FROM CLASSICAL THETA FUNCTIONS TO TOPOLOGICAL QUANTUM FIELD THEORY

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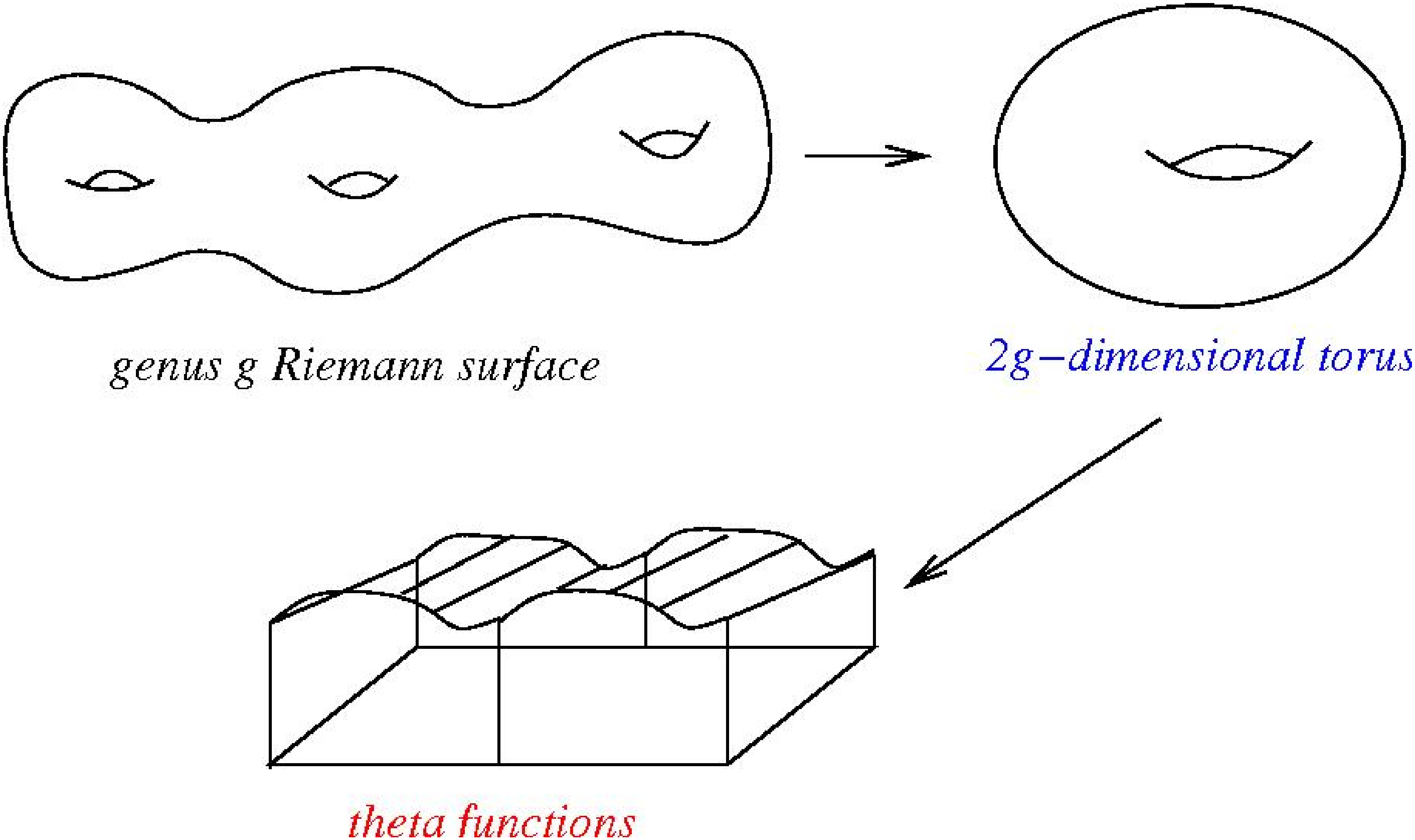
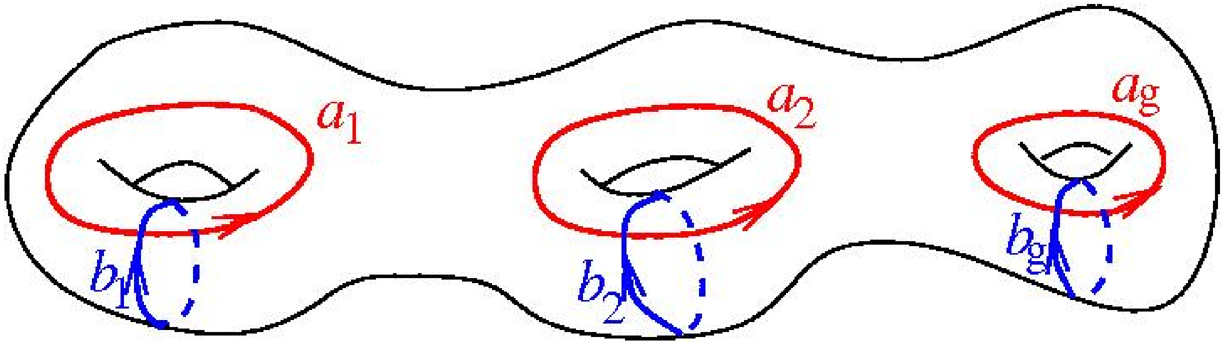
WE WILL CONSTRUCT THE ABELIAN CHERN-SIMONS TOPOLOGICAL QUANTUM FIELD THEORY DIRECTLY FROM THE THEORY OF CLASSICAL THETA FUNCTIONS.

CLASSICAL THETA FUNCTIONS





To a

- *CLOSED GENUS g RIEMANN SURFACE Σ_g, endowed with a canonical basis $a_1, a_2, \ldots, a_g$, $b_1, b_2, \ldots, b_g$ of $H_1(\Sigma_g, \mathbb{R})$, associate the*
- *PERIOD MATRIX (I_g, Π) defined by $\zeta_1, \zeta_2, \ldots, \zeta_g$ holomorphic 1-forms such that*

$$\int_{a_k} \zeta_j = \delta_{jk}, \quad \pi_{jk} = \int_{b_k} \zeta_j, \quad j, k = 1, \ldots, g.$$

The columns of (I_g, Π) span a lattice $L(\Sigma_g)$ in $\mathbb{C}^g = \mathbb{R}^{2g}$. The complex torus

$$\mathcal{J}(\Sigma_g) = \mathbb{C}^g / L(\Sigma_g) = H_1(\Sigma_g, \mathbb{R}) / H_1(\Sigma_g, \mathbb{Z})$$

is the

JACOBIAN VARIETY OF Σ_g.

- *$\mathcal{J}(\Sigma_g)$ has an associated holomorphic line bundle, whose sections can be identified with holomorphic functions on $\mathbb{C}^g$ satisfying certain periodicity conditions. These are the classical theta functions (Jacobi).*

- *The mapping class group (modular group) of Σ_g acts on theta functions (Hermite-Jacobi action).*

- *There is an action of a finite Heisenberg group on theta functions which induces the Hermite-Jacobi action via a Stone-von Neumann theorem (Weil).*

We use geometric quantization for constructing the Hilbert space of theta functions and Weyl quantization for constructing the quantum observables that form the Heisenberg group.

Endow $\mathcal{J}(\Sigma_g)$ with the symplectic form $\omega = \sum_{j=1}^{g} dx_j \wedge dy_j$ *where* $z = x + \Pi y$. *Fix Planck's constant* $\hbar = \frac{1}{N}$ *(Weil's integrality condition),* N *even integer.*

The states are the holomorphic sections of a line bundle $\Lambda_1 \otimes \Lambda_2$*, where* Λ_1*: holomorphic line bundle of curvature* $-2\pi i N\omega$ *and* $\Lambda_2 = K^{1/2}$*:*

classical theta functions.

An orthonormal basis is given by the theta series

$$\theta_\mu(z) = \sum_{n\in\mathbb{Z}^g} e^{2\pi i N[\frac{1}{2}(\frac{\mu}{N}+n)^T \Pi(\frac{\mu}{N}+n)+(\frac{\mu}{N}+n)^T z]}, \quad \mu \in \mathbb{Z}_N^g.$$

For a function f on $\mathcal{J}(\Sigma_g)$, the quantum observable $Op(f)$ is the Toeplitz operator with symbol $e^{-\frac{\hbar\Delta_\Pi}{4}}f$, where Δ_Π is the Laplace-Beltrami operator.

Proposition: The Weyl quantization of the exponentials is given by

$$Op\left(e^{2\pi i(p^Tx+q^Ty)}\right)\theta_\mu^\Pi(z)=e^{-\frac{\pi i}{N}p^Tq-\frac{2\pi i}{N}\mu^Tq}\theta_{\mu+p}^\Pi(z).$$

Proposition: Quantized exponentials satisfy the multiplication rule

$$Op\left(e^{2\pi i(p^Tx+q^Ty)}\right)Op\left(e^{2\pi i(p'^Tx+q'^Ty)}\right)=e^{\frac{\pi i}{N}\begin{vmatrix}p & q\\ p' & q'\end{vmatrix}}Op(e^{2\pi i((p+p')^Tx+(q+q')^Ty)}$$

Quantized exponentials form a finite Heisenberg group $\mathbf{H}(\mathbb{Z}_N^g)$, *which is a* $\mathbb{Z}_{2N}$-*extension of* $\mathbb{Z}_N^g \times \mathbb{Z}_N^g$ *and is a quotient of*

$$\mathbf{H}(\mathbb{Z}^g) = \{(p,q,k),\ (p,q) \in \mathbb{Z}^{2g}(= H_1(\Sigma_g,\mathbb{Z})), k \in \mathbb{Z}\}$$

with multiplication

$$(p,q,k)(p',q',k') = (p+p', q+q', k+k' + \begin{vmatrix} p\, q \\ p' q' \end{vmatrix}).$$

The representation of $\mathbf{H}(\mathbb{Z}_N^g)$ *on theta functions is called the*

SCHRÖDINGER REPRESENTATION

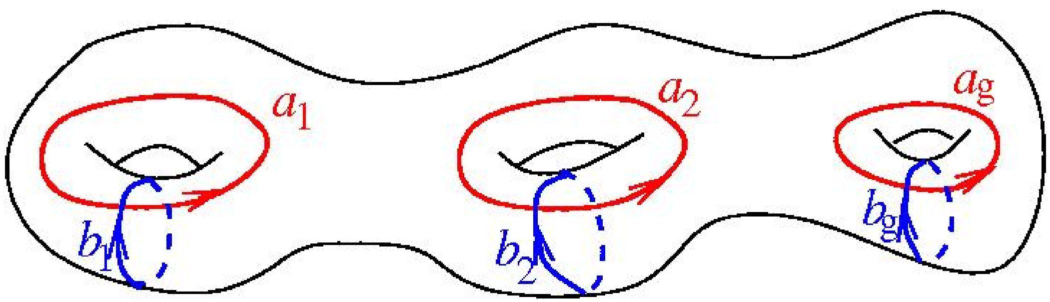
(by analogy with the representation of $\mathbf{H}(\mathbb{R})$ *on* $L^2(\mathbb{R})$ *for a free particle).*

Theorem (Stone-von Neumann) The Schrödinger representation of $\mathbf{H}(\mathbb{Z}_N^g)$ *is the unique irreducible unitary representation of* $\mathbf{H}(\mathbb{Z}_N^g)$ *with the property that* $(0,0,1)$ *acts as* $e^{\frac{\pi i}{N}} Id$.

Extend Schrödinger representation to a representation of $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$.

An element h of the mapping class group of Σ_g induces a linear symplectomorphism $\tilde{h}$ on $\mathcal{J}(\Sigma_g)$

$$\tilde{h} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$



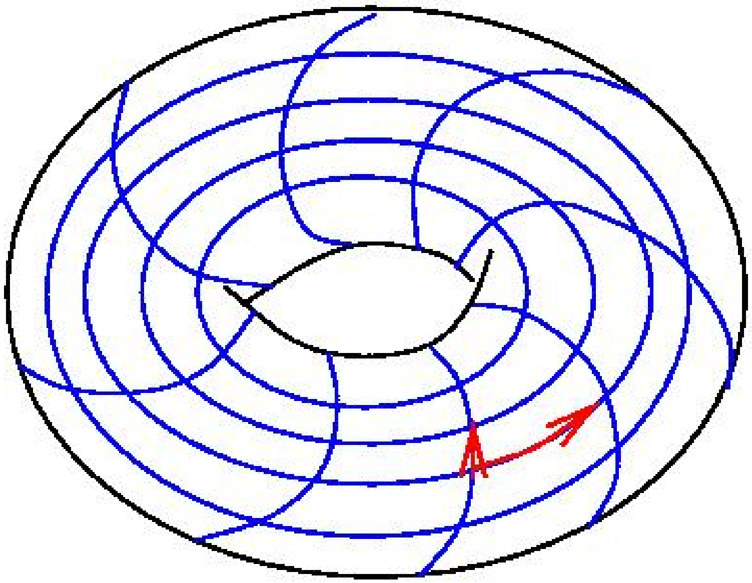
$$h \cdot \mathit{Op}\left(e^{2\pi i(p^T x + q^T y)}\right) = \mathit{Op}\left(e^{2\pi i[(Ap+Bq)^T x + (Cp+Dq)^T y]}\right).$$

By Stone-von Neumann $u \cdot \theta_\mu^\Pi = (h \cdot u)\theta_\mu^\Pi$ is equivalent to the Schrödinger representation; there is a unique automorphism $\rho(h)$ of the space of theta functions satisfying the exact Egorov identity

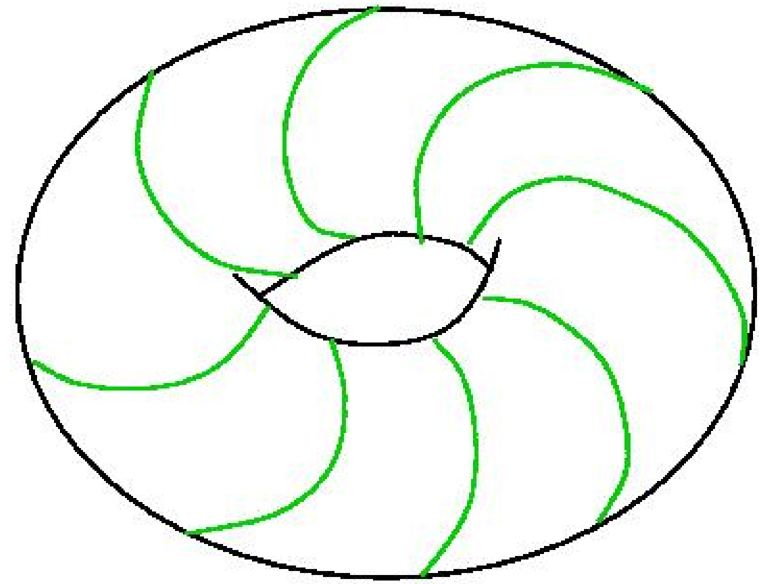
$$h \cdot \mathit{Op}\left(e^{2\pi i(p^T x + q^T y)}\right) = \rho(h)\mathit{Op}\left(e^{2\pi i(p^T x + q^T y)}\right)\rho(h)^{-1}.$$

The map $h \to \rho(h)$ is a projective representation of the mapping class group, called the Hermite-Jacobi action. $\rho(h)$ is a discrete Fourier transform.

Weyl quantization in a real polarization yields a more combinatorial picture:



finite Heisenberg group
(quantized exponentials)



theta functions
(Bohr-Sommerfeld leaves)

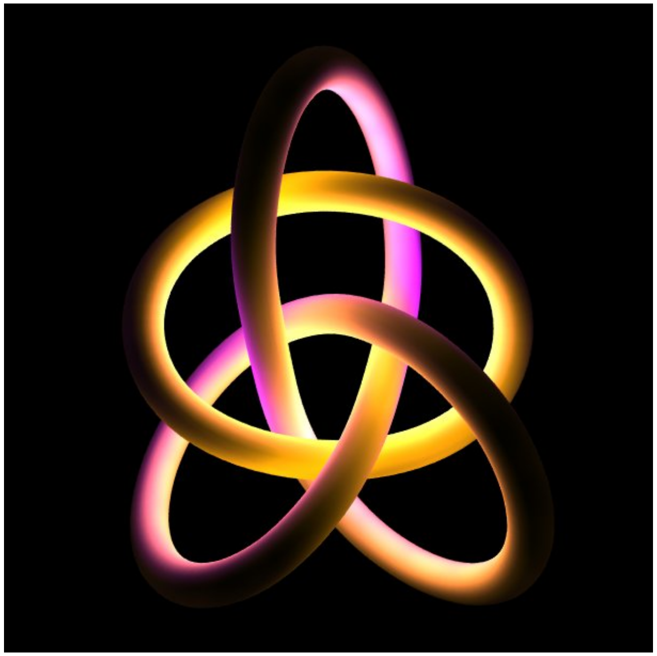
This gives rise to the abstract version of the Schrödinger representation as the left regular action of the group algebra of the finite Heisenberg group on a quotient of itself.

To summarize, *we have the following situation (A. Weil, Acta Math. 1964):*

- *the space of theta functions,*
- *a representation of the group algebra of the finite Heisenberg group on theta functions together with an action of the mapping class group of the surface on the Heisenberg group,*
- *a projective representation of the mapping class group on theta functions.*

The two representations are related by the exact Egorov identity.

WE WILL GIVE THESE A KNOT THEORETICAL INTERPRETATION!

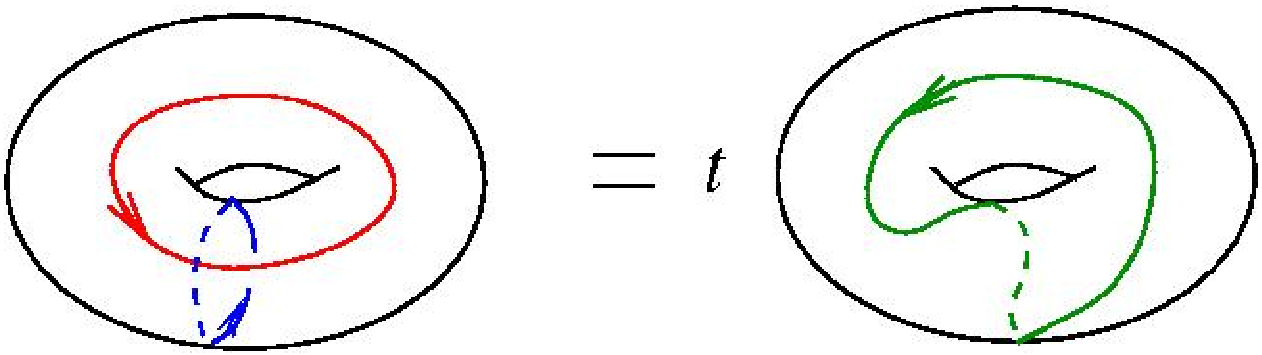


from Rob Scharein's KnotPlot

The multiplication rule for quantized exponentials on the torus:

$$Op\left(e^{2\pi i(px+qy)}\right) Op\left(e^{2\pi i(p'x+q'y)}\right) = t^{\begin{vmatrix} p & q \\ p' & q' \end{vmatrix}} Op(e^{2\pi i((p+p')x+(q+q')y)})$$

where $t = e^{\frac{i\pi}{N}}$. *The determinant is the algebraic intersection number of the curve* (p,q) *of slope* p/q *with the curve* (p',q') *of slope* p'/q'.

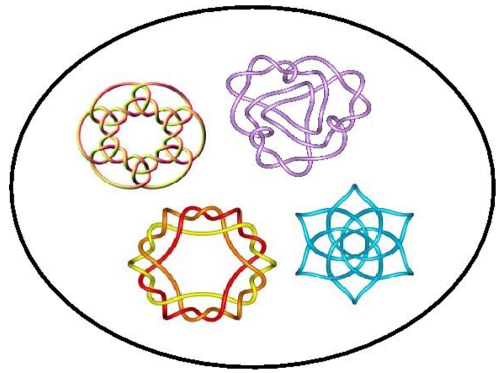


$$(1,0)\cdot(0,1) = t(1,1)$$

We will model the algebra $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ *(of quantized exponentials) using certain skein modules first introduced by J. Przytycki. These are the:*

SKEIN MODULES ASSOCIATED TO THE LINKING NUMBER.

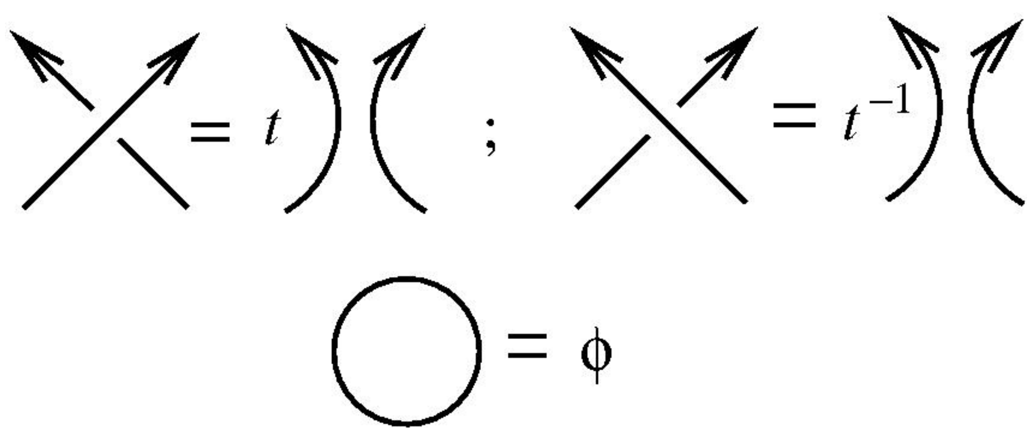
Linking number skein modules (algebraic topology based on knots)



M: *orientable* 3*-manifold,* t: *free variable.*

Take the free $\mathbb{C}[t, t^{-1}]$*-module with basis the isotopy classes of framed oriented links in* M*, including the empty link* $\emptyset$*.*

Factor it by the relations below, where the terms in each relation depict framed links that are identical except in an embedded ball, in which they look as shown. In other words, we are allowed to smooth each crossing, and to replace trivial link components by the empty link.



The result of the factorization is

$$\mathcal{L}_t(M) = \text{linking number skein module of } M.$$

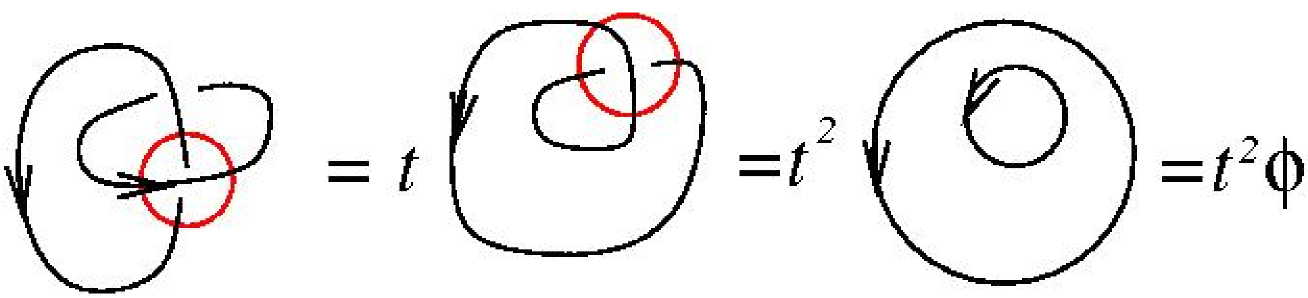
The elements of $\mathcal{L}_t(M)$ are called skeins.

In $\mathcal{L}_t(S^3) = \mathbb{C}$,

$$L = t^{lk(L)}\emptyset$$

where $lk(L)$ is the sum of the linking numbers of ordered pairs of components plus the sum of the writhes of the components.

Example:



- *Reduced linking number skein modules:*

$$\widetilde{\mathcal{L}}_t(M) = \mathcal{L}_t(M)/(t = e^{\frac{i\pi}{N}}, \gamma^N = \emptyset) \quad \forall \gamma \text{ oriented simple closed curve}$$

(Nth power means N parallel copies).

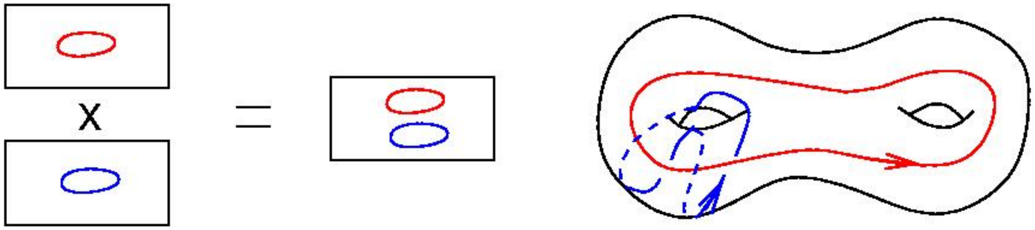
Proposition. $\widetilde{\mathcal{L}}_t(M) = \mathbb{C}^{\frac{N}{2}\dim H_1(\partial M,\mathbb{R})}$.

• *Linking number skein algebras:*

The equality

$$\Sigma_g \times [0,1] \cup \Sigma_g \times [0,1] \approx \Sigma_g \times [0,1]$$

induces a multiplication in $\mathcal{L}_t(\Sigma_g \times [0,1])$ *and in* $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$.



The equality

$$\partial M \times [0,1] \cup M \approx M$$

induces an action • *of* $\mathcal{L}_t(\partial M \times [0,1])$ *on* $\mathcal{L}_t(M)$

• *and of* $\widetilde{\mathcal{L}}_t(\partial M \times [0,1])$ *on* $\widetilde{\mathcal{L}}_t(M)$.

We have the following topological descriptions of the group algebra of the finite Heisenberg group and the Schrödinger representation.

THEOREM. The group algebra of the finite Heisenberg group $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ *is isomorphic to the reduced linking number skein algebra* $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$.

THEOREM. The Schrödinger representation of $\mathbb{C}[\mathbf{H}(\mathbb{Z}_N^g)]$ *on theta functions coincides with the action of* $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$ *on* $\widetilde{\mathcal{L}}_t(H_g)$, *where* H_g *is the genus* g *handlebody.*

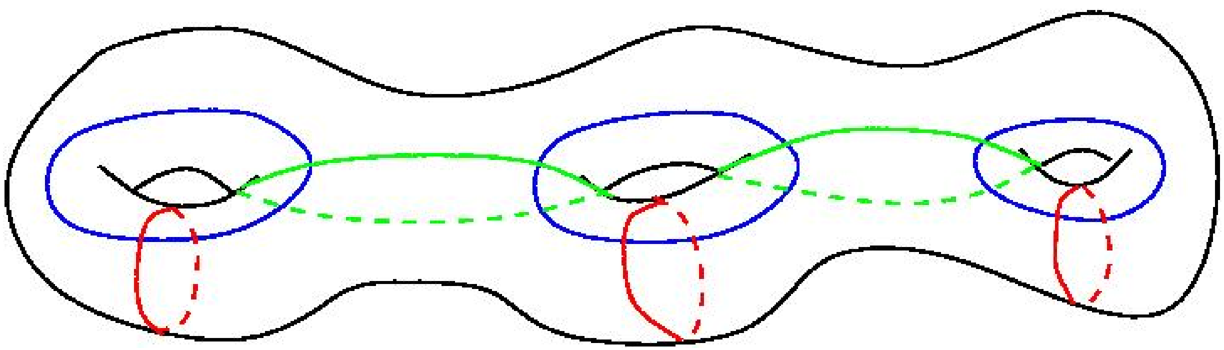
THE DISCRETE FOURIER TRANSFORM

Proposition. $\widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$ coincides with the algebra of all linear operators on $\widetilde{\mathcal{L}}_t(H_g)$.

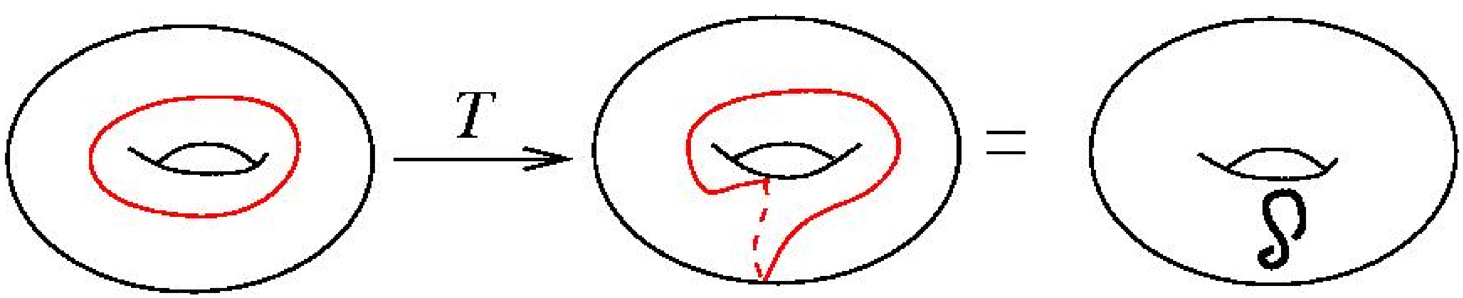
Corollary. The discrete Fourier transform defined by an element of the mapping class group can be represented as multiplication by a skein.

WHICH SKEIN?

Recall that every element of the mapping class group can be represented as a product of Dehn twists along the following curves:



Each Dehn twist can be represented as surgery along a framed curve:



Hence each element of the mapping class group can be represented as surgery on a framed link.

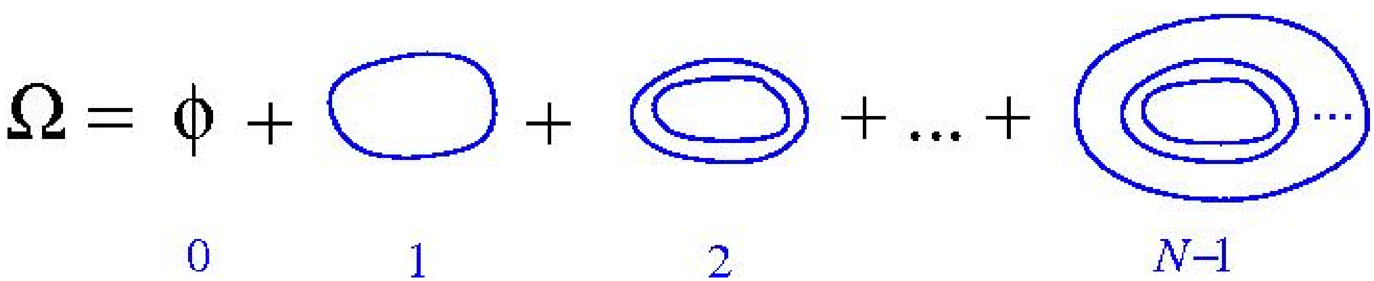
THEOREM. Let h be an element of the mapping class group of Σ_g defined by surgery on the framed link L_h in $\Sigma_g \times [0,1]$. Then the discrete Fourier transform

$$\rho(h) : \widetilde{\mathcal{L}}_t(H_g) \longrightarrow \widetilde{\mathcal{L}}_t(H_g)$$

is given by

$$\rho(h)\beta = \Omega(L_h)\beta$$

where $\Omega(L_h)$ is the skein obtained from L_h by replacing each link component by



THE EXACT EGOROV IDENTITY AS A HANDLE SLIDE

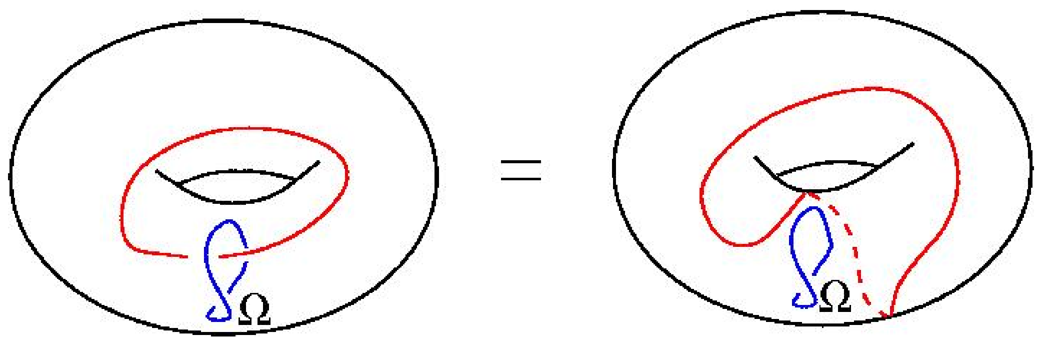
The exact Egorov identity

$$\mathit{Op}\left(f \circ h^{-1}\right) = \rho(h)\mathit{Op}(f)\,\rho(h)^{-1}.$$

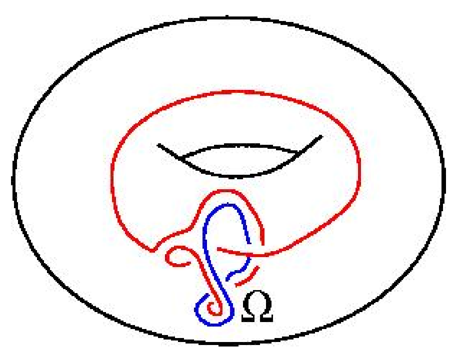
can be rewritten for skeins as

$$\rho(h)\sigma = h(\sigma)\rho(h)$$

where $\sigma \in \widetilde{\mathcal{L}}_t(\Sigma_g \times [0,1])$. *Here is an example:*

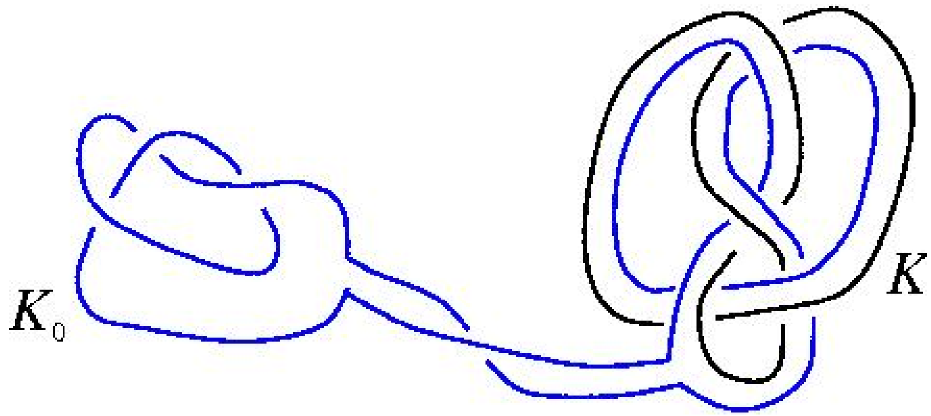


The diagram on the right is the same as



which is the slide of the red curve along the blue curve.

Here is the slide $K_0\#K$ of the "trefoil" knot K_0 along the "figure 8" knot K.



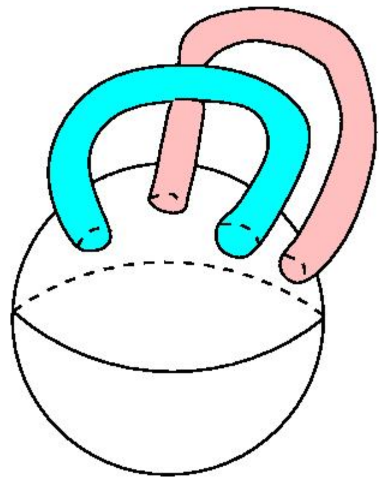
The Egorov identity is a particular situation of a slide. In fact any slide can be obtained from the Egorov identity. Thus, as a corollary of the Stone-von Neumann theorem, we obtain

THEOREM. Let M be a 3-dimensional manifold, σ an arbitrary skein in $\widetilde{\mathcal{L}}_t(M)$ and K_0 and K two oriented framed knots in M disjoint from σ. Then, in $\widetilde{\mathcal{L}}_t(M)$,

$$\sigma \cup K_0 \cup \Omega(K) = \sigma \cup (K_0\#K) \cup \Omega(K).$$

Knot slides are related to handle slides in a 4-manifold!

• *Any* 3*-manifold is the boundary of a* 4*-manifold obtained by adding* 2*-handles* $D^2 \times D^2$ *along* $S^1 \times D^2$ *to a* 4*-dimensional ball.*

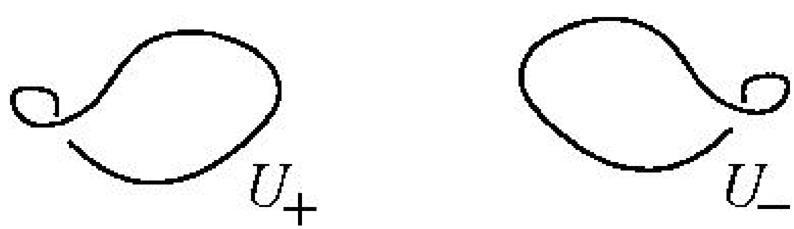


• *The operation of adding a handle corresponds to Dehn surgery with integer coefficients on the boundary* S^3 *of the* 4*-dimensional ball.*

• *This surgery is encoded by a framed link (the cores of the attaching solid tori).*

• *Handle slides correspond to slides of the link components along one another.*

• *Adding a trivial handle, which does not change the* 3*-manifold, corresponds to adding to the surgery link one of these:*



THEOREM. (R. Kirby) Any two surgery descriptions of a 3-manifold are related by handle-slides and addition and deletion of trivial handles.

As a corollary of the Stone-von Neumann theorem we obtain:

THEOREM. Given a manifold M obtained as surgery on the framed link L, the number

$$Z(M) = \Omega(U_+)^{-b_+}\Omega(U_-)^{-b_-}\Omega(L)$$

is a topological invariant of the manifold M, where b_+ and b_- are the number of the positive, respectively negative eigenvalues of the linking matrix of L.

REMARK. One recognizes this to be a Reshetikhin-Turaev formula associated to the linking number.

RELATION TO ABELIAN CHERN-SIMONS THEORY:

E. Witten (CMP 1989) quantized the space of all of $u(1)$*-connections on a 3-dimensional manifold* M *with the Chern-Simons Lagrangian:*

$$L(A) = \frac{1}{4\pi}\int_M \mathit{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right)$$

Partition function:

$$Z(M) = \int e^{\frac{i}{\hbar} L(A)} \mathcal{D}A$$

where the integral is over gauge equivalence classes of $u(1)$*-connections in* M*.*

$$Z(M) = Z(M)$$

EXPLANATION:

$\mathcal{J}(\Sigma_g)$ = *Div*$^{(0)}$/*principal*=*stable line bundles over* Σ_g
= *Hom*$(\pi_1(\Sigma_g), U(1))$= *moduli space of flat* $u(1)$*-connections on* Σ_g.

Quantizing the Jacobian variety means quantizing the moduli space of flat $u(1)$*-connections on* Σ_g.

- *Method 1. Weyl quantization*
- *Method 2. Witten's path integral quantization*

Witten's method has many symmetries (gauge, topological).

Weyl quantization is symmetric under the action of the mapping class group.

Weyl quantization and the path integral quantization coincide (gauge group $SU(2)$ *on the torus G-Uribe, CMP 2003, gauge group* $U(1)$ *on general surfaces Andersen, CMP 2005).*