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The Weyl Quantization and the Quantum Group Quantization of the Moduli Space of Flat $SU(2)$-Connections on the Torus are the Same

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Abstract: We prove that, for the moduli space of flat $SU(2)$-connections on the 2-dimensional torus, the Weyl quantization and the quantization performed using the quantum group of $SL(2, \mathbf{C})$ are the same. This is done by comparing the matrices of the operators associated through the two quantizations to cosine functions. We also discuss the $*$-product of the Weyl quantization and show that it satisfies the product-to-sum formula for noncommutative cosines on the noncommutative torus.

1. Introduction

Quantization is a procedure for replacing functions on the phase space of a physical system (classical observables) by linear operators. While understood in many general situations, this procedure is far from being algorithmic. Some more exotic spaces whose quantizations are of interest to mathematicians are the moduli spaces of flat connections on a surface. Among them the case of the moduli space of flat $SU(2)$-connections on a torus is a particularly simple example of an algebraic variety that fails to be a manifold, yet is very close to being one.

In this paper we compare two methods of quantizing the moduli space of flat $SU(2)$-connections on the torus. The first uses the quantum group of $SL(2, \mathbf{C})$. This quantization scheme arose when Reshetikhin and Turaev constructed a topological quantum field theory associated to the Jones polynomial of a knot. It describes both the quantum observables and the Hilbert spaces in terms of knots and links colored by representations of the quantum group of $SL(2, \mathbf{C})$. Heuristically, the operators of the quantization were defined by Witten using path integrals for the Chern-Simons action.

On the other hand, the moduli space of flat $SU(2)$-connections on the torus is the same as the character variety of $SU(2)$-representations of its fundamental group, so it admits a covering by the plane. Therefore one can apply a classical quantization procedure of the plane, in an equivariant manner, to obtain a quantization of the moduli space.

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The first such procedure was introduced by Hermann Weyl. It consists of a simple rule for assigning differential operators to exponential functions, then this is extended to all smooth functions via the inverse Fourier transform.

Our main result is the following

Theorem. *The Weyl quantization and the quantum group quantization of the moduli space of flat $SU(2)$-connections on the torus are unitary equivalent.*

The paper is structured as follows. In Sect. 2 we describe the geometric realization and the Kähler structure of the moduli space of flat $SU(2)$-connections on the torus. In Sect. 3 we review Witten's description of the quantization for the particular case of the torus with a path integral of the Chern-Simons action, then explain the rigorous construction of Reshetikhin and Turaev using quantum groups. We also describe the matrices of the operators associated to cosine functions in the basis of the Hilbert space consisting of the colorings of the core of the solid torus by irreducible representations.

We then explain the Weyl quantization of the moduli space (Sect. 4). This is done in the holomorphic setting, which can be related to the classical, real setting through the Bargmann transform. A distinguished basis of the Hilbert space is introduced in terms of odd theta functions. Section 5 contains the main result of the paper (Theorem 5.3). It shows that the two quantizations are unitary equivalent. The unitary equivalence maps the distinguished basis consisting of odd theta functions to the basis consisting of the colored cores of the solid torus. In Sect. 6 we discuss the $*$-product that arises from this quantization. We conclude with some final remarks about the quantization scheme based on the Kauffman bracket skein module, for which the result does not hold due to a sign obstruction.

2. The Phase Space We are Quantizing

Throughout the paper $\mathbf{T}^2$ will denote the 2-dimensional torus. The moduli space of flat $SU(2)$-connections on a surface is the same as the character variety of $SU(2)$-representations of the fundamental group of the surface [2], i.e. the set of the morphisms of the fundamental group of the surface into $SU(2)$ modulo conjugation. This is a complex algebraic variety. In the case of the torus, morphisms from $\pi_1(\mathbf{T}^2) = \mathbf{Z}\oplus\mathbf{Z}$ to $SU(2)$ are parameterized by the images of the two generators of $\mathbf{Z}\oplus\mathbf{Z}$, i.e. by two commuting matrices. The two matrices can be simultaneously diagonalized. Moreover conjugation can permute simultaneously the entries in the two diagonal matrices. So the moduli space of flat $SU(2)$-connections on the torus can be described geometrically as

$$\mathcal{M} = \{(s,t) \quad | \quad |s| = |t| = 1\}/(s,t) \sim (\bar{s},\bar{t}).$$

This set is called the "pillow case". It has a 2-1 covering by the torus, with branching points $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$. We will think of the moduli space as the quotient of the complex plane by the group Λ generated by the translations $z \to z+1$ and $z \to z+i$, and by the symmetry with respect to the origin $\sigma(z) = -z$.

Off the four singularities $\mathcal{M}$ is a Kähler manifold, with Kähler form ω induced by $-\pi dz\wedge d\bar{z} = 2\pi i dx\wedge dy$ on $\mathbf{C}$. Note that

$$\pi dz\wedge d\bar{z} = \partial\bar{\partial}\ln h_0(z,\bar{z}),$$

where the Kähler potential $h_0(z,\bar{z}) = e^{-\pi|z|^2}$ is nothing but the weight of the Bargmann measure on the plane. Also, note that the Kähler form ω is the genus one case of Goldman's symplectic form defined in [11].

The classical observables are the C^∞ functions on this variety. Using the covering map we identify the algebra of observables on the character variety with the algebra of functions on $\mathbf{C} = \mathbf{R} \oplus \mathbf{R}$ generated by $\cos 2\pi(px + qy)$, $p, q \in \mathbf{Z}$.

Another family of important functions on $\mathcal{M}$ are $\sin 2\pi n(px+qy)/\sin 2\pi(px+qy)$ where p and q are coprime. As functions of connections, these associate to a connection the trace in the n-dimensional irreducible representation of $SU(2)$ of the holonomy of the connection around the curve of slope p/q on the torus.

To quantize $\mathcal{M}$ means to replace classical observables f by linear operators $op(f)$ on some Hilbert space, satisfying Dirac's conditions:

1. $op(1) = Id$,
2. $op(\{f, g\}) = \frac{1}{i\hbar}[op(f), op(g)] + O(\hbar)$.

Here $\{f, g\}$ is the Poisson bracket induced on $\mathcal{M}$ by the form ω, which is

$$\{f, g\} = \frac{1}{2\pi i}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right).$$

Also $[A, B]$ is the commutator of operators, and $\hbar$ is Planck's constant.

Since in our case the phase space is an orbifold covered by the plane, and this orbifold is Kähler off singularities, it is natural to perform equivariant quantization of the plane. We do this using Weyl's method and then compare the result with the quantization from [17] which was done using the quantum group of $SL(2, \mathbf{C})$.

3. Review of the Quantum Group Approach

The quantization of the moduli space of flat $SU(2)$-connections on a surface performed using the quantum group of $SL(2, \mathbf{C})$ at roots of unity is an offspring of Reshetikhin and Turaev's construction of quantum invariants for 3-manifolds [17]. Their work was inspired by Witten's heuristic explanation of the Jones polynomial using Chern-Simons topological quantum field theory. We present below Witten's idea for the particular case of the torus, and then show how the Reshetikhin-Turaev construction yields a quantization of $\mathcal{M}$.

3.1. Path integrals. In [22], Witten outlined a way of quantizing the moduli space of flat connections on a trivial principal bundle with gauge group a simply connected compact Lie group. This space is identified with the symplectic quotient of the total space of connections under the action of the group of gauge transformations [2]. Let us recall how this is done when the group is $SU(2)$ and the principal bundle lies over the cylinder over the torus $M = \mathbf{T}^2 \times [0, 1]$.

For A an $SU(2)$-connection on M define the Chern-Simons Lagrangian to be

$$\mathcal{L} = \frac{1}{4\pi}\int_M \mathrm{tr}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A),$$

where tr is the trace on the 2-dimensional irreducible representation of $su(2)$. The Lagrangian is invariant under gauge transformations up to the addition of an integer.

Now, using Witten's idea we associate operators to the observables of the form $2\cos 2\pi(px + qy)$ and $\sin 2\pi n(px + qy)/\sin 2\pi(px + qy)$ on the torus $p, q, n \in \mathbf{Z}$.

Let N be some fixed integer called the level of the quantization. As such, Planck's constant is $\hbar = \frac{1}{N}$. Assume that p and q are arbitrary integers, and let n be their greatest common divisor. Denote $p' = p/n$, $q' = q/n$. Consider the cylinder over the torus $\mathbf{T}^2 \times [0, 1]$ and let C be the curve of slope p'/q' in $\mathbf{T}^2 \times \{\frac{1}{2}\}$. The operator associated by the quantization to the function $\sin 2\pi n(p'x + q'y)/\sin 2\pi(p'x + q'y)$ and denoted shortly by $S(p, q)$ (S from sine) is the integral with kernel defined by the following path integral:

$$< A_1|S(p,q)|A_2 >= \int_{\mathcal{M}_{A_1,A_2}} e^{iN\mathcal{L}(A)} \mathrm{tr}_{V^n}(\mathrm{hol}_C(A))\mathcal{D}A,$$

where A_1, A_2 are conjugacy classes of connections on $\mathbf{T}^2$, A is a conjugacy class of connections under the action of the gauge group on $\mathbf{T}^2 \times [0, 1]$ such that $A|_{\mathbf{T}^2\times\{0\}} = A_1$ and $A|_{\mathbf{T}^2\times\{1\}} = A_2$, and $\mathrm{tr}_{V^n}(\mathrm{hol}_C(A))$, known as the Wilson line, is the trace of the n-dimensional irreducible representation of $SU(2)$ evaluated on the holonomy of A around C of slope p'/q'. Here the "average" is taken over all conjugacy classes of connections modulo the gauge group.

With the same notations one defines the operator $C(p, q)$ (C from cosine) representing the quantization of the function $2\cos 2\pi(px + qy)$ by

$$< A_1|C(p,q)|A_2 >= \int_{\mathcal{M}_{A_1,A_2}} e^{iN\mathcal{L}(A)} (\mathrm{tr}_{V^{n+1}} - \mathrm{tr}_{V^{n-1}})(\mathrm{hol}_C(A))\mathcal{D}A.$$

This is so because of the formula

$$2\cos nx = \frac{\sin(n+1)x}{\sin x} - \frac{\sin(n-1)x}{\sin x}.$$

We only discuss briefly the Hilbert space of the quantization, assuming the reader is familiar with [1] and [22]. The next section will make these ideas precise. The Hilbert space is spanned by the quantum invariants (i.e. partition functions) of all 3-manifolds with boundary equal to the torus. Since any 3-manifold can be obtained by performing surgery on a link that lies in the solid torus, it follows that the Hilbert space of the quantization is spanned by the partition functions of pairs of the form $(S^1 \times \mathbf{D}^2, L)$, where L is a (colored) link in the solid torus $S^1 \times \mathbf{D}^2$.

3.2. Quantization using the quantum group of $SL(2,\mathbf{C})$. The quantization of the character variety of the torus using the quantum group of $SL(2, \mathbf{C})$ is a particular consequence of the topological quantum field theory constructed in [17]. Let us discuss its essential features.

Fix a level $r \geq 3$ of the quantization, and let $t = e^{\frac{\pi i}{2r}}$. Comparing with the previous section, $N = 2r$. Quantized integers are defined by the formula $[n] = (t^{2n} - t^{-2n})/(t^2 - t^{-2})$. The quantum algebra of $sl(2, \mathbf{C})$, denoted by $\mathbf{U}_t$ is a deformation of its universal enveloping algebra and has generators X, Y, K subject to the relations

$$KX = t^2XK, \quad KY = t^{-2}YK, \quad XY - YX = \frac{K^2 - K^{-2}}{t^2 - t^{-2}},$$
$$X^r = Y^r = 0, \quad K^{4r} = 1.$$

This algebra is Hopf, so its representations form a ring under the operations of direct sum and tensor product. Reducing modulo summands of quantum trace zero, this ring contains a subring generated by finitely many irreducible representations $V^1, V^2, \ldots, V^{r-1}$. Here V^k has dimension k, basis $e_{-(k-1)/2}, e_{-(k-3)/2} \ldots, e_{(k-1)/2}$ and the action of $\mathbf{U}_t$ is defined by

$$\begin{aligned} Xe_j &= [m+j+1]e_{j+1}, \\ Ye_j &= [m-j+1]e_{j-1}, \\ Ke_j &= t^{2j}e_j. \end{aligned}$$

The idea originating in [15] and further developed in [17] is to color any link in a 3-dimensional manifold by such irreducible representations. For a knot K we denote by $V^n(K)$ its coloring by V^n. Motivated by the representation theory of $\mathbf{U}_t$ we extend formally the definition of the $V^n(K)$ to all integers n by the rules $V^r(K) = 0$, $V^{n+2r}(K) = V^n(K)$ and $V^{r+n}(K) = -V^{r-n}(K)$. Here the negative sign means that we color the knot by V^{r-n}, then consider the vector with opposite sign in $V(\mathbf{T}^2)$.

There is a rule, for which we refer the reader to [15] and [17], for associating numerical invariants to colored links in the 3-sphere. Briefly, the idea is to use a link diagram such as the one in Fig. 1, with the local maxima, minima and crossings separated by horizontal lines, and then define an automorphism of $\mathbf{C}$ by associating to minima maps of the form $\mathbf{C} \to V^n \otimes V^n$, to maxima maps of the form $V^n \otimes V^n \to \mathbf{C}$, to crossings the quasitriangular R matrix of the quantum algebra in the n-dimensional irreducible representation, and to parallel strands tensor products of representations. The automorphism is then the multiplication by a constant and the link invariant is equal to that constant. For K a knot in the 3-sphere, $V^2(K)$ is its Jones polynomial [14] evaluated at the specific root of unity, and for $n \geq 2$, $V^n(K)$ is called the colored (or generalized) Jones polynomial.

We are now able to describe the quantization of the torus. First consider the vector space freely spanned by all colored links in the solid torus. On this vector space define the pairing $[\cdot,\cdot]$ induced by the operation of gluing two solid tori such that the meridian of the first is identified with the longitude of the second and vice versa, as to obtain a 3-sphere (Fig. 2). The pairing of two links $[L_1, L_2]$ is equal to the quantum invariant of the resulting link in the 3-sphere. The Hilbert space of the quantization is obtained by factoring the vector space by all linear combinations of colored links λ such that $[\lambda, \lambda'] = 0$ for all λ' in the vector space. This quotient, denoted by $V(\mathbf{T}^2)$ by quantum topologists, is finite dimensional. It is the Hilbert space of the quantization.

Let α be the core $S^1 \times \{0\}$ of the solid torus $S^1 \times \mathbf{D}$, $\mathbf{D} = \{z, |z| \leq 1\}$. A basis of $V(\mathbf{T}^2)$ is given by $V^k(\alpha)$, $k = 1, 2, \ldots, r-1$. The inner product is determined by requiring that this basis is orthonormal. The pairing $[\cdot,\cdot]$ is not the inner product.

These basis elements play an important role in the construction of the Reshetikhin-Turaev invariants of closed 3-manifolds. As suggested in [22] and rigorously done in [17], the quantum invariant of a 3-manifold obtained by performing surgery on a link is calculated by gluing to the complement of the link solid tori with cori colored by irreducible representations, computing the colored link invariants and then summing over all possible colorings.

The choice of the inner product is not accidental. The orthonormal basis $V^k(\alpha)$, $k = 1, 2, \ldots, r-1$ arises, by applying Walker's axioms for a topological quantum field theory with corners [20], from the basis β_k^k of the vector space of the annulus with boundary labeled by k, with the inner product $< \beta_k^k, \beta_k^k >= \sqrt{\sum_m [m]}/[k]$. This inner product on the annulus is defined using the trace pairing for representations of $\mathbf{U}_t$ [8].

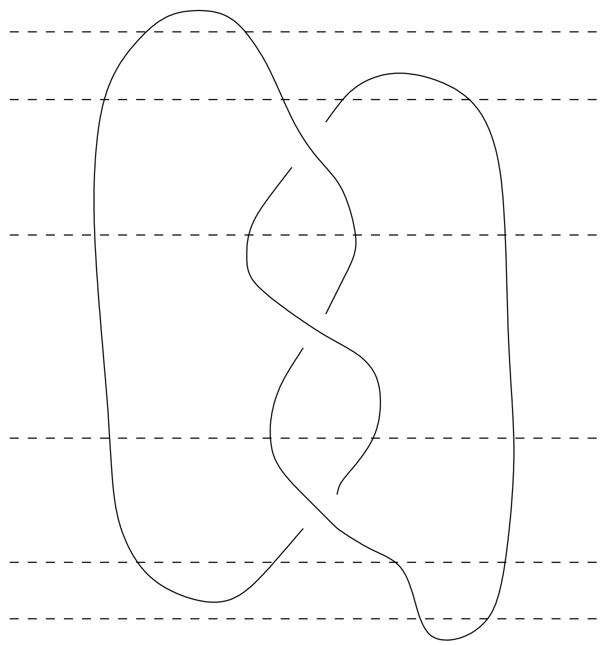

Fig. 1. Knot diagram

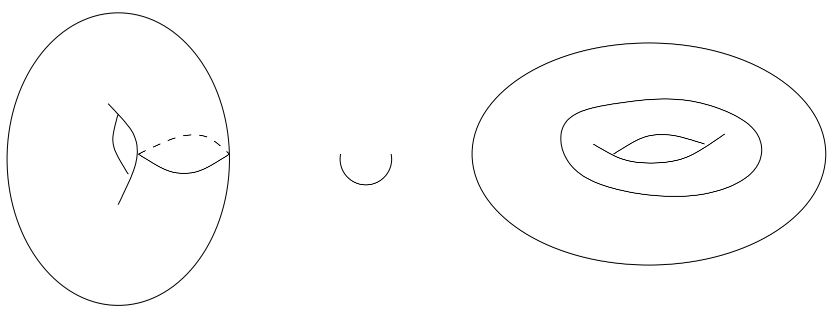

Fig. 2. Heegaard decomposition of the sphere

Consider now two integers p and q, let n be their greatest common divisor, and let also $p' = p/n$, $q' = q/n$. The operator $S(p, q)$ associated to the classical observable $\sin 2\pi n(p'x + q'y)/\sin 2\pi(p'x + q'y)$ is obtained by coloring the curve of slope p'/q' in the cylinder over the torus by the representation V^n. As explained in the previous section, the operator that quantizes $2\cos 2\pi(px + qy)$ is

$$C(p, q) = C(np', nq') = S((n + 1)p', (n + 1)nq') - S((n - 1)p', (n - 1)q').$$

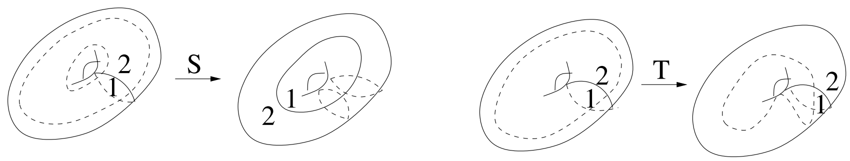

Fig. 3. S and T maps

Their action on the Hilbert space of the solid torus is defined by gluing the cylinder over the torus to the solid torus. The operator associated to an arbitrary function in $C^\infty(\mathcal{M})$ is defined by approximating the function with trigonometric polynomials in cosines, quantizing those, then passing to the limit.

3.3. The matrices of observables for the quantum group quantization. In this section we compute the matrices of the quantum group quantization of the observables $\cos 2\pi(px+qy)$, $p, q \in \mathbf{Z}$.

Theorem 1. *In any level r and for any integers p, q and k the following formula holds*

$$C(p,q)V^k(\alpha) = t^{-pq}\left(t^{2qk}V^{k-p}(\alpha) + t^{-2qk}V^{k+p}(\alpha)\right).$$

Proof. The proof uses topological quantum field theory with corners. This is a concept introduced by K. Walker which refines Atiyah's axioms of a TQFT [1] to allow gluings with corners. Details about this can be found in [20, 10]. The idea of the proof is to reduce the problem to the computation of the quantum invariant of a link complement, then construct the link complement from simple manifolds and use Walker's axioms to compute its invariant.

In Walker's theory a key role is played by decompositions of the boundaries of manifolds into disks, annuli and pairs of pants together with some curves (seams) that keep track of twistings. These structures on the boundary are called DAP-decompositions. At the level of the vector space they correspond to choices of a basis. Gluings along subsurfaces (i.e. along a subset of the disks, annuli and pairs of pants of the decomposition) yield contractions of the Hilbert space of the quantization.

Here are some facts needed below. Let $X = \sqrt{\sum_{j=1}^{r-1}[j]^2}$ (not to be confused with the generator of $\mathbf{U}_t$). The vector space of an annulus has basis β_j^j, $j = 1, 2, \ldots, r-1$, with pairing given by

$$<\beta_j^j, \beta_k^k> = \delta_{j,k}X/[j].$$

Gluing the boundaries of an annulus we obtain a torus, with the same basis for the vector space. The pairing on the torus will make β_j^j an orthonormal basis, but we don't need this now. The moves S and T on the torus are described in Fig. 3. The (m,n)-entry of the matrix of S is $[mn]$. The move T is diagonal and its j^{th} entry is t^{j^2-1}.

The quantum invariant of the cylinder over a surface is the identity matrix. So the quantum invariant of the cylinder over an annulus is $\sum_{n=1}^{r-1}\frac{[n]}{X}\beta_n^n \otimes \beta_n^n \otimes \beta_n^n \otimes \beta_n^n$ (when taking the cylinder over the annulus the boundary of the solid torus will be canonically decomposed into four annuli). If the DAP-decomposition of a 3-manifold involves two disjoint annuli, its invariant can be written in the form

$$\sum_{k,j=1}^{r-1} \beta_k^k \otimes \beta_j^j \otimes v_{k,j}.$$

Gluing the two annuli produces a 3-manifold that in the newly obtained DAP-decomposition has the invariant equal to $\sum_{k=1}^{r-1} \frac{X}{[k]} v_{k,k}$.

In the proof of the theorem we use the following formula, which can be checked using the fact that the sum of the roots of unity is zero.

Lemma 1. *Let a, b, c, d and e be integers. Then*

$$\sum_{x,y=1}^{r-1} [ax]t^{bx^2}[cy]([x(y+d)]t^{2ey} + [x(y-d)]t^{-2ey})$$
$$= X^2 t^{bc^2+be^2-2de}([a(c+e)]t^{2(be-d)c} + [a(c-e)]t^{-2(be-d)c}).$$

It suffices to check that the two sides of the equality yield the same results when paired with all $V^m(\alpha)$. Recalling the pairing $[\cdot,\cdot]$ on the solid torus, we must show that

$$[C(p,q)V^k(\alpha), V^m(\alpha)]$$
$$= \frac{t^{-pq}}{t^2-t^{-2}}\left(t^{2(qk-pm+km)} - t^{2(qk+pm-km)} + t^{2(-qk+pm+km)} - t^{2(-qk-pm-km)}\right).$$

Let d to be the greatest common divisor of p and q, $p' = p/d$ and $q' = q/d$. We concentrate first on the computation of

$$[S((d+1)p', (d+1)q')V^k(\alpha), V^m(\alpha)] \tag{1}$$

and

$$[S((d-1)p', (d-1)q')V^k(\alpha), V^m(\alpha)]. \tag{2}$$

Here is the place where we use the topological quantum field theory with corners from [10].

The expression in (1) is the invariant of the link that has one component equal to the (p', q')-curve on a torus colored by V^{d+1}, and the other two components the cores of the two solid tori that lie on one side and the other of the torus knot (see Fig. 4), colored by V^k (the one inside) and V^m (the one outside). The expression in (2) is the invariant of the same link but with the (p', q')-curve colored by V^{d-1}. It was shown in [10] that this number is equal to X^{-1} times the coordinate of $\beta_{d+1}^{d+1} d \otimes \beta_k^k \otimes \beta_m^m$ of the vector that is the quantum invariant of the link complement.

Let us produce the complement of this link by gluing together two simple 3-manifolds, whose quantum invariants are easy to compute. Consider first the cylinder over an annulus A and glue its ends to obtain the manifold $A \times S^1$. In the basis of the vector space of $V(\mathbf{T}^2 \times \mathbf{T}^2)$ determined by the DAP-decomposition $\partial A \times \{1\}$ the invariant of this manifold is $\sum_k \beta_k^k \otimes \beta_k^k$. Take another copy of the same manifold. Change the decomposition curves of the exterior torus of the first manifold to the p'/q'-curve and of the exterior torus of the second manifold to the longitude. Of course, to do this on

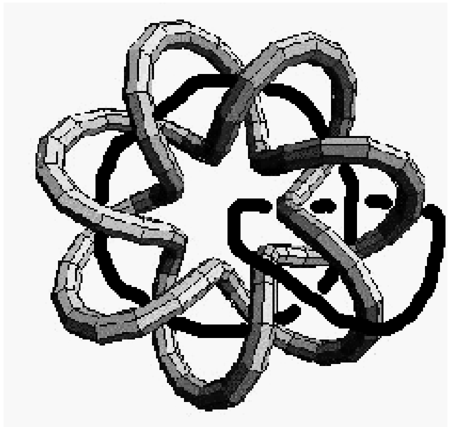

Fig. 4. Link complement

the second manifold we apply the S-move and so the invariant of the second manifold changes to

$$\frac{1}{X}\sum_{\delta, j_{n+1}} [dj_{n+1}]\beta_\delta^\delta \otimes \beta_{j_{n+1}}^{j_{n+1}}.$$

With the first manifold the story is more complicated. Consider the continued fraction expansion

$$\frac{q'}{p'} = -\frac{1}{-a_1 - \frac{1}{-a_2 - \cdots \frac{1}{-a_n}}}.$$

The required move on the boundary is then $ST^{-a_n}ST^{-a_{n-1}}S\cdots ST^{-a_1}S$. So the invariant of the first manifold in the new DAP-decomposition is

$$X^{-n-1}\sum_{j_1,\cdots,j_{n+1}} [j_{n+1}j_n]t^{-a_n(j_n^2-1)}[j_nj_{n-1}]\cdots[j_2j_1]t^{-a_1(j_1^2-1)}[j_1k]\beta_k^k \otimes \beta_{j_{n+1}}^{j_{n+1}}.$$

Now expand one annulus in the exterior tori of each of the two manifolds. Then glue just one annulus from the the first manifold to one annulus from the second. This way we obtain the complement of the link in the discussion. One of its boundary tori is decomposed into two annuli. Contract one of them. Since the gluing introduced a factor of $X/[j_{n+1}]$, the invariant of the manifold is

$$X^{-n-1}\sum_{j_1,\cdots,j_{n+1},\delta} \frac{[\delta j_{n+1}]}{[j_{n+1}]}[j_{n+1}j_n]t^{-a_n(j_n^2-1)}[j_nj_{n-1}]\cdots[j_2j_1]t^{-a_1(j_1^2-1)}$$
$$\times [j_1k]\beta_{j_{n+1}}^{j_{n+1}} \otimes \beta_\delta^\delta \otimes \beta_k^k.$$

At this moment we have the right 3-manifold but with the wrong DAP-decomposition. We need to fix the DAP-decomposition of the torus that corresponds to the basis element $\beta_{j_{n+1}}^{j_{n+1}}$ (the boundary of the regular neighborhood of the (p', q')-curve) such as to transform the decomposition curve into the meridian of the link component. For this we apply the move $(ST^{-a_n}ST^{-a_{n_1}}S\cdots ST^{-a_1}S)^{-1}$. We obtain the following expression for the invariant of the extended manifold

$$X^{-2n-2}\sum_{j_1,\ldots,j_{2n+2},\delta,k,m}[mj_{2n+2}][j_{2n+2}j_{2n+1}]t^{a_1j_{2n+1}^2}\cdots[j_{n+2}j_{n+1}]\frac{[\delta j_{n+1}]}{[j_{n+1}]}$$
$$\times [j_{n+1}j_n]t^{-a_nj_n^2}[j_nj_{n-1}]\cdots[j_2j_1]t^{-a_1j_1^2}[j_1k]\beta_\delta^\delta\otimes\beta_k^k\otimes\beta_m^m$$

(in this formula we already reduced t^{a_k} and t^{-a_k}, $1\le k\le n$).

It is important to observe that after performing the described operations the seams came right, so no further twistings are necessary.

Now fix k and m, let $\delta=d\pm1$ and focus on the coefficients of $\beta_{d\pm1}^{d\pm1}\otimes\beta_k^k\otimes\beta_m^m$. Multiplied by X these are the colored Jones polynomial of the link [10] with the (p',q')-curve colored by the $d+1$-, respectively $d-1$-dimensional irreducible representation of $\mathbf{U}_t$. Since $C(p,q)=S((d+1)p',(d+1)dq')-S((d-1)p',(d-1)q')$ and also

$$\frac{[(d+1)j_{n+1}]}{[j_{n+1}]}-\frac{[(d-1)j_{n+1}]}{[j_{n+1}]}=t^{2j_{n+1}}+t^{-2j_{n+1}},$$

we deduce that the value of $[C(p,q)V^k(\alpha),V^m(\alpha)]$ is equal to

$$X^{-2n-1}\sum_{j_1,\ldots,j_{2n+2}}[mj_{2n+2}][j_{2n+2}j_{2n+1}]t^{a_1j_{2n+1}^2}\cdots[j_{n+2}j_{n+1}](t^{2dj_{n+1}}+t^{-2dj_{n+1}})$$
$$\times[j_{n+1}j_n]t^{-a_nj_n^2}[j_nj_{n-1}]\cdots[j_2j_1]t^{-a_1j_1^2}[j_1k].$$

We want to compute these iterated Gauss sums. We apply successively Lemma 1 starting with $x=j_n$, $y=j_{n+1}$, then $x=j_{n-1}$, $y=j_{n+2}$ and so on to obtain

$$\begin{aligned}
&X^{-2n+1}\sum_{j_1,\ldots,j_{2n+2}}t^{-a_nd^2}[mj_{2n+2}][j_{2n+2}j_{2n+1}]t^{a_1j_{2n+1}^2}\cdots[j_{n+3}j_{n+2}]\\
&\quad\times([j_{n-1}(j_{n+2}+d)]t^{-2a_ndj_{n+2}}+[j_{n-1}(j_{n+2}-d)]t^{2a_ndj_n+2})\\
&\quad\times t^{-a_{n-1}j_{n-1}^2}[j_{n-1}j_{n-2}]\cdots[j_2j_1]t^{-a_1j_1^2}[j_1k]\\
&=X^{-2n+3}\sum_{j_1,\ldots,j_{2n+2}}t^{-a_n(a_na_{n-1}-1)d^2}[mj_{2n+2}][j_{2n+2}j_{2n+1}]t^{a_1j_{2n+1}^2}\cdots[j_{n+4}j_{n+3}]\\
&\quad\times([j_{n-2}(j_{n+3}+a_nd)]t^{-2(a_na_{n-1}-1)dj_n+3}+[j_{n-2}(j_{n+3}-a_nd)]t^{2(a_na_{n-1}-1)dj_n+3})\\
&\quad\times t^{-a_{n-2}j_{n-2}^2}[j_{n-2}j_{n-3}]\cdots[j_2j_1]t^{-a_1j_1^2}[j_1k]=\cdots\\
&=X^{-1}\sum_{j_{2n+1},j_{2n+2}}t^{-p'q'd^2}[mj_{2n+2}][j_{2n+2}j_{2n+1}]\\
&\quad\times([kj_{2n+1}+kdq']t^{-2dp'j_{2n+1}}+[kj_{2n+1}-kdq']t^{2dp'j_{2n+1}})\\
&=X^{-1}t^{-pq}\sum_{j_{2n+1},j_{2n+2}=1}^{r-1}[mj_{2n+2}][j_{2n+2}j_{2n+1}]\\
&\quad\times([kj_{2n+1}+kq]t^{-2pj_{2n+1}}+[kj_{2n+1}-kq]t^{2pj_{2n+1}}).
\end{aligned}$$

This sum is equal to

$$t^{-pq}([k(m+q)]t^{-2mp}+[k(m-q)]t^{2mp})$$

and the theorem is proved.

4. The Weyl Quantization

The first general quantization scheme was introduced by Weyl in 1931. This scheme applies to functions on $\mathbf{R}^{2n}$ and postulates that the function $e^{2\pi i(px_j+qy_j)}$ corresponds to the operator $e^{2\pi(pX_j+qD_j)}$, where X_j is multiplication by the variable x_j and $D_j=\frac{1}{2\pi i}\frac{\partial}{\partial x_j}$. In general, the operator associated to a function is a pseudo-differential operator with symbol equal to the function. Since we are quantizing a Kähler manifold, we will convert to the complex picture using the Bargmann transform.

Here and throughout the paper we choose for Planck's constant $\hbar=\frac{1}{N}$, where $N=2r$ is an even integer. This is done so that Weil's integrality condition is satisfied, and so that the Reshetikhin-Turaev topological quantum field theory is well defined. We would like the Hilbert space of the quantization to be the space of square integrable holomorphic sections of a line bundle with first Chern class equal to $-N\omega=-2\pi iNdx\wedge dy$. It is finite dimensional since the phase space is compact and the Heisenberg uncertainty principle shows that particles occupy boxes of positive volume. It suffices to find the line bundle $\mathcal{L}$ for $N=1$, and then let the bundle for an arbitrary N be $\mathcal{L}^{\otimes N}$.

4.1. The line bundle. Recall that the moduli space $\mathcal{M}$ is obtained by factoring the complex plane by the group Λ generated by $z\to z+1$, $z\to z+i$ and $z\to\sigma(z)=-z$. The line bundle on $\mathcal{L}$ then lifts to a trivial line bundle on the complex plane. This shows that the line bundle $\mathcal{L}$ is defined by some cocycle

$$\chi:\mathbf{C}\times\Lambda\to\mathbf{C}\backslash\{0\}$$

as the quotient $\mathbf{C}\times\mathbf{C}/\sim$ under the equivalence $(z,a)\sim(w,b)$ if there is $\lambda\in\Lambda$ such that $(w,b)=(\lambda z,\chi(z,\lambda)a)$. We use the multiplicative notation since the group Λ is not commutative. The cocycle condition is

$$\chi(z,\lambda)\chi(\lambda z,\mu)=\chi(z,\mu\lambda).$$

Now we want the cocycle to be holomorphic off the singular points of the character variety, to get a holomorphic line bundle. Also we want it to be compatible with the hermitian structure. Brian Hall suggested us to work with a different hermitian structure than the one induced by the Bargmann measure. This simplifies computations, and makes the relationship with theta functions more transparent. Thus we consider the Kähler potential $h(z)=e^{-2\pi|\mathrm{Im}\,z|^2}$ which still has the property that

$$-i\pi dz\wedge d\bar{z}=\partial\bar{\partial}\ln h(z).$$

Requiring the line bundle to have curvature $i\pi dz\wedge d\bar{z}$ yields $h(z)=|\chi(z,\lambda)|^2h(\lambda^{-1}z)$. To find the cocycle χ we first determine $\chi(z,m+in)$.

Since $h(z+m+in)=\exp(-4\pi(-inz+in\bar{z}+n^2))h(z)$, it follows that

$$|\chi(z,m+in)|=\exp 2\pi(-inz+in\bar{z}+n^2).$$

From the fact that χ is holomorphic, it follows that

$$\chi(z, m+in) = \exp\pi\left(-2inz + n^2\right) \cdot \exp(i\alpha(m,n)).$$

The cocycle condition yields

$$\exp(i\alpha(m,n) + i\alpha(p,q) - i\pi(mq-np)) = \exp(i\alpha(m+p, n+q)).$$

This shows that $\exp(\alpha(m,n) - i\pi mn)$ is a morphism from $\mathbf{Z}\times\mathbf{Z}$ to S^1. We obtain

$$\chi(z, m+in) = (-1)^{mn}\exp\pi(-2inz+n^2)\exp(-2\pi i(\mu m + \nu n))$$

for some real numbers μ and ν.

Recall that σ denotes the symmetry of the complex plane with respect to the origin. We have $|\chi(z,\sigma)| = h(z)/h(-z) = 1$. Since χ is holomorphic in z it follows that $\chi(z,\sigma) = \exp(i\pi\beta)$ for some β. We want to determine β.

Use the model of the torus obtained by identifying opposite sides of a square. The action of σ maps $\frac{1}{2}$ to $-\frac{1}{2}$, and the two correspond to the same point on the character variety. So

$$\chi\left(\frac{1}{2},\sigma\right) = \chi\left(\frac{1}{2},-1\right),$$

and hence $e^{2\pi i\mu} = e^{i\pi\beta}$. This shows that $\mu = \beta/2$. The same argument with $\chi(\frac{1}{2}i,\sigma)$ and $\chi(\frac{1}{2}i,-i)$ shows that $\nu = \beta/2$. Also

$$\chi\left(\frac{1}{2}+\frac{1}{2}i,\sigma\right) = \chi\left(\frac{1}{2}+\frac{1}{2}i,-1-i\right) = e^{\pi i[2(\mu+\nu)-1]},$$

which implies that modulo 2, $2(\mu+\nu)-1=\beta$. Therefore $\beta = 1$.

We conclude that

$$\begin{aligned}\chi(z,(m+in)) &= (-1)^{mn}\exp\pi(-2inz+n^2)\\ \chi(z,\sigma) &= -1.\end{aligned}$$

4.2. The Hilbert space of the quantization. Recall that the line bundle $\mathcal{L}$ corresponds to the case where the Planck's constant is equal to 1. To get the general case with $\hbar = 1/N$, we consider the line bundle $\mathcal{L}^{\otimes N}$. This bundle has the hermitian metric defined by $h_N(z) = (h(z))^N = \exp(N\pi|\mathrm{Im}\, z|^2)$, and is given by the cocycle $\chi_N = (\chi)^N$. The Hilbert space of the quantization consists of the sections of the line bundle over $\mathbf{C}$ that are holomorphic and whose pull-backs to the plane are square integrable with respect to the measure $\exp(2N\pi y^2)dx\,dy$ and satisfy $f(\lambda z) = \chi_N(z,\lambda)f(z)$, $\lambda\in\Lambda$. The Hilbert space can thus be identified with that of holomorphic functions on the plane subject to the conditions

$$f(z+m+in) = (-1)^{mnN}e^{N\pi(n^2-2inz)}f(z)$$

and

$$f(-z) = -f(z).$$

This is nothing but the space of odd theta functions. If we drop the second condition, then we get the space of classical theta functions

$$\Theta_N = \{f \quad | \quad f(z+m+in) = e^{N\pi(n^2-2inz)} f(z)\}$$

It can be seen in [5] that the torus has other possible quantization spaces, which arise by twisting the line bundle with flat bundles. This is not the case with $\mathcal{M}$.

The Hilbert space of the quantization is

$$\mathcal{H}_N = \{f \in \Theta_N \quad | \quad f(z) = -f(-z)\}.$$

Both Θ_N and $\mathcal{H}_N$ are endowed with the inner product

$$< f, g >= \int_{\mathbf{T}^2} f(z)\overline{g(z)} e^{-2N\pi|\mathrm{Im}\, z|^2} dxdy.$$

To be more accurate, the integral that defines the inner product on $\mathcal{H}_N$ should be performed over a fundamental domain of the group Λ, however it is more convenient to compute on the torus.

Recall that an orthogonal basis of Θ_N is given by θ_j, $j = 0, 1, \ldots, 2r-1$, where

$$\theta_j(z) = \sum_{n=-\infty}^{\infty} e^{-\pi(Nn^2+2jn)+2\pi iz(j+Nn)}.$$

The formula makes sense for all j. We have $\theta_{j+N}(z) = e^{\pi(N+2j)}\theta_j(z)$ and $\theta_{-j}(-z) = \theta_j(z)$, where the second equality follows by replacing n by $-n$.

An orthonormal basis of $\mathcal{H}_N$ is given by

$$\zeta_j = \sqrt[4]{\frac{N}{2}} e^{-\pi j^2/N}(\theta_j - \theta_{-j}), \quad j = 1, 2, \ldots, r-1.$$

To see why these vectors are indeed orthonormal note that

$$\begin{aligned}< \theta_k - \theta_{-k}, \theta_j - \theta_{-j} > &= < \theta_k, \theta_j > + < \theta_{-k}, \theta_{-j} > - < \theta_{-k}, \theta_j > - < \theta_k, \theta_{-j} > \\ &= \delta_{jk} \parallel \theta_j \parallel^2 .\end{aligned}$$

Here we used the fact that the θ_j's can be shifted to have indices equal to one of the numbers $0, 1, \ldots, 2r-1$ and the latter form an orthonormal basis in the Hilbert space associated to the torus.

Using the same formula we extend the definition of ζ_k for all $k \in \mathbf{Z}$. Clearly ζ_r has to be equal to zero, since

$$\theta_r(z) - \theta_{-r}(z) = \theta_r(z) - e^{-\pi(2r-2r)}\theta_r(z) = 0.$$

Also, for $1 \le k \le r-1$ we have

$$\begin{aligned}\theta_{r+k} - \theta_{-r-k} &= e^{\pi(2r-2(-r+k))}\theta_{-r+k} - e^{-\pi(2r+2(r-k))}\theta_{r-k} \\ &= -e^{2rk\pi}(\theta_{r-k} - \theta_{-r+k}).\end{aligned}$$

Thus normalizing we get that $\zeta_{r+k} = -\zeta_{r-k}$. Finally, since θ_{j+2r} is a multiple of θ_j it follows that $\zeta_{j+2r} = \zeta_j$, for all integers j.

4.3. The operators of the quantization. Each observable $f : \mathcal{M} \to \mathbf{R}$ yields a sequence of operators indexed by the level $N = 2r$. Whenever there is no danger of confusion we omit the index N. We relate Weyl quantization to Toeplitz quantization and then work with Toeplitz operators, for which computations are easier.

Let us first consider the case of the complex plane. Modulo some adjustments to suit the notations of this paper, Propositions 2.96 and 2.97 in [7] (see also [13]) show that the operator associated by the Toeplitz quantization to a function f on $\mathbf{C}$ is equal to the operator associated by the Weyl quantization to the function

$$\sigma(z, \bar{z}) = \frac{1}{N} \int_{\mathbf{C}} e^{-2\pi |z-u|^2 N} f(u, \bar{u}) du d\bar{u}.$$

Note that

$$\sigma(z, \bar{z}) = e^{\frac{\Delta}{4N}} f(z, \bar{z}),$$

where

$$\Delta f = \frac{1}{2\pi} \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right).$$

Here we used the notation $z = x + iy$, $\bar{z} = x - iy$. Hence doing Weyl quantization with symbol f is the same as doing Toeplitz quantization with symbol $e^{-\frac{\Delta}{4N}} f$. Thinking equivariantly, we now define the operators of the quantization of the character variety. Let

$$\Pi_N : L^2(\mathcal{M}, \mathcal{L}^{\otimes N}) \to \mathcal{H}_N$$

be the orthogonal projection from the space of square integrable sections with respect to the measure $e^{-2N\pi y^2} dx dy$ onto the space $\mathcal{H}_N$.

To a sufficiently regular function f on the character variety we associate the operator $op_N(f)$ (in level N) given by

$$op_N(f) : \mathcal{H}_N \to \mathcal{H}_N, \quad g \to \Pi_N \left(\left(e^{-\frac{\Delta}{4N}} f \right) g \right).$$

The operator $g \to \Pi_N(fg)$ is the Toeplitz operator of symbol f, denoted by T_f.

An important family of operators are the ones associated to the functions $2\cos 2\pi(px + qy)$, which we denote by $\underline{C}(p, q)$. These are the same as the Toeplitz operators with symbols

$$2e^{-\frac{\Delta}{4N}} \cos 2\pi(px + qy) = 2e^{\frac{p^2+q^2}{2N}\pi} \cos 2\pi(px + qy).$$

5. Weyl Quantization Versus Quantum Group Quantization

To simplify the computation, we pull back everything to the line bundle on the torus. Hence we do the computations in Θ_N. We start with two lemmas that hold on the torus. They were inspired by [5].

Let j, k, p be integers such that $-r + 1 \le j, k \le r - 1$, $p = p_0 + \gamma N$ with γ an integer. There are two possibilities $-r+1 \le j+p_0 < N = 2r$ or $N = 2r \le j+p_0 < N+1$. Let also $u(y)$ be a bounded continuous function.

Lemma 2. *Assume that $j + p_0 < N$. Then $< e^{2\pi i p x} u(y)\theta_j, \theta_k >$ is different from zero if and only if $k = j + p_0$ and in this case it is equal to*

$$e^{-\frac{\pi}{2N} p^2 + (j+p_0/2)^2/N} \sum_{m=-\infty}^{\infty} e^{-\pi m^2/2N} e^{-2\pi i (j+p/2)m/N} \hat{u}(m),$$

where $\hat{u}(m)$ is the m^{th} Fourier coefficient of u.

Proof. Separating the variables we obtain

$$\begin{aligned} < \exp(2\pi p x) u(y)\theta_j, \theta_k > &= \int_{\mathbf{T}^2} \exp(2\pi p x) u(y) \theta_j \overline{\theta_k} e^{-2N\pi y^2} dx dy \\ &= \sum_{m,n} e^{-\pi(Nm^2 + 2jm + Nn^2 + 2kn)} \int_0^1 e^{2\pi i x (N(m-n) + p + j - k)} dx \\ &\quad \times \int_0^1 e^{-2\pi y (j + Nn + k + Nm) - 2\pi N y^2} u(y) dy. \end{aligned}$$

The first integral is equal to zero unless $n = m + \gamma$ and $k = p_0 + j$. If $k = p_0 + j$ the expression becomes

$$\begin{aligned} & e^{-\pi N \gamma^2 - 2\pi j \gamma - 2\pi p_0 \gamma} \int_0^1 \Bigg(e^{-2\pi (N y^2 + (2j + p_0 + N\gamma) y)} \\ & \quad \times \sum_m e^{-2\pi \left(Nm^2 + Nm\gamma + 2m\left(j + \frac{p_0}{2} + Ny\right)\right)} \Bigg) u(y) dy. \end{aligned}$$

After completing the square in the exponent of the third exponential we obtain that this is equal to

$$e^{-\frac{\pi}{2N} p^2 + (j+p_0/2)^2/N} \int_0^1 \sum_m e^{-2\pi N \left(m + y + \frac{j+p/2}{N}\right)^2} u(y) dy.$$

Using the Poisson formula ($\sum_m f(m) = \sum_m \hat{f}(m)$) for the function e^{-x^2} we transform the sum of the exponentials into

$$\sum_m e^{-\pi m^2/2N} e^{2\pi i \left(y + \frac{j+p/2}{N}\right) m}.$$

It follows that the inner product we are computing is equal to

$$e^{-\frac{\pi}{2N} p^2 + \pi (j+p_0/2)^2/N} \sum_m e^{-\pi m^2/2N} e^{2\pi i (j+p/2)m/N} \int_0^1 e^{2\pi i m y} u(y) dy$$

which proves the lemma.

Lemma 3. *Assume that* $j + p_0 \geq N$ *and denote* $p_1 = p - (\gamma + 1)N$. *Then* $< e^{2\pi ipx}u(y)\theta_j, \theta_k >$ *is different from zero if and only if* $k = j + p_1$ *and in this case it is equal to*

$$e^{-\frac{\pi}{2N}p^2+\pi(j+p_1/2)^2/N}\sum_{n=-\infty}^{\infty} e^{-\pi m^2/2N}e^{-2\pi i(j+p/2)m/N}\hat{u}(m),$$

where $\hat{u}(m)$ *is the* m^{th} *Fourier coefficient of* u.

Proof. We start with a computation like the one in the proof of Lemma 2 to conclude that $k = p_0 + j - N = p_1 + j$ and $m = n - \gamma - 1$. From here the same considerations apply mutatis mutandis to yield the conclusion.

Theorem 2. *The quantum group and the Weyl quantizations of the moduli space of flat* $SU(2)$*-connections on the torus are unitary equivalent. The unitary isomorphism establishing this equivalence maps the orthonormal basis* $V^j(\alpha)$, $j = 1, 2, \ldots, r-1$, *of* $V(\mathbf{T}^2)$ *to the orthonormal basis* ζ_j, $j = 1, 2, \ldots, r-1$, *of* $\mathcal{H}_N$.

Proof. One has to show that the matrices of the operators associated by the two quantizations to a given function are the same. Since linear combinations of cosines are dense in the algebra of functions on the moduli space, it suffices to check this property for cosines. The quantum group quantization associates to $\cos 2\pi(px + qy)$ the operator $C(p, q)$, while Weyl quantization associates to it the operator $\underline{C}(p, q)$. We verify that the matrix of the operator $\underline{C}(p, q)$ in the basis ζ_j is the same as the matrix of the operator $C(p, q)$ in the basis $V^j(\alpha)$. We have

$$\underline{C}(p, q) = e^{\frac{p^2+q^2}{2N}}T_{2\cos 2\pi(px+qy)},$$

where $T_{2\cos 2\pi(px+qy)}$ is the Toeplitz operator of symbol $2\cos 2\pi(px + qy)$. Let us pull back everything to the torus using the covering map $\mathbf{T}^2 \to \mathcal{M}$ so that we can work with exponentials.

We do first the case $j + p_0 < N$. If in Lemma 2 we let $u(y) = e^{2\pi iqy}$ we obtain

$$T_{e^{2\pi ipx+2\pi iqy}}\theta_j = e^{-\frac{\pi}{2N}p^2+(j+p_1/2)^2/N}e^{-\pi q^2/2N}e^{-2\pi i(j+p/2)q/N}\theta_{j+p_0}.$$

Using this formula and the fact that

$$\zeta_j = \sqrt[4]{\frac{N}{2}}e^{-\pi j^2/N}(\theta_j - \theta_{-j})$$

after doing the algebraic computations we arrive at

$$e^{\frac{p^2+q^2}{2N}\pi}T_{2\cos 2\pi(px+qy)}\zeta_j = t^{-pq}\left(t^{2jq}\zeta_{j-p_0} + t^{-2jq}\zeta_{j+p_0}\right).$$

If $j + p_0 \geq N$, we let $p_1 = p - (\gamma + 1)N$, and an application of Lemma 3 shows that in this case

$$e^{\frac{p^2+q^2}{2N}\pi}T_{2\cos 2\pi(px+qy)}\zeta_j = t^{-pq}\left(t^{2jq}\zeta_{j-p_1} + t^{-2jq}\zeta_{j+p_1}\right).$$

But we have seen that $\zeta_{j+N} = \zeta_j$ for all j, so in the above formulas p_0 and p_1 can be replaced by p. It follows that for all j, $1 \leq j \leq r-1$, and all integers p and q we have

$$e^{\frac{p^2+q^2}{2N}\pi} T_{2\cos 2\pi(px+qy)}\zeta_j = t^{-pq}\left(t^{2jq}\zeta_{j-p} + t^{-2jq}\zeta_{j+p}\right).$$

Theorem 1 shows that

$$C(p,q)V^j(\alpha) = t^{-pq}\left(t^{2jq}V^{j-p}(\alpha) + t^{-2jq}V^{j+p}(\alpha)\right).$$

Hence the unitary isomorphism defined by $V^j(\alpha) \to \zeta_j$ transforms the operator $C(p,q)$ into the operator $\underline{C}(p,q)$, and the theorem is proved.

From now on we identify the two quantizations and use the notation $C(p,q)$ for the operators. Recall the notation $t = e^{i\pi/N}$. As a byproduct of the proof of the theorem we obtain the following product-to-sum formula for $C(p,q)$'s (see also [9]).

Proposition 1. *For any integers m, n, p, q one has*

$$C(m,n) * C(p,q) = t^{\left|{mn \atop pq}\right|} C(m+p, n+q) + t^{-\left|{mn \atop pq}\right|} C(m-p, n-q),$$

where $\left|{mn \atop pq}\right|$ is the determinant.

We conclude this section by noting that in Witten's picture the operator associated by Weyl quantization to the function $\sin 2\pi(n+1)(p'x+q'y)/\sin 2\pi(p'x+q'y)$ (p', q' relatively prime) is the same as the quantum group quantization of the Wilson line around the curve of slope p'/q' on the torus in the n-dimensional irreducible representation of $SU(2)$.

6. The Star Product

6.1. Definition of the star product. Let (M, ω) be a symplectic manifold. A $*$-product on M is a binary operation on

$$C^\infty(M)[[N^{-1}]]$$

which is associative, and for all $f, g \in C^\infty(M)$ satisfies $N^{-k} f * g = f * N^{-k} g = N^{-k}(f * g)$ and also

$$f * g = \sum_{k=0}^{\infty} N^{-k} B_k(f,g).$$

The operators $B_k(f,g)$ are bi-differential operators from $C^\infty(M) \times C^\infty(M)$ to $C^\infty(M)$, such that $B_0(f,g) = fg$, and such that Dirac's correspondence principle

$$B_1(f,g) - B_1(g,f) = \{f,g\}$$

is satisfied. Here $\{f,g\}$ stands for the Poisson bracket induced by the symplectic form. One says that $C^\infty(M)[[N^{-1}]]$ is a deformation of $C^\infty(M)$ in the direction of the given Poisson bracket. We use N for the variable of the formal series to be consistent with the rest of the paper.

The character variety $\mathcal{M}$ is a symplectic manifold off the four singularities.

Proposition 2. *The formula*

$$\begin{aligned} &2\cos 2\pi(mx+ny) * 2\cos 2\pi(px+qy) \\ &\quad = t^{\left|\begin{smallmatrix} m n \\ p q\end{smallmatrix}\right|} 2\cos 2\pi((m+p)x+(n+q)y) \\ &\qquad + t^{-\left|\begin{smallmatrix} m n \\ p q\end{smallmatrix}\right|} 2\cos 2\pi((m-p)x+(n-q)y) \end{aligned}$$

defines a $*$-product on $C^{\infty}(\mathcal{M})[[N^{-1}]]$, which is a deformation quantization in the direction of the Kähler form $i\pi\, dz \wedge d\bar{z}$.

In these formulas the exponentials should be expanded formally into power series in N^{-1}.

Proof. We have

$$\begin{aligned} &2\cos 2\pi(mx+ny) * 2\cos 2\pi(px+qy) - 2\cos 2\pi(mx+ny) * 2\cos 2\pi(px+qy) \\ &\quad = \pi(imq-inp)N^{-1} 2\cos 2\pi((m+p)x+(n+q)y) \\ &\qquad + \pi(inp-imq)N^{-1} 2\cos 2\pi((m-p)x+(n-q)y) \\ &\qquad - \pi(ipn-iqm)N^{-1} 2\cos 2\pi((p+m)x+(q+n)y) \\ &\qquad - \pi(imq-inp)N^{-1} 2\cos 2\pi((p-m)x+(q-n)y) + O(N^{-2}) \\ &\quad = 2\pi i N^{-1}(mq-np) 2\cos 2\pi((m+p)x+(n+q)y) \\ &\qquad - 2\pi i N^{-1}(mq-np) 2\cos 2\pi((m-p)x+(n-q)y) + O(N^{-2}) \\ &\quad = N^{-1}\{2\cos 2\pi(mx+ny), 2\cos 2\pi(px+qy)\} + O(N^{-2}) \end{aligned}$$

so the correspondence principle is satisfied. The coefficients $B_k(f,g)$ are bidifferential operators since

$$B_1(f,g) = \frac{1}{4\pi i} \det \begin{vmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2} \end{vmatrix} f(x_1,y_1) g(x_2,y_2)\Big|_{\substack{x_1=x_2=x \\ y_1=y_2=y}}$$

and $B_k = B_1^k/k!$.

We would like to point out that this $*$-product is different from the one that would arise if we applied the quantization methods outlined in [3] since Berezin's ideas correspond to the anti-normal (respectively normal) ordering of the operators.

Thinking now of the deformation parameter as a fixed natural number we see that the $*$-algebra defined in Proposition 6.1 is a subalgebra of Rieffel's noncommutative torus [18] with rational Planck's constant. That is, our $*$-product is the restriction of Rieffel's $*$-product to trigonometric series in cosines.

6.2. The generalized Hardy space. In [12] Guillemin has shown that for each compact symplectic prequantizable manifold M there exists a $*$-product and a circle bundle $S^1 \hookrightarrow P \to M$ with a canonical volume μ such that $L^2(P,\mu)$ contains finite dimensional vector spaces $\mathcal{H}_N$ of N-equivariant functions (mod the action of S^1) satisfying for all $f, g \in C^{\infty}(M)$:

$$\Pi_N M_f \Pi_N M_g \Pi_N = \Pi_N M_{(f*g)_N} \Pi_N + O(N^{-\infty}),$$

where Π_N is the orthogonal projection onto $\mathcal{H}_N$, M_f is the multiplication by f and $(f*g)_N$ is to be understood as the $*$-product for a certain fixed integer value of N.

Our operators are not Toeplitz so they won't fit exactly Guillemin's construction. However, along the same lines we will construct a representation of the $*$-algebra from the previous section onto an infinite dimensional Hilbert space that contains all $\mathcal{H}_N$'s as direct summands. The reader can find a detailed account on how these things are done in general in [6].

Let $\mathcal{L}$ be the line bundle constructed in 4.1, and let $Z \subset \mathcal{L}^*$ be the unit circle bundle in the dual of $\mathcal{L}$. Z is an S^1-principal bundle. A point in Z is a pair (x, ϕ), where $x \in \mathbf{C}$ and ϕ is a complex valued functional with $|\phi(x)| = \|x\|$ (the length of x being given by the hermitian structure). More precisely

$$Z = \{(z, \xi); \quad |\xi| = e^{-\pi|z|^2/2}\}.$$

The map $(z, \xi) \to (z, \xi/|\xi|)$ identifies Z with $\mathbf{C} \times S^1$. Let θ be the argument of ξ and consider the volume form $d\theta dz$ on Z. Using this volume form we can define the space $L^2(Z)$.

Now let N be an even integer. Note that

$$\mathcal{L}^{\otimes N} \simeq Z \times_N \mathbf{C},$$

where $Z \times_N \mathbf{C}$ is the quotient of $Z \times \mathbf{C}$ by the equivalence

$$(p \cdot e^{iN\theta}, z) \sim (p, e^{iN\theta} z).$$

A complex valued smooth function f on Z is called N-equivariant if for all $(x, \phi) \in Z$,

$$f(x, \phi \cdot e^{i\theta}) = e^{iN\theta} f(x, \phi).$$

The set of all N-equivariant functions is denoted by $C^\infty(Z)_N$. There exists an isomorphism

$$C^\infty(\mathcal{M}, \mathcal{L}^{\otimes N}) \simeq C^\infty(Z)_N,$$

which transforms a section $s \in C^\infty(\mathcal{M}, \mathcal{L}^{\otimes N})$ to a function f, with $f(z, \xi) = \xi^N s(z)$. Here of course s has to be viewed as a $\mathbf{C}$-valued function subject to the equivariance conditions from Sect. 4. It is easy to see that this map gives rise to a unitary isomorphism between the space of L^2-sections of $\mathcal{L}^{\otimes N}$ and the L^2-completion of $C^\infty(Z)_N$. A little Fourier analysis (involving integrals of the form $\int e^{im\theta} e^{-in\theta} d\theta$) shows that for different N's, the images of the corresponding L^2 spaces are orthogonal.

As a result, the spaces $\mathcal{H}_N$ are embedded as mutually orthogonal subspaces of $L^2(Z)$. Define

$$\mathcal{H} = \bigoplus_{N \text{ even}} \mathcal{H}_N.$$

This space is the generalized version of the classical Hardy space.

We denote by Π the orthogonal projection of $L^2(Z)$ onto $\mathcal{H}$. Let f be a smooth function on $\mathcal{M}$, which can be viewed as the limit (in the C^∞ topology) of a sequence of trigonometric polynomials in $\cos(px + qy)$, $p, q \in \mathbf{Z}$. Define the operator

$$op(f) : \mathcal{H} \to \mathcal{H}, \quad g \to \Pi\left(\left(e^{-\frac{\Delta}{4N}} f\right) g\right).$$

If $e^{-\frac{\Delta}{4N}} f$ is a C^∞ function on the character variety, this operator is bounded. The restriction of $op(f)$ to $\mathcal{H}_N$ coincides with $op_N(f)$, for all $N \geq 1$. Moreover, the product-to-sum formula from Proposition 5.4 shows that the $*$-product on $C^\infty(\mathcal{M})[[N^{-1}]]$ defined by the multiplication of these operators is the same as the $*$-product introduced in Sect. 6.1.

7. Final Remarks

An alternative approach to the Reshetikhin-Turaev theory was constructed in [4] using Kauffman bracket skein modules. This approach also leads to a quantization of the moduli space of flat $SU(2)$-connections on the torus. Do we obtain the Weyl quantization in that situation as well? The answer is no.

Indeed, the analogues of the basis vectors $V^n(\alpha)$ are the colorings of the core of the solid torus by Jones-Wenzl idempotents. More precisely, to the vector $V^n(\alpha)$ corresponds the vector $S_{n-1}(\alpha)$, where $S_{n-1}(\alpha)$ is the coloring of α by the $n-1^{\text{st}}$ Jones-Wenzl idempotent. It follows from Theorem 5.6 and the discussion preceding it in [9] that the action of the operator associated to $2\cos 2\pi(px+qy)$, denoted by $(p,q)_T$, on these basis elements is

$$(p,q)_T S_{n-1}(\alpha) = (-1)^q t^{-pq}(t^{2qk} S_{k-p-1}(\alpha) + t^{-2qk} S_{k+p-1}(\alpha)).$$

The factor $(-1)^q$ does not appear in the formula from Theorem 1, proving that this quantization is different. However, both quantizations yield the same $*$-algebra, as shown in [9].

Finally, although quantum field theory is intimately related to Wick quantization, the present paper shows that this is not the case with the topological quantum field theory of Witten, Reshetikhin and Turaev.

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