Homework 7

1. Prove that not all zeros of the polynomial

$$P(x) = x^4 - \sqrt{7}x^3 + 4x^2 - \sqrt{22}x + 15$$

are real.

2. Prove that $e^x \geq x^e$ for all $x > 0$.

3. Let $\alpha > 1$. Prove that for all $x > -1$,

$$(1 + x)^\alpha \geq 1 + \alpha x.$$

4. Let $a > b > 0$ and let $n \in \mathbb{N}$, $n \geq 2$. Prove that

$$a^{1/n} - b^{1/n} < (a - b)^{1/n}.$$

5. Prove that for $x > 1$,

$$\frac{x - 1}{x} < \ln x < x - 1.$$

6. Use the Mean Value Theorem to prove that

$$|\sin x - \sin y| \leq |x - y|$$

for all $x, y \in \mathbb{R}$.

7. Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$|f(x) - f(y)| < |x - y|^2$$

for all $x, y \in \mathbb{R}$.