COMPACT PERTURBATIONS OF FREDHOLM n-TUPLES, II

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We study compact perturbations of Fredholm n-tuples of index zero. We prove that if the operators in such an n-tuple acting on a Hilbert space satisfy certain functional relations then the n-tuple cannot be perturbed with compact operators to an invertible one.

The study of commuting n-tuples of operators has been initiated by J. L. Taylor in [5] and [6]. Since then several properties of a single operator have been generalized to n-tuples. In [2] R. Curto asked if the fact that a Fredholm operator of index zero acting on a Hilbert space can be made invertible by adding a compact operator remains true for commuting pairs. In [3] it has been shown that pairs of the form (T,T) with T Fredholm and $indT \neq 0$ cannot be perturbed to invertible pairs. The aim of this paper is to extend this result to n-tuples.

We shall start by reviewing some important facts about commuting n-tuples. We consider only operators on a certain infinite dimensional Hilbert space H. Following [5] we attach to each commuting n-tuple $T = (T_1, T_2, \cdots, T_n)$ a complex of Hilbert spaces $(K^p(T,H), \delta_T)$, called the Koszul complex, by defining $K^p(T,H) := H \bigotimes \Lambda^p$, and $\delta_T : K^p(T,H) \to K^{p+1}(T,H)$, $\delta_T := T_1 \otimes E_1 + \cdots + T_n \otimes E_n$, where $\Lambda^p = \Lambda^p[e_1, e_2, \cdots, e_n]$ are the p-forms on $\mathbf{C}^n$ and $E_i\omega = e_i\omega$, $\omega \in \Lambda^p, i = 1, 2, \cdots, n$.

The n-tuple T is called invertible if its Koszul complex is exact. The spectrum of T, denoted by $\sigma(T)$, is the set of all $z = (z_1, z_2, \cdots, z_n) \in \mathbf{C}^n$ such that $z - T = (z_1 - T_1, z_2 - T_2, \cdots, z_n - T_n)$ is not invertible. In [5] it is proved that the spectrum is a compact nonvoid set. For any holomorphic map f on an open neighborhood of $\sigma(T)$ one can define $f(T)$ (cf. [6]). The spectral mapping theorem asserts that $f(\sigma(T) = \sigma(f(T))$ (cf. [6]). A general overview of the properties of the spectrum can be found in [1].

Let $H^p(T)$, $0 \leq p \leq n$, be the cohomology spaces of the Koszul complex. The n-tuple T is called Fredholm if all these spaces are finite dimensional, in this case we define the index of T to be $indT = \sum_{p=0}^{n}(-1)^p dimH^p(T)$. It is known that the index is preserved under compact perturbations (cf. [4]). Given two commuting tuples $T = (T_1, T_2, \cdots, T_n)$ and $T' = (T_1, T_2, \cdots, T_n, S)$ one has the long exact sequence in cohomology

$$0 \to H^0(T') \to H^0(T) \stackrel{\hat{S}_0}{\to} H^0(T) \to H^1(T') \to H^1(T) \to \cdots$$

$$\cdots \to H^{p-1}(T) \to H^p(T') \to H^p(T) \stackrel{\hat{S}_p}{\to} H^p(T) \to \cdots$$

where $\hat{S}_p$ is the operator induced by $S \otimes 1 : K^p(T,H) \to K^p(T,H)$, $0 \leq p \leq n$. We observe that the tuple T' is invertible if and only if all the operators $\hat{S}_p$ are isomorphisms. As a consequence of this long exact sequence if T is Fredholm then T' is Fredholm of index zero; this is a method of obtaining Fredholm n-tuples of index zero. Given a subspace $\mathcal{H}$ of H we denote by $P_{\mathcal{H}}$ the orthogonal projection on $\mathcal{H}$.

Lemma. *Let T be such that for any n, $dimkerT^n < \infty$ and $dimkerT^n \to \infty$. If S commutes with T and the sequence $dim(kerS \bigcap kerT^n)$, $n \in \mathbf{N}$ is bounded then there exists a sequence of nontrivial orthogonal subspaces $\mathcal{H}_n$ in H such that $P_{\mathcal{H}_n}S|\mathcal{H}_n$ is invertible, and for every m and n, $P_{\mathcal{H}_n}S|\mathcal{H}_n$ is similar to $P_{\mathcal{H}_m}S|\mathcal{H}_m$.*

Proof. Let $\mathcal{K}_n = kerT^n \ominus kerT^{n-1}$. Since $dimkerT^n \to \infty$, the spaces $\mathcal{K}_n$ are nontrivial. Moreover, the operator

$$P_{\mathcal{K}_n}T|\mathcal{K}_n : \mathcal{K}_n \to \mathcal{K}_{n-1}$$

is injective, therefore $dim\mathcal{K}_n \leq dim\mathcal{K}_{n-1}$. This shows that the sequence $dim\mathcal{K}_n$, $n \in \mathbf{N}$ is a decreasing sequence of natural numbers, so it becomes stationary. It follows that there exists a number n_0 such that for $n \geq n_0$, the operator $P_{\mathcal{K}_{n-1}}T|\mathcal{K}_n$ is an isomorphism. Since for every n, $kerT^n \subset kerT^{n+1}$ and the sequence $dim(kerS \bigcap kerT_n)$, $n \in \mathbf{N}$ is bounded, there exists a number $n_1 > n_0$ such that for $n \geq n_1$, the operator $P_{\mathcal{K}_n}S|\mathcal{K}_n$ is injective, hence invertible. Moreover, the operator $P_{\mathcal{K}_n}T|\mathcal{K}_n$ defines a similarity between $P_{\mathcal{K}_n}S|\mathcal{K}_n$ and $P_{\mathcal{K}_{n+1}}S|\mathcal{K}_{n+1}$ for every $n \geq n_1$. Taking $\mathcal{H}_n = \mathcal{K}_{n+n_1}$, $n \geq 0$, we obtain a sequence of spaces with the desired property.

Theorem. *Let $(T_1, T_2, \cdots, T_n)$ be a commuting n-tuple with T_1 Fredholm and $indT_1$ different from zero. If there exists for each k, $2 \leq k \leq n$ an analytic function of two variables f_k such that*

1. $f_k(0, w) = 0$ *implies* $w = 0$,
2. $f_k(T_1, T_k) = L_k$, L_k *compact,*

then the n-tuple $(T_1, T_2, \cdots, T_n)$ cannot be perturbed with compact operators to an invertible n-tuple.

Proof. Suppose that such compacts $K_1, K_2, \cdots, K_n$ exist. Denote $S_i = T_i + K_i$. Then S_1 is Fredholm of nonzero index, we may assume $indS_1 > 0$. We remark that for every k, $2 \leq k \leq n$ the operator $N_k = f_k(S_1, S_k)$ is compact. Consider the analytic function $f : \mathbf{C}^n \to \mathbf{C}^n$, $f(z_1, z_2, \cdots, z_n) = (z_1, f_2(z_1, z_2), \cdots, f_n(z_1, z_n))$. Then $f^{-1}(0) = 0$, and since $(S_1, S_2, \cdots S_n)$ is invertible, from the spectral mapping theorem it follows that $(S_1, N_2, \cdots, N_n)$ is also invertible. Let us show that this is not possible.

Let us consider k to be the smallest integer with the property that the sequence $dim(kerS_1^m \bigcap kerN_2 \bigcap \cdots \bigcap kerN_k)$, $m \in \mathbf{N}$, is bounded. Such a k exists, for by the spectral mapping theorem $(S_1^m, N_2, \cdots, N_n)$ is invertible for every m, hence $kerS_1^m \bigcap kerN_2 \bigcap \cdots \bigcap kerN_n$ $= 0$. Consider the subspace $H_0 = kerN_2 \bigcap \cdots \bigcap N_{k-1}$ (in case $k = 2$ take $H_0 = H$). Since the operators $S_1, N_2, \cdots, N_k$ commute, H_0 is invariant for S_1 and N_k. Moreover, because of the minimality of k, the operators $S_1|H_0$ and $N = N_k|H_0$ satisfy the hypothesis of the previous lemma.

Let $\mathcal{H}_m$ be the spaces obtained by applying the lemma. Since $P_{\mathcal{H}_1}N|\mathcal{H}_1$ is invertible its spectral radius r is nonzero, so because of the similarity we have $\|P_{\mathcal{H}_m}N|\mathcal{H}_m\| \geq r > 0$ for every m, which contradicts the fact that N is compact. This proves the theorem.

We remark that from the proof it follows that there is an obstruction to making the n-tuple either left or right invertible. As a consequence of the theorem, if T is Fredholm of nonzero index and $k_1, k_2, \cdots, k_n$ are positive integers then the n-tuple $(T^{k_1}, T^{k_2}, \cdots, T^{k_n})$ has the index equal to zero, but cannot be perturbed with compact operators to an invertible n-tuple. The next example will show that the obstruction to making tuples invertible can be provided by the index of a subtuple. We still don't know if this works in general.

Let $\mathbf{H}^2(\mathbf{D}^2)$ be the Hardy space on the bidisk, and T_{z_1} and T_{z_2} the two shifts defined by $T_{z_1}f(z_1, z_2) = z_1 f(z_1, z_2)$, $T_{z_2}f(z_1, z_2) = z_2 f(z_1, z_2)$ for $f \in \mathbf{H}^2(\mathbf{D}^2)$. It is well known that the pair (T_{z_1}, T_{z_2}) is Fredholm of index 1. Therefore the triple $(T_{z_1}, T_{z_2}, 0)$ is Fredholm of index zero. Let us show that it cannot be perturbed with compact operators to a commuting invertible triple.

Suppose that there exist compact operators K_1, K_2 and K_3 such that the triple $(T_{z_1}+K_1, T_{z_2}+K_2, K_3)$ is invertible. Let $S_1 = T_{z_1}+K_1$ and $S_2 = T_{z_2}+K_2$. By theorem 3.8 in [4] $ind(S_1^n, S_2) = n \cdot ind(S_1, S_2) = n$, which shows that $dimH^0(S_1^n, S_2) + dimH^2(S_1^n, S_2) \to \infty$ for $n \to \infty$. So there is a sequence of positive integers $\{n_k\}_k$ such that either $dimH^0(S_1^{n_k}, S_2) \to \infty$ or $dimH^2(S_1^{n_k}, S_2) \to \infty$. Without loss of generality we may assume that $dimH^0(S_1^{n_k}, S_2) \to \infty$. Since $H^0(S_1^n, S_2) = kerS_1^n \cap kerS_2$ and $kerS_1^n \subset kerS_1^{n+1}$ we get that $dim(kerS_1^n \cap kerS_2) \to \infty$ for $n \to \infty$. From the spectral mapping theorem it follows that (S_1^n, S_2, K_3) is invertible for any positive integer n hence $kerS_1^n \cap kerS_2 \cap kerK_3 = 0$. Therefore we can apply the lemma to the space $kerS_2$, and to the operators $S_1|kerS_2$ and $K_3|kerS_2$. Using the same idea as in the proof of the theorem we contradict the compactness of K_3, which proves the claim.

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