{a^2 - 1} \right) = \frac{f(-a + \sqrt{a^2 - 1})}{g'(-a + \sqrt{a^2 - 1})} = \frac{1}{2a} \sqrt{a^2 - 1}. \]

So the value of the integral is

\[ \frac{2}{i} \int_\gamma \frac{1}{z^2 + 2az + 1} \, dz = \frac{2\pi i}{2a} \sqrt{a^2 - 1} = \frac{2\pi}{\sqrt{a^2 - 1}}. \]

As a corollary we obtain

\[ \int_0^{2\pi} \frac{1}{w + \cos \theta} \, d\theta = \frac{2\pi}{\sqrt{w^2 - 1}}, \quad w \in \mathbb{C} \setminus [-1, 1] \]

where \( \sqrt{w^2 - 1} \) is defined so that it is positive on the real axis. Indeed, both sides are holomorphic functions and they coincide on the set \( \mathbb{R} \setminus [-1, 1] \) which contains accumulation points, so they must coincide everywhere.

Example 13. Let us compute

\[ \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx. \]

We compute instead

\[ \int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} \, dx. \]

Now let \( f(z) = \frac{e^{iz}}{1 + z^2} \), and consider the contour \( \gamma_1 \cup \gamma_2 \), where \( \gamma_1(t) = t, \quad -R \leq t \leq R \) and \( \gamma_2(t) = Re^{it}, \quad 0 \leq t \leq \pi \). Note that on \( \gamma_2(t) \), \( |f(z)| \leq 1/(R^2 - 1) \), so

\[ \int_{\gamma_2} f(z) \, dz \to 0, \quad \text{when} \ R \to \infty \]

On the other hand,

\[ \int_{\gamma_1 \cup \gamma_2} f(z) \, dz = 2\pi i \text{Res} f(i) = 2\pi i \frac{e^{-1}}{2i} = \frac{\pi}{e}. \]

Letting \( R \to \infty \) we obtain that

\[ \int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} \, dx = \frac{\pi}{e}. \]
Example 14. Compute
\[ \int_0^1 \frac{dx}{\sqrt{x(1-x)}}. \]
Note that this is an improper integral. Let \( \epsilon, \theta > 0 \) be small, and consider the contour which is the union of the curves:
\[
\begin{align*}
\gamma_1(t) &= t + \epsilon \sin \theta, \epsilon \cos \theta \leq t \leq 1 - \epsilon \cos \theta, \\
\gamma_2(t) &= 1 + e^{it}, -\pi + \theta \leq t \leq \pi - \theta, \\
\gamma_3(t) &= 1 - t - \epsilon \sin \theta, \epsilon \cos \theta \leq t \leq 1 - \epsilon \cos \theta, \\
\gamma_4(t) &= e^{-it}, \theta \leq t \leq 2\pi - \theta.
\end{align*}
\]
Then \( \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \) is a closed contour. Consider the analytic function
\[ f : \mathbb{C} \setminus [0,1] \to \mathbb{C}, f(z) = \frac{1}{\sqrt{z(1-z)}}. \]
Note that \( f \) is analytic at infinity, meaning that at infinity it has a Laurent series expansion
\[ \sum_{n \leq -1} a_n z^n. \]
By letting \( \epsilon, \theta \to 0 \), we obtain that
\[ \int_0^1 \frac{dx}{\sqrt{x(1-x)}} + (-1) \int_1^0 \frac{dx}{\sqrt{x(1-x)}} = 2\pi i \text{Res}(f; \infty). \]
Note that the \(-1\) factor in front of the second integral comes from that fact that we pick a phase of \(-1\) in the square root as we go around the zero. Note also that
\[ \frac{1}{\sqrt{z(1-z)}} = \frac{1}{z} \left( \frac{1}{z} - 1 \right)^{-1}. \]
The residue at \( \infty \) is
\[ \lim_{z \to \infty} z \frac{1}{z} \left( \frac{1}{z} - 1 \right)^{-1/2}. \]
This is \(-i\) because we pick the branch of the square root which has \( \sqrt{-1} = i \). Thus the integral is
\[ \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \frac{1}{2} 2\pi i (-i) = \pi. \]

Remark 3. Given this example, we point out that the residue at infinity of \( f(z) \) is the residue at zero of \(-\frac{1}{z^2} f(1/z)\). This is because if \( \gamma_1 \) is oriented clockwise, and we let \( \gamma_2(z) = 1/\gamma_1(z) \), then \( \gamma_2 \) is oriented counterclockwise. The residue of \( f \) at \( \infty \) is
\[ \frac{1}{2\pi i} \int_{\gamma_1} f(z)dz \]
and by changing the variable \( z = 1/w \),
\[ \frac{1}{2\pi i} \int_{\gamma_1} f(z)dz = \frac{1}{2\pi i} \int_{\gamma_2} f(1/w)(-1/w^2)dw, \]
which is the residue at 0 of \( f(1/z)(-1/z^2) \).
Example 15. Compute
\[
\int_1^\infty \frac{dx}{x\sqrt{x^2 - 1}}
\]
First, note that the integral converges absolutely, namely that
\[
\lim_{a \to 1^+} \lim_{b \to \infty} \int_a^b \frac{dx}{x\sqrt{x^2 - 1}}
\]
exists. Now consider a contour that runs around $-1$ clockwise, runs parallel and close to $[-1, -R]$, runs clockwise on $|z| = R$ until close to $R$, runs parallel and close to $[R, 1]$, runs clockwise around $1$, runs parallel and close to $[1, R]$, then runs counterclockwise from $R$ to $-R$ on $|z| = R$, and finally returns to $-1$ running parallel to $[-R, -1]$.

Now use the branch cut of $\sqrt{z}$ that removes the positive real semiaxis. This choice allows us to define $f(z)$ on $\mathbb{C}\setminus((-\infty, -1] \cup [1, \infty))$. The only singularity of $f$ inside the contour is $z = 0$, and $f$ runs counterclockwise around it. We compute $\text{Res}(f; 0) = -i/2\sqrt{\pi}$.

Consequently the residue theorem gives us that the integral of $f$ on the contour is $2\pi i\text{Res}(f; 0) = 2\pi i(-i) = \sqrt{\pi}$.

Proposition 12. The Gaussian integral formula holds:
\[
\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.
\]
Proof. (cf. R. Remmert, Theory of Complex Functions, Springer) Set
\[
f(z) = \frac{e^{-z^2}}{1 + e^{-2az}}, \quad a = (1 + i)\frac{\sqrt{\pi}}{2} = \sqrt{\pi}e^{i\pi/4}.
\]
It is not hard to see that $f(z) - f(z + a) = e^{-z^2}$, and that the poles of $f$ are located at $\frac{a}{2} + na$, $n \in \mathbb{Z}$. Moreover, we can see that $g(z) = 1 + e^{-2az}$ has a simple zero at $a/2$ because $g'(a/2) = 2a$.

Hence the residue of $f$ at $z = a/2$ is
\[
\lim_{z \to a/2} \frac{z - a/2}{e^{-2az} - (-1)} e^{-z^2} = \frac{1}{g'(a/2)} e^{-a^2/4} = \frac{e^{-i\pi/4}}{2\sqrt{\pi} e^{i\pi/4}} = \frac{1}{2\sqrt{\pi}} e^{-i\pi/2} = -i/2\sqrt{\pi}.
\]
Let $r, s > 0$, and integrate $f$ on the parallelogram with corners $-r, s, s + a, -r + a$. For sufficiently large $r, s$ this parallelogram $f$ has only one residue at $a/2$, and it is $-i/(2\sqrt{\pi})$. Note also that the integral of $f$ on the short sides of the parallelogram converges to zero. And integrating $f$ on the long sides is the same as integrating the Gaussian on $(-r, s)$. Passing to the limit with $r, s \to \infty$ we obtain
\[
\int_{-\infty}^{\infty} e^{-t^2} dt = 2\pi i\text{Res}_{a/2}f = 2\pi i\left(\frac{-i}{2\sqrt{\pi}}\right) = \sqrt{\pi}.
\]

Corollary 3. The Fresnel integral formulas hold:
\[
\int_0^\infty \cos t^2 dt = \int_0^\infty \sin t^2 dt = \frac{\sqrt{2\pi}}{4}.
\]
Proof. Consider the contour $\gamma$ consisting of the union of the curves $\gamma_1(t) = t, 0 \leq t \leq R$, $\gamma_2(t) = Re^{it}, 0 \leq t \leq \pi/4$ and $\gamma_3(t) = (R - t)e^{i\pi/4}, 0 \leq t \leq R$. Note that $\int_\gamma e^{-z^2}dz = 0$, because the integrand is holomorphic inside the contour. Also, $\lim_{R \to \infty} \int_{\gamma_2} e^{-z^2}dz = 0$. So
\[
\int_0^\infty e^{-t^2}dt = \int_0^\infty e^{-it^2}e^{i\pi/4}dt.
\]
Consequently
\[
\frac{\sqrt{\pi}}{2} = \int_0^\infty (\cos t^2 - i \sin t^2) \frac{1 + i}{\sqrt{2}} dt.
\]
Equating the real and imaginary parts and solving the system implies that each of the Fresnel integrals equals $\frac{\sqrt{2\pi}}{4}$.

4.5 Computation of integrals - a different perspective

I find it easier to understand the integrals from the previous section in the context of Riemann surfaces (for the definition of a Riemann surface, see Chapter 6).

First, the idea that we integrate functions is not quite correct. We integrate forms:

- we integrate 1-forms over curves;
- we integrate 2-forms over surfaces;
- in general, we integrate n-forms over n-dimensional manifolds.

Why is a form better than a function? Because it carries with it the information over the integration measure, which is a must when the curve, surface, etc. lives inside a more general space than the plane (such as a manifold).

Let us discuss just the case of 1-form on a Riemann surface $X$. In local coordinates, a 1-form looks like $f(z)dz$. But when you change from one coordinates system to another, that is from one chart $\phi_a : U_a \cap U_b \to \mathbb{C}$ to $\phi_b : U_a \cap U_b \to \mathbb{C}$, with the change of coordinates $\phi = \phi_b \circ \phi_a^{-1}$, then the form changes by $f(\phi(z))\phi'(z)dz$. So the 1-form is defined everywhere (with maybe some singularities), and it has a concrete formula in every local chart, with this formula changing from one system of coordinates to another by

\[
f(z)dz \mapsto f(\phi(z))d\phi(z) = f(\phi(z))\phi'(z)dz.
\]

This formula is chosen so that the integral does not change when changing the local coordinates, since by the first substitution we have
\[
\int_\gamma f(z)dz = \int_\gamma f(\phi(z))\phi'(z)dz.
\]
In this formula we think of $\gamma$ as the physical curve on $X$ (and not of its formula in local coordinates, which changes when you change coordinates). Because of this we have a well defined integral of the form on any (compact) curve in $X$: simply decompose the curve into pieces such that each piece lies in a coordinate chart, integrate on each piece in local coordinates, and then sum the results.
Example 16. Compute

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4)^2}.$$

We are supposed to compute the integral of the function

$$f : \mathbb{C} \to \mathbb{C}, \quad f(z) = \frac{1}{(z^2 + 4)^2}$$

over the real axis. We can add the point at infinity to the plane, and now we work on the Riemann sphere. Thus we have to integrate the 1-form

$$\alpha = f(z) = \frac{dz}{(z^2 + 4)^2}$$

over a circle of the Riemann sphere that passes through the point at infinity. There is one problem, the form is not yet defined at the point at infinity. Let us show that we can define it there, too. To work in local coordinates, use the chart

$$\phi_\infty : (\mathbb{C} \cup \{\infty\}) \setminus \{0\} \to \mathbb{C}, \phi_\infty(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C}, \\ 0 & \text{if } z = \infty. \end{cases}$$

In local coordinates near $\infty$,

$$\omega = \frac{1}{((1/z)^2 + 4)^2} \frac{d}{dz}(1/z)dz = -\frac{z^2}{(4z^2 + 1)^2}dz.$$ 

This clearly can be extended to $z = 0$. So our 1-form is defined on the entire Riemann sphere. On each side it bounds a disk, and the value of the integral equals $2\pi i$ times the sum of the residues on one side. And these residues can be computed in local coordinates. We compute the integral as the sum of the residues that are on the left side of the contour, and this is the residue at $z = 2i$. This residue is

$$\frac{d}{dz}[(z - 2i)^2 \frac{1}{(z - 2i)^2(z + 2i)^2}]|_{z=2i} = -\frac{2}{(z + 2i)^3} |_{z = 2i} = -\frac{2}{(4i)^3} = \frac{1}{32i}.$$ 

Hence

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 4)^2} = 2\pi i \frac{1}{32i} = \frac{\pi}{16}.$$
Chapter 5

The maximum modulus principle

5.1 The Maximum modulus theorem

Theorem 30. (The Maximum Modulus Theorem) Let $G$ be a bounded open set in $\mathbb{C}$ and let $f$ be a nonconstant function that is continuous on $\overline{G}$ and holomorphic in $G$. Then there is $a \in \partial G$ such that $|f(z)| \leq |f(a)|$ for all $z \in G$.

Proof. We will give two proofs:

1. Because $f$ is continuous on $\overline{G}$, it has a maximum on $\overline{G}$. But this maximum cannot be in $G$ because $f(G)$ is open in $\mathbb{C}$, so $|f(G)|$ is open in $\mathbb{R}$.

2. Because $f$ is continuous on $\overline{G}$, it has a maximum on $\overline{G}$. Assume that there is a maximum $a \in G$. Choose $\epsilon > 0$ such that $B(a, \epsilon) \subset G$. Then for $\gamma(t) = a + \epsilon e^{it}$, using Cauchy’s formula we have

$$f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{z-a} dz = \frac{1}{2\pi} \int_{0}^{2\pi} f(a + \epsilon e^{it}) dt.$$ 

So

$$|f(a)| \leq \frac{1}{2\pi} \int_{0}^{2\pi} |f(a + \epsilon e^{it})| dt \leq |f(a)|$$

with equality only if $f$ is constantly equal on $\gamma$. But then the zeros of the holomorphic function $f(z) - f(a)$ contain a circle, so this function is identically zero, a contradiction. The conclusion follows. \square

Corollary 4. If $f$ is analytic on the open set $G$ and $f$ has a maximum in $G$, then $f$ is constant on $G$.

Corollary 5. Let $f$ be a holomorphic function on $G \subset \mathbb{C} \cup \infty$. If there is $M > 0$ such that for every point $a$ on the boundary of $G$ in $\mathbb{C} \cup \infty$, $\limsup_{z \to a} |f(z)| \leq M$, then $|f(z)| \leq M$ for all $z \in G$.

As a corollary, we have the following result.

Theorem 31. (Schwarz’s Lemma) Suppose $f : B(0, 1) \to \mathbb{C}$ is analytic with

(a) $|f(z)| \leq 1$ for all $z$;

(b) $f(0) = 0$.

Then $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$ for all $z \in B(0, 1)$. Moreover if $|f'(0)| = 1$ or if $|f(z)| = |z|$ for some $z$, then there is $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} z$.  

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CHAPTER 5. THE MAXIMUM MODULUS PRINCIPLE

**Theorem 32.** The only analytic maps that map bijectively the open unit disk onto itself are of the form

\[ f(z) = e^{i\theta} \frac{z - a}{-\bar{a}z + 1}, \quad \theta \in \mathbb{R}, |a| < 1. \]

*Proof.* We have seen as a homework exercise that the above are the only Möbius transformations satisfying the property from the statement, and moreover that these maps extend to the closed unit disk, mapping the boundary to the boundary.

Consider the Möbius transformation

\[ \phi_a(z) = \frac{z - a}{-\bar{a}z + 1}. \]

For \( f \) an automorphism of the unit disk onto itself, as specified in the statement, let \( g = f \circ \phi_a \). Then \( g \) is also an automorphism of the unit disk and it satisfies \( g(0) = 0 \). If there is a \( z \) such that \(|g(z)| = |z|\), then we are done, because by the Schwarz’s lemma \( g(z) = e^{i\theta}z \), and so \( f \) is a Möbius transformation. If \(|g(z)| < |z| \) on \(|z| < 1\), then, again by Schwarz’s lemma,

\[ |z| = |g^{-1}(g(z))| \leq |g(z)| < |z|, \]

impossible. We are done.

**Theorem 33.** (Phragmén-Lindelöf) Let \( G \) be a simply connected region and let \( f \) be an analytic function on \( G \). Suppose that there is an analytic function \( \phi : G \to \mathbb{C} \) which never vanishes and is bounded on \( G \). If \( M \) is a constant and if the boundary of \( G \) in the Riemann sphere can be partitioned into two sets \( A \) and \( B \) such that

(a) for every \( a \in A \), \( \limsup_{z \to a} |f(z)| \leq M \);

(b) for every \( b \in B \) and for every \( \eta > 0 \), \( \limsup_{z \to b} |f(z)||\phi(z)|^{\eta} \leq M \);

then \(|f(z)| \leq M \) for all \( z \in G \).

*Proof.* Let \(|\phi(z)| \leq \kappa \) for all \( z \in G \). Because \( G \) is simply connected, there is analytic branch of \( \log \phi(z) \) on \( G \). Hence \( g(z) = \exp(\eta \log \phi(z)) \) is an analytic branch of \( \phi(z)^{\eta} \), and \(|g(z)| = |\phi(z)|^{\eta} \). Define \( F : G \to \mathbb{C}, F(z) = f(z)g(z)\kappa^{-\eta} \). Then \(|F(z)| \leq |f(z)| \) for all \( z \in G \). But then by the Maximum Modulus Theorem \(|F(z)| \leq \max(M, \kappa^{-\eta}M) \) for all \( z \in G \). Then

\[ |f(z)| \leq |\kappa/\phi(z)|^{\eta} \max(M, \kappa^{-\eta}M). \]

Now let \( \eta \to 0 \) to get the conclusion.
Chapter 6

Convergence and compactness in spaces of holomorphic functions

6.1 Constructing topologies on spaces of continuous and holomorphic functions

6.1.1 Some introductory remarks

We will first define a topology on the space of continuous functions on an open set, then we will embed the space of holomorphic functions in the space of continuous functions and consider the induced topology.

This topology will turn both the space of continuous functions and the space of holomorphic functions on an open set into a Fréchet space, and in particular into a metric space. The choice of the topology is standard, it is motivated by the choice of the topology on the space of continuous functions on a compact set, and by our discussion on uniform convergence of power series, which is a particular example of convergence of holomorphic functions.

Sorry folks, but we do need some point set topology and again we need some real, and even functional analysis. I have lecture notes for both on my web page in case you need more information.

The main idea is that we treat functions as points in an infinite dimensional space and we have a notion of distance between two functions which allows us to address problems of convergence of sequences of functions.

We mostly care about holomorphic functions whose domain and range are subsets of $\mathbb{C}$, but we also care about maps between more general Riemann surfaces such as maps from the Riemann sphere into itself (like Möbius transformations) and maps from a subset of $\mathbb{C}$ to a Riemann surface.

The notion of convergence is a direct extension of that for power series, so we will see uniform convergence on closed bounded sets. It also extends the notion of convergence of a sequence of continuous functions on a closed bounded interval, which is usually defined using the sup norm (the $L^\infty$ norm).

Since this course is suppose to help you build your skills for advancing in mathematics, I feel obliged to tell you the correct general framework in which convergence is phrased for spaces of functions.

6.1.2 Elements of topology; Riemann surfaces

Recall the notion of convergence of a sequence of complex numbers: $\lim z_n = z^*$ if for every $\epsilon > 0$, there is $N \in \mathbb{N}$ such that for $n \geq N$, $z_n \in B(z^*, \epsilon)$. The open disks of radius $\epsilon$ allow us to define the
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notion of neighborhood, namely what it means for \( z_n \) to be near \( z^* \). In this class we have already encountered the notion of an open set in \( \mathbb{C} \) as a set that can be written as a union of open disks. And it is not hard to see that we can rephrase the notion of convergence to say that \( \lim z_n = z^* \) if for every open set \( G \) containing \( z^* \), there is \( N \in \mathbb{N} \) such that for \( n \geq N \), \( z_n \in G \). Note that arbitrary unions of open sets are open, and also finite intersections of open sets are open. The collection of all open sets is called a topology on \( \mathbb{C} \). Together with open sets come closed sets, which are by definition the complements of open sets.

**Definition.** A topology on a set \( X \) is a collection of open sets such that

- \( X \) and \( \emptyset \) are open;
- if \( U_1, U_2, \ldots, U_n \) are open, then \( \cap_{j=1}^{n} U_n \) is open;
- if \( U_\alpha, \alpha \in A \) are open, then \( \bigcup_{\alpha \in A} U_\alpha \) is open.

Now we want to treat functions as points in a space and have a similar notion of convergence for functions (or maps), and for that we need open sets in the space of functions. In order for this to be a nice notion of convergence, the open sets should satisfy the same two properties: arbitrary unions of open sets are open, and also finite intersections of open sets are open. What kind of functions do we have in mind? First, let us discuss the playground where these function live.

**Definition.** A Riemann surface is a set \( X \) that can be written as a union of subsets \( \{ U_\alpha \}_{\alpha \in A} \) for which there exist bijective maps \( \phi_\alpha : U_\alpha \to G_\alpha \) (with \( G_\alpha \) an open set of \( \mathbb{C} \)) such that for every \( \alpha, \beta \) such that \( U_\alpha \cap U_\beta \neq \emptyset 

\[ \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta) \]

is holomorphic.

The maps \( \phi_\alpha : U_\alpha \to G_\alpha \) are called coordinate charts, and together they form an atlas on \( X \). In this class we only consider those Riemann surfaces that admit an atlas with *countably many charts*. In other words, a Riemann surface is a set that locally looks like an open subset of \( \mathbb{C} \).

**Example 17.** The Riemann sphere is a Riemann surface. Indeed, we let \( U_0 = \mathbb{C} \), and \( U_\infty = (\mathbb{C} \setminus \{0\}) \cup \{\infty\} \), with

\[ \phi_0 : U_0 \to \mathbb{C}, \quad \phi_0(z) = z, \]

\[ \phi_\infty : U_\infty \to \mathbb{C}, \quad \phi_\infty(z) = \begin{cases} \frac{1}{z} & \text{if } z \neq \infty \\ \infty & \text{if } z = \infty. \end{cases} \]

Then

\[ \phi_0 \circ \phi_\infty^{-1} : \mathbb{C} \setminus \{0\} \to \mathbb{C}, \quad \phi_0 \circ \phi_\infty^{-1}(z) = \frac{1}{z}. \]

**Example 18.** The Riemann surface of the logarithm also satisfies these properties. For every point \( w \neq 0 \) we can define an open set \( U_w \) on this surface that is in bijection with the set of the form \( G_\alpha = \mathbb{C} \setminus \{ \lambda a \mid \lambda \geq 0 \} \). Compositions of maps \( \phi_\beta \circ \phi_\alpha^{-1} \) are defined on \( \mathbb{C} \) without two rays starting at the origin, and on each of the two connected components they are the identity map.

We declare any set of the form \( \phi^{-1}(G) \) with \( G \subset G_\alpha \) open in \( \mathbb{C} \) to be open in \( X \), and we consider arbitrary unions and finite intersections of such sets to be open. This defines a topology on \( X \). The complements of open sets are called closed sets.
Now we the concepts of open and closed sets at hand, we can talk about compactness. Let $X$ be a topological space, namely a set endowed with a collection of open sets that have the property that both $X$ and the empty set are open, the finite intersection of open sets is open, and the infinite union of open sets is open.

**Definition.** We say that $K$ is compact in $X$ if every family of open sets whose union contains $K$ has a finite subfamily whose union still covers $K$.

This is phrased by saying that every open cover has a finite subcover.

**Definition.** We say that $K$ is sequentially compact in $X$ if every sequence in $K$ contains a convergent subsequence.

We recall that a distance (or metric) on a set $X$ is a function $d : X \to [0, \infty)$ such that $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$ and $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$. The set $X$ endowed with this function is called a metric space. The topology on a metric space is the smallest topology in which all balls $B(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ are open. Every open set is actually a union of balls (no need to worry about intersections).

**Theorem 34.** For metric spaces the two notions of compactness are equivalent.

**Example 19.** A set $K \subset \mathbb{C}$ is compact if and only if it is closed and bounded. This is the content of the Heine-Borel theorem.

**Example 20.** The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is compact. Indeed, if $\{G_\alpha\}_\alpha$ is a family of open sets whose union is the Riemann sphere, then for some $\alpha_0$, $\infty \in G_{\alpha_0}$. But $G_{\alpha_0}$ is open if and only if $\phi_\infty(G_{\alpha_0})$ is open in $\mathbb{C}$. It is not hard to see that this forces $(\mathbb{C} \cup \{\infty\}) \setminus G_{\alpha_0}$ to be both closed and bounded. So there are finitely many of the $G_\alpha$ that cover it, which together with $G_{\alpha_0}$ form a finite cover of the Riemann sphere.

For $\varepsilon > 0$, we say that a set $K$ in a metric space has an $\varepsilon$-net if there are $x_1, x_2, \ldots, x_n \in K$ such that every point $x$ in $K$ is at distance less than $\varepsilon$ from one of $x_1, x_2, \ldots, x_n$.

**Proposition 13.** A closed set is compact if and only if for every $\varepsilon > 0$ there is an $\varepsilon$-net.

Now we turn to maps and functions. We are interested in holomorphic maps between Riemann surfaces.

**Definition.** Let $X$ and $Y$ be Riemann surfaces defined respectively by the atlases $\phi_\alpha : U_\alpha \to G_\alpha$ and $\psi_\mu : V_\mu \to D_\mu$. A map $f : X \to Y$ is called holomorphic if $\psi_\mu \circ f \circ \phi_\alpha^{-1}$ is holomorphic whenever the composition makes sense.

**Example 21.** Recall holomorphic maps $f : G \subset \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, which we have called meromorphic functions.

Holomorphic maps are a subset of the much larger set of continuous functions.

**Definition.** A function $f : X \to Y$ is called continuous if the preimage through $f$ of every open set is open.

Now we have two spaces: that of continuous functions: $C(X, Y)$ and of holomorphic functions: $H(X, Y)$. It is not hard to see that $H(X, Y) \subset C(X, Y)$. We introduce now a standard topology on $C(X, Y)$ that will induce a topology on $H(X, Y)$. 


**Definition.** The compact-open topology on $C(X,Y)$ is the smallest topology for which all sets of continuous functions of the form

$$\{ f : X \to Y \mid f(K) \subset V \}, \quad K \text{ compact in } X, V \text{ open in } Y$$

are open.

Note that finite intersection of sets of this form are not necessarily of this form, but they are nevertheless postulated to be open. The induced topology on $H(X,Y)$ is the smallest topology such that all sets of holomorphic functions of the same form are open.

**Proposition 14.** Let $X,Y$ be Riemann surfaces.

(i) Assume that $X'$ is a Riemann surface such that $X' \subset X$, and let us consider the subset $S$ of $C(X',Y)$ consisting of the restrictions of functions in $C(X,Y)$ to $X'$. Then the topology of $S$ induced by the compact-open topology of $C(X,Y)$ (in which the open sets are the images of open sets in $C(X,Y)$) is finer than the topology induced by the inclusion of $S$ into $C(X',Y)$. (This means that the first topology has more open sets than the second).

(ii) Assume that $Y'$ is a Riemann surfaces such that $Y' \subset Y$. Then the compact-open topology of $C(X,Y')$ coincides with the topology induced by the inclusion $C(X,Y') \subset C(X,Y)$.

**Proof.** For (i) note that a compact set in $X'$ is also compact in $X$. For (ii) note that a set is open in $Y'$ if and only if it is open in $Y$. 

6.1.3 **The case $f : X \to Y$ where $Y$ is a subset of the Riemann sphere**

All Riemann surfaces of interest in this class are naturally metric spaces, but the plane and the Riemann sphere have metrics that are easy to describe.

The plane is a one dimensional $\mathbb{C}$-vector space and it has a norm given by the absolute value. We recall that a norm on a $\mathbb{C}$-vector space $V$ is a function $| \cdot | : V \to [0, \infty)$ such that $|v| = 0$ if and only if $v = 0; |\lambda v| = |\lambda||v|$ for every scalar $\lambda \in \mathbb{C}$ and $|v + w| \leq |v| + |w|$. The norm on $\mathbb{C}$ defines a distance function by $d(z,w) = |z - w|$.

We can define on the Riemann sphere the following distance

$$\rho : \mathbb{C} \cup \{\infty\} \to [0,1], d(z,w) = \begin{cases} 
\frac{|z - w|}{1 + |z - w|}, & \text{if } z, w \in \mathbb{C} \\
1 & \text{if } z = \infty \text{ or } w = \infty.
\end{cases}$$

- If $X$ is a compact Riemann surface, such as the Riemann sphere, and $Y = \mathbb{C}$ we can turn $C(X,Y)$ and implicitly $H(X,Y)$ into normed vector spaces by using the norm

$$\|f\| = \sup_{z \in X} |f(z)|.$$  

This norm introduces a notion of distance: $\rho(f,g) = \|f - g\|$, and the distance introduces a topology on $C(X,Y)$ (and hence on $H(X,Y)$) where the open sets are unions of balls of the form $B(f,r) = \{g \mid \|f - g\| < r\}$. This turns out to coincide with the compact-open topology. This topology turns $C(X,Y)$ into a Banach space, meaning that every Cauchy sequence is convergent. We will see that $H(X,Y)$ is also a Banach space.

- If $X$ is compact and $Y$ is the Riemann sphere, or any open subset of it, then $C(X,Y)$ and $H(X,Y)$ are metric spaces with the metric

$$\rho(f,g) = \sup_{z \in X} d(f(z), g(z)).$$
Again it is standard that $C(X, Y)$ is a complete metric space, meaning that every Cauchy sequence is convergent. But we will see that $H(X, Y)$ is also complete.

- If $X$ can be written as a countable union of compact subsets: $X = \cup_{n=1}^{\infty}K_n$, and $Y = \mathbb{C}$, then $C(X, Y)$ and $H(X, Y)$ each are endowed with a countable collection of what are called seminorms (like norms except that $\|f\|$ can be zero without necessarily $f$ being zero). These are defined by

$$\|f\|_n = \sup_{z \in K_n} |f(z)|.$$  

There is a metric on $C(X, Y)$ and $H(X, Y)$ defined by

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}.$$  

Again $C(X, Y)$ is complete, and we will see that $H(X, Y)$ is also complete. These two linear spaces are therefore what we call Fréchet spaces.

- If $X$ can be written as a countable union of compact subsets: $X = \cup_{n=1}^{\infty}K_n$, and $Y$ is the Riemann sphere or an open subset of it, then $C(X, Y)$ and $H(X, Y)$ are endowed with a metric defined by

$$\rho(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{z \in K_n} d(f(z), g(z)).$$  

Again $C(X, Y)$ is complete, and we will see that $H(X, Y)$ is also complete.

Here we consider the case where $X$ is compact, or can be covered in a nice way by a sequence of compact sets (such as when $X$ is $\mathbb{C}$ or an open subset of $\mathbb{C}$) and $Y$ a metric space, such as $\mathbb{C}$, an open subset of $\mathbb{C}$, or the Riemann sphere. As we have seen above, $C(X, Y)$ admits a metric. But it also has the compact-open topology. In this section we will prove that the two topologies: the one induced by the metric, and the compact-open topology coincide. We thus conclude that the compact-open topology is metrizable.

**Proposition 15.** Let $G$ be an open subset of $\mathbb{C}$ (which can be $\mathbb{C}$ itself. There is a sequence $\{K_n\}_n$ of compact subsets of $G$ such that

(a) $G = \cup_{n=1}^{\infty}K_n$;
(b) $K_n \subset \text{int}K_{n+1}$;
(c) Every component of $(\mathbb{C} \cup \{\infty\})\setminus K_n$ contains a component of $(\mathbb{C} \cup \{\infty\})\setminus G$.

Here int$K$ is the interior of $K$ namely the largest open set contained in $K$.

**Proof.** Page 143.

**Corollary 6.** If $\{K_n\}_n$ is such a sequence of compact sets, then for every compact set $K \subset G$ there is an $n$ such that $K \subset K_n$.

**Proof.** Indeed, the interiors of $K_n$ cover $K$ so finitely many of them cover $K$. As they are nested, one of them covers $K$.

Here is a first consequence of this construction.
Proposition 16. The compact-open topology on $C(X,Y)$ is generated by countably many open sets. In other words, there are countably many open subsets of $C(X,Y)$ such that any other open set is an arbitrary union of finite intersections of such sets.

Proof. Let $\phi_\alpha : U_\alpha \to G_\alpha$, $\alpha \in A \subset \mathbb{N}$ and $\psi_\mu : V_\mu \to D_\mu$, $\mu \in B \subset \mathbb{N}$ be atlases of $X$ and $Y$, respectively. For each $G_\alpha$, consider the compact sets $K_\alpha^n$ provided by Proposition 15. For each $\mu$, let $B_\mu^n$ be a countable collection of open subsets of $D_\mu$ such that any other open set of $D_\mu$ is a union of some of the $B_\mu^n$ (in other words a countable basis). Then the family of open sets in $C(X,Y)$,

$$\{ f : X \to Y \mid f(K) \subset V \},$$

where $K$ ranges over all finite unions of sets of the form $\phi_\alpha^{-1}(K_\alpha^n)$ and $V$ ranges over all arbitrary unions (which must be countable unions) of sets of the form $\psi_\mu^{-1}(B_\mu^n)$ is a countable union of open sets with this property.

Indeed, if $K'$ is a compact set in $X$, we will show that $K'$ lies inside of such a compact $K$. Indeed, being compact it lies in finitely many charts, $\phi_{\alpha_1}, \phi_{\alpha_2}, \ldots, \phi_{\alpha_k}$. Let us prove the property by induction on $k$. If $k = 1$, we are done by the Proposition. If not, the set $K \setminus (U_{\alpha_1} \cup \cdots \cup U_{\alpha_{k-1}})$ is compact (being a closed subset of a compact set). It therefore lies in the interior of some $\phi_1^{-1}(K_1^{\alpha_1})$. But $K \setminus \text{int}\phi_1^{-1}(K_1^{\alpha_1})$ is compact, and lies in $k - 1$ charts, thus it is inside a compact of the desired form. Add $\phi_1^{-1}(K_1^{\alpha_1})$ to this compact to conclude that $K'$ itself lies inside such a compact $K$.

On the other hand, it is clear that any open set of $Y$ is in the union of open sets of the described form, because this is how we define the topology of $Y$.

Corollary 7. Every point in $C(X,Y)$ has a countable system of neighborhoods, meaning that for every $f$ in $C(X,Y)$ there is a countable family of open sets $O_n, n \geq 1$ such that for every open set $O$ containing $f$, there is $n$ such that $O_n \subset O$.

The sequence of compact sets constructed above allows us to define a metric $\rho$ on $C(G,Y)$ when $G$ is an open subset of $\mathbb{C}$ and $Y$ is a metric space. We will now prove that the metric topology and the compact-open topology coincide.

Lemma 4. For very $\epsilon > 0$ there is $\delta > 0$ and $K \subset G$ compact such that for $f, g \in C(G,Y)$

$$\sup\{d(f(z), g(z)) \mid z \in K\} < \delta \text{ implies } \rho(f, g) < \epsilon.$$  

Conversely, if $\delta > 0$ and a compact set $K$ are given, there is $\epsilon > 0$ such that for $f, g \in C(G,Y)$,

$$\rho(f, g) < \epsilon \text{ implies } \sup\{d(f(z), g(z)) \mid z \in K\} < \delta.$$  

Proof. Page 144.

Proposition 17. (a) A set $O \subset C(G,Y)$ is open if and only if for each $f$ in $O$, there is $K \subset G$ compact and $\delta > 0$ such that

$$\{g \mid d(f(z), g(z)) < \delta, z \in K\}.$$

(b) A sequence $\{f_n\}_n$ in $C(G,Y)$ converges to $f$ if and only if it converges to $f$ uniformly on all compact subsets of $G$.

Consequently, the compact-open topology is metrizable.
6.1. CONSTRUCTING TOPOLOGIES ON SPACES OF CONTINUOUS AND HOLOMORPHIC FUNCTIONS

Note that the metric depends on the sequence of compacts that cover $X$, but the topology that it defines is independent of it.

An important observation is that the sets

$$B_K(f, \epsilon) = \{ g \mid \sup_{z \in K} d(f(z), g(z)) < \epsilon \}$$

play the same role that the $\epsilon$-neighborhoods (i.e. $B(z, \epsilon)$ play in $\mathbb{C}$).

Now let $X$ be an arbitrary Riemann surface, and $Y$ a Riemann surface that is a subset of the Riemann sphere, and thus $Y$ is a metric space. We can then define a metric on $C(X, Y)$ as well. Let $\phi_m : U_m \to G_m$, $m \in \mathbb{N}$ be an atlas of $X$. For each $G_m$, consider the compact sets $K^m_n$ provided by Proposition 15. Define the compacts

$$K_m = \bigcup_{j=1}^{m} \phi_j^{-1}(K^m_j).$$

One can show that for every compact subset $K$ of $X$, there is $m \in \mathbb{N}$ such that $K \subset K_m$. Then the metric

$$\rho(f, g) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{\sup_{z \in K_m} d(f(z), g(z))}{1 + \sup_{z \in K_m} d(f(z), g(z))}$$

defines the same topology as the compact-open topology. The proof is similar to that of Lemma 4.

**Theorem 35.** The space $C(X, Y)$ is a complete metric space.

**Proof.** If $\{f_n\}_n$ is Cauchy, then its restriction to any compact set $K$ is Cauchy. Then the sequence $f_n|K$ is pointwise convergent, and being on a compact set, it is uniformly convergent. But then its limit is a continuous function. \(\square\)

**Definition.** A set $\mathcal{F} \subset C(G, Y)$ is normal if its closure is compact.

**Proposition 18.** A set $\mathcal{F} \subset C(G, Y)$ is normal if and only if for every compact set $K \subset G$ and $\delta > 0$ there are functions $f_1, f_2, \ldots, f_n \in \mathcal{F}$ such that for every $f \in \mathcal{F}$ there is at least one $k$, $1 \leq k \leq n$ with

$$\sup_{z \in K} d(f(z), f_k(z)) < \delta.$$

**Proof.** Since the closure of $\mathcal{F}$ is compact, for every $\epsilon > 0$, $\mathcal{F}$ has an $\epsilon$-net. Now use Lemma 4. \(\square\)

**Definition.** A set $\mathcal{F} \subset C(G, Y)$ is equicontinuous at a point $z_0$ in $G$ if for every $\epsilon > 0$ there is $\delta > 0$ such that for $|z - z_0| < \delta$, $\rho(f(z), f(z_0)) < \epsilon$ for every $f \in \mathcal{F}$.

**Theorem 36.** (Arzela-Ascoli) A set $\mathcal{F} \subset C(G, Y)$ is normal if and only if the following two conditions are satisfied:

(a) for each $z \in G$, $\{f(z) \mid f \in \mathcal{F}\}$ has compact closure in $Y$.
(b) $\mathcal{F}$ is equicontinuous at every point in $G$.

**Proof.** Pages 149-150. \(\square\)

All the above considerations apply when $X$ has such a sequence of compact sets (and this is the case for the Riemann surface of a function $w(z)$ defined implicitly by a polynomial equation $R(z, w) = 0$). If $X$ itself is compact, the sup norm already defines a metric on $C(X, Y)$, and in this metric $C(X, Y)$ is complete. As such, $C(X, Y)$ is a Banach space.
6.2 Spaces of analytic functions

6.2.1 Closeness of $H(X,Y)$

Let us consider the general case of $X,Y$ Riemann surfaces, and let us look at the subspace $H(X,Y)$ of $C(X,Y)$. As we have seen above, the topology of $C(X,Y)$, and hence of $H(X,Y)$, is generated by countably many open sets. Consequently, every point in $H(X,Y)$ has a countable system of neighborhoods.

**Theorem 37.** $H(X,Y)$ is a closed subspace of $C(X,Y)$.

**Proof.** Because in the topology of $C(X,Y)$ every point has a countable system of neighborhoods, it suffices to show that every sequence in $H(X,Y)$ that converges in $C(X,Y)$ converges in $H(X,Y)$.

So let us assume that the sequence of holomorphic maps $f_n : X \to Y$ converges to a continuous function $f$ in the compact-open topology, and let us show that $f$ is holomorphic.

Let $p \in X$ be an arbitrary point. We want to show that $f$ is holomorphic in a neighborhood of $p$. Choose charts $\phi : U \subset X \to \mathbb{C}$ and $\psi : V \subset Y \to \mathbb{C}$ such that $f(U) \subset V$, and choose a neighborhood $W$ of $p$ such that $K = \overline{W}$ is a compact subset of $U$. Let $\mathcal{O} = \{g \mid g(K) \subset V\}$. Then there is $N$ such that for $n \geq N$, $f_n \in \mathcal{O}$, that is $f_n(K) \subset V$. Set $W' = \phi(W)$. Because $f_n$ converges to $f$ in the compact-open topology, $f_n|K$ converges to $f|K$ in the compact-open topology. But then the sequence of holomorphic functions $h_n = \psi \circ f_n \circ \phi^{-1} : W' \to \mathbb{C}$, $n \geq 1$ converges in the compact-open topology to $h = \psi \circ f \circ \phi^{-1} : W' \to \mathbb{C}$. By Lemma 4, $\{h_n\}$ converges uniformly to $h$.

We will show that $h$ is holomorphic.

Let $T$ be a triangle in $W'$. Then by Cauchy’s theorem,

$$\int_T h_n(z)dz = 0.$$  

Then

$$0 = \lim_{n \to \infty} \int_T h_n(z)dz = \int_T \lim_{n \to \infty} h_n(z)dz = \int_T h(z)dz.$$ 

So $\int_T h(z)dz = 0$ for any triangle $T$ in $W'$. By Morera’s theorem, $h$ is holomorphic in $W'$. Thus, by definition, $f$ is holomorphic in $W$. We are done.

**Remark 4.** Note that the situation is quite the opposite from real analysis. The analytic functions on an interval of the real axis are dense in the space of continuous functions, as a consequence of Weierstrass’ Theorem which shows that polynomials are dense in the space of continuous functions in the compact-open topology.

**Theorem 38.** If $Y$ is a Riemann surface that is a subset of the Riemann sphere, then $H(X,Y)$ is a complete metric space.

**Proof.** By Theorem 35, $C(X,Y)$ is a complete metric space. By Theorem 37, $H(X,Y)$ is a closed subspace of $C(X,Y)$. Thus $H(X,Y)$ is a complete metric space.

In all that follows $G$ is an open subset of $\mathbb{C}$.

**Theorem 39.** If $f_n \to f$ in the compact-open (i.e. uniformly in the metric) topology of $H(G, \mathbb{C})$ then $f'_n \to f'$ in the compact-open topology of $H(G, \mathbb{C})$. 

6.2. SPACES OF ANALYTIC FUNCTIONS

Proof. Let $B(a, r) \subset G$. Choose $R > r$ such that $B(a, R) \subset G$. If $\gamma(t) = a + Re^{it}$, then Cauchy’s integral formula gives

$$f'_n(z) - f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_n(w) - f(w)}{(w-z)^2} \, dw.$$

Hence

$$|f'_n(z) - f'(z)| \leq \frac{R \sup_{w \in \gamma} |f_n(w) - f(w)|}{(R-r)^{k+1}}, \quad |z - a| \leq r.$$

Since $f_n$ converges uniformly to $f$ on the curve $\gamma$ (which is compact), $f'_n$ converges uniformly to $f'$ on $|z - a| \leq r$. By Theorem 38, $f'$ is holomorphic. \hfill \Box

Corollary 8. If $f_n : G \to \mathbb{C}$ converges to $f$ in the compact-open topology, then $f_n^{(k)}$ converges to $f^{(k)}$ in the compact-open topology. In other words, if $f_n : G \to \mathbb{C}$ converges to $f$ uniformly on compact sets, then $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on compact sets. In particular, if $f_n : G \to \mathbb{C}$ and $\sum_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ then

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z).$$

Note that this last fact is a generalization of the result about differentiating a power series.

Theorem 40. (Hurwitz’s Theorem) Let $G \subset \mathbb{C}$ be open and suppose that the sequence $f_n$ in $H(G, \mathbb{C})$ converges to $f$. If $f$ is not identically equal to zero, and if $B(a, R) \subset G$ and $f(z) \neq 0$ for $|z - a| = R$, then there is an integer $N$ such that for $n \geq N$, $f$ and $f_n$ have the same number of zeros in $B(a, R)$.

Proof. Page 152. \hfill \Box

Corollary 9. If $f_n \in H(G, \mathbb{C})$ converges to $f \in H(G, \mathbb{C})$, and if each $f_n$ never vanishes, then $f$ is either identically equal to zero, or it is never zero. Indeed, if $f$ had an isolated zero $a$, then by Hurwitz’s Theorem, for sufficiently large $n$, $f_n$ would have an isolated zero in some neighborhood of $a$. But this does not happen. So either $f$ is never zero or it is identically equal to zero.

Let $X$ be a Riemann surface. Let $f_\infty : X \to \mathbb{C} \cup \{\infty\}$, $f_\infty(z) = \infty$.

Proposition 19. $H(X, \mathbb{C}) \cup \{f_\infty\}$ is a closed subset of $H(X, \mathbb{C} \cup \{\infty\})$. In other words a sequence of holomorphic functions converges in the compact-open topology of the space of meromorphic functions either to a holomorphic function or to a function that is constantly equal to $\infty$.

Proof. Let us assume that $f \neq f_\infty$. We will show that $f$ is holomorphic. It suffices to show that $f$ is holomorphic in every chart, so let $\phi : U \to G$ be a chart. Define $g = f \circ \phi^{-1}$ and $g_n = f_n \circ \phi^{-1}$, $n \geq 1$. Because $f_n \to f$ in the compact-open topology of $H(X, \mathbb{C} \cup \{\infty\})$, $g_n \to g$ in the compact-open topology of $H(G, \mathbb{C} \cup \{\infty\})$.

The inversion $z \mapsto 1/z$ is an automorphism of the Riemann sphere, so it maps compact sets to compact sets and open sets to open sets. Since $g_n \to g$ in the compact-open topology, $1/g_n \to 1/g$ in the compact-open topology of $H(G, \mathbb{C} \cup \{\infty\})$. Since all function $g_n$ are analytic, the holomorphic function $1/g_n$ is never zero.

Assume that $(1/g)(a) = 0$. Because zeros and poles are isolated, there is a disk $B(a, R)$ such that $a$ is the only zero of $1/g$ in $B(a, R)$, and $1/f$ has no poles in $B(a, R)$. Because $1/g_n$ converges
to \(1/g\) in the metric of \(H(G, \mathbb{C} \cup \{\infty\})\) (as the metric and the compact-open topology are the same), there is \(N\) such that for \(n \geq N\), \(1/g_n\) has no poles in \(B(a, R)\) either.

We therefore have a sequence of holomorphic functions \(1/g_n\) on \(B(a, R)\) that converges to the holomorphic function \(1/g\) in the topology of \(H(B(a, R), \mathbb{C} \cup \{\infty\})\). But notice that the metrics \(d(z, w) = |z - w|\) and \(d'(z, w) = \frac{|z - w|}{1 + |z - w|}\) induce the same topology on \(\mathbb{C}\). Consequently, if we work only with holomorphic functions, the compact-open topology of \(H(B(a, R), \mathbb{C})\) and the topology induced by the compact-open topology of \(H(B(a, R), \mathbb{C} \cup \{\infty\})\) are the same. Consequently, \(1/g_n\) converges to \(1/g\) in \(H(B(a, R), \mathbb{C})\). Now we can apply Hurwitz’ Theorem and concluded that for large \(n\), \(1/g\) and \(1/g_n\) have the same number of zeros in \(B(a, R)\). But this contradicts the fact that \(1/g_n\) has no zeros. Hence \(1/g\) has no zeros either, showing that \(g\) is holomorphic. Hence \(f\) is holomorphic in the local chart \(U\), and consequently it is holomorphic everywhere.

**Remark 5.** Convergence in \(H(X, \mathbb{C})\) and \(H(X, \mathbb{C} \cup \{\infty\})\) are not the same thing. For example \(f_n(z) = n\) does not converge in the first topology, but it converges to \(f_\infty\) in the second. What we have used in the proof is that if \(f_n \to f\) in \(H(X, \mathbb{C} \cup \{\infty\})\) and if \(f_n, f \in H(X, \mathbb{C})\) then \(f_n \to f\) in \(H(X, \mathbb{C})\).

At the heart of this lies the fact that if \(f \in H(X, \mathbb{C})\), then in \(H(X, \mathbb{C} \cup \{\infty\})\) \(f\) has a countable system of neighborhoods that is also a countable system of neighborhoods in \(H(X, \mathbb{C})\).

### 6.2.2 Theta functions

There are holomorphic functions on \(\mathbb{C}\) that are periodic: \(e^z, \sin z, \cos z\). But are there any double periodic functions? Liouville’s theorem proves that this is impossible, because a double periodic function is bounded. The closest that we can get is to have two complex numbers \(a\) and \(b\) that are not a real multiple of each other and a function \(f\) such that \(f(z + a) = f(z)\) and \(f(z + b) = \mu(z)f(b)\) where \(\mu(z)\) is a correction factor that is as simple as possible. Theta functions are of this form. They were introduced by Jacobi in relation to elliptic integrals. More precisely, he has shown that the inverse function of an elliptic integral, an so called elliptic function, can be written as a rational expression in theta functions. Theta functions were further studied and generalized by Riemann in his treatise on elliptic functions and Riemann surfaces.

**Definition.** For every \(\tau\) with \(\text{Im} \tau > 0\), the Riemann theta function is defined by

\[
\theta(z, \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z), \quad z \in \mathbb{C}.
\]

**Theorem 41.** (a) For fixed \(\tau\), the series \(\sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n z)\) converges uniformly and absolutely on every compact subset of \(\mathbb{C}\) and therefore defines a holomorphic function of the variable \(z\) on \(\mathbb{C}\).

(b) For fixed \(z\), the series converges uniformly and absolutely on every compact subset of the upper half plane and therefore defines a holomorphic function of the variable \(\tau\) on the upper half-plane.

**Proof.** We can prove (a) and (b) simultaneously, by noticing that if \(|\text{Im} z| \leq \alpha\) and \(\text{Im} \tau \geq \beta > 0\) then

\[
|\exp(\pi i n^2 \tau + 2\pi i n z)| \leq \exp(-\pi n^2 \beta + 2\pi n \alpha) = [\exp(-|n|\pi \beta + 2\pi \alpha)]^{|n|}.
\]

If we choose \(N > 2\alpha/\beta\), then \(r = \exp(-|n|\pi \beta + 2\pi \alpha) < 1\), so for \(|n| \geq N\), \([\exp(-|n|\pi \beta + 2\pi \alpha)]^{|n|} < r^{|n|}\), showing that the series that defines theta functions is bounded from above by a power series, hence converges uniformly. So for fixed \(\tau\), the series converges uniformly on \(A_\alpha = \{z | -\alpha \leq \text{Im} z \leq \alpha\}\), for all \(\alpha \geq 0\), and since every compact in \(\mathbb{C}\) is contained in some set \(A_\alpha\), as a function
of \( z \), the series converges uniformly on compacts. For fixed \( z \), the series converges uniformly on \( B_\beta = \{ \tau \mid \text{Im} \tau \geq \beta \} \), for all \( \beta > 0 \), and since every compact in \( \text{Im} \tau > 0 \) is contained in some \( B_\beta \), as a function of \( \tau \) the series converges uniformly on compacts.

**Proposition 20.** Theta functions satisfy the identities

\[
\theta(z + 1, \tau) = \theta(z, \tau), \quad \theta(z + \tau) = e^{-\pi i \tau - 2\pi i z} \theta(z, \tau).
\]

**Proof.** The first identity follows from

\[
\exp(\pi i n^2 \tau + 2\pi i n(z + 1)) = e^{2\pi i} \exp(\pi i n^2 \tau + 2\pi i n z) = \exp(\pi i n^2 \tau + 2\pi i n z).
\]

For the second identity, we have

\[
\theta(z + \tau, \tau) = \sum_{n=-\infty}^{\infty} \exp(\pi i n^2 \tau + 2\pi i n(z + \tau)) = \sum_{n=-\infty}^{\infty} \exp(\pi i (n+1)^2 \tau - \pi i n^2 \tau + 2\pi i n(z + 1) - 2\pi i z)
\]

\[
= \sum_{m=-\infty}^{\infty} \exp(\pi i m^2 \tau - \pi i n^2 \tau + 2\pi i m z - 2\pi i z) = e^{-\pi i \tau - 2\pi i z} \theta(z, \tau).
\]

Let now \( x, t \in \mathbb{R} \) and consider the function of 2 real variables \( \theta(x, it) \).

**Proposition 21.** The function \( \theta(x, it) \) satisfies the heat equation

\[
\frac{\partial}{\partial t} \theta(x, it) = \frac{1}{4\pi} \frac{\partial^2}{\partial x^2} \theta(x, it).
\]

**Proof.** We compute

\[
\theta(x, it) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 t) \exp(2\pi i n x) = 1 + 2 \sum_{n=1}^{\infty} \exp(-\pi n^2 t) \cos(2\pi n x).
\]

Hence

\[
\frac{\partial}{\partial t} \theta(x, it) = 2 \sum_{n=1}^{\infty} (-\pi n^2) \exp(-\pi n^2 t) \cos 2\pi n x;
\]

\[
\frac{\partial^2}{\partial x^2} \theta(x, it) = 2 \sum_{n=1}^{\infty} (-4\pi^2 n^2) \exp(-\pi n^2 t) \cos 2\pi n x.
\]

The proposition is proved.

And now let us apply the Argument Principle to find the number of zeros of the theta function in the fundamental domain, which is the parallelogram with vertices \( 0, 1, 1 + \tau, \tau \), which includes the sides from 0 to 1 and from 0 to \( \tau \), but misses the other sides. Choose a parallelogram \( P \) that is the translate of this parallelogram and does not pass through the zeros, and let \( \gamma_1, \gamma_2, \gamma_3, \gamma_4 \) be the sides \( 0 \to 1, 1 \to 1 + \tau, 1 + \tau \to \tau, \tau \to 0 \).

The number of zeros of \( \theta(z, \tau) \) inside this parallelogram is

\[
\frac{1}{2\pi i} \int_P \frac{\theta'(z, \tau)}{\theta(z, \tau)} \, dz = \frac{1}{2\pi i} \left( \int_{\gamma_1} \frac{\theta'(z, \tau)}{\theta(z, \tau)} \, dz + \int_{\gamma_2} \frac{\theta'(z, \tau)}{\theta(z, \tau)} \, dz + \int_{\gamma_3} \frac{\theta'(z, \tau)}{\theta(z, \tau)} \, dz + \int_{\gamma_4} \frac{\theta'(z, \tau)}{\theta(z, \tau)} \, dz \right).
\]
Because $\theta(z+1, \tau) = \theta(z, \tau)$,
\[\int_{\gamma_2} \frac{\theta'(z, \tau)}{\theta(z, \tau)} dz + \int_{\gamma_4} \frac{\theta'(z, \tau)}{\theta(z, \tau)} dz = 0.\]

Because $\theta(z + \tau, \tau) = e^{-\pi i \tau - 2\pi i z} \theta(z, \tau)$,
\[\theta'(z + \tau, \tau) = (-2\pi i)e^{-\pi i \tau - 2\pi i z} \theta(z, \tau) + e^{-\pi i \tau - 2\pi i z} \theta'(z, \tau),\]
so
\[\int_{\gamma_1} \frac{\theta'(z, \tau)}{\theta(z, \tau)} dz + \int_{\gamma_3} \frac{\theta'(z, \tau)}{\theta(z, \tau)} dz = \int_{\gamma_1} \frac{\theta'(z, \tau)}{\theta(z, \tau)} dz - \int_{\gamma_1} \frac{\theta'(z + \tau, \tau)}{\theta(z + \tau, \tau)} dz\]
\[= \int_{\gamma_1} \frac{\theta'(z, \tau)}{\theta(z, \tau)} dz - \int_{\gamma_1} \frac{e^{-\pi i \tau - 2\pi i z} \theta'(z, \tau)}{e^{-\pi i \tau - 2\pi i z} \theta(z, \tau)} dz - (-2\pi i) \int_{\gamma_1} \frac{e^{-\pi i \tau - 2\pi i z} \theta(z, \tau)}{e^{-\pi i \tau - 2\pi i z} \theta(z, \tau)} dz = 2\pi i.\]

Thus
\[\frac{1}{2\pi i} \int_P \frac{\theta'(z, \tau)}{\theta(z, \tau)} dz = 1\]

showing that the theta function has a single zero in a fundamental domain. In fact because the theta function is even, this zero can only be $0, 1/2, \tau/2$, or $(1 + \tau)/2$. It is $(1 + \tau)/2$.

6.2.3 Compactness in $H(X, Y)$