Oscillation of second-order dynamic equations

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Abstract: In this paper, we consider the second-order nonlinear dynamic equations

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) = 0 \text{ and } (p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0$$

on an isolated time scale $\mathbb{T}$. Our first goal is to establish a relationship between the oscillatory behaviour of these equations. Here we assume that $\tau : \mathbb{T} \to \mathbb{T}$. We also give two results about the behaviour of the linear form of the latter equation on a general time scale that is unbounded above. We use the Riccati transformation technique to obtain our results.

Keywords: oscillation; dynamic equations; time scale; functional equation.

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Biographical notes: Raegan Higgins is an Assistant Professor in the Department of Mathematics and Statistics at Texas Tech University. She received her Bachelor's Degree in Mathematics from Xavier University of Louisiana in 2002 and her Doctorate in Mathematics from the University of Nebraska-Lincoln in 2008. Her current research is in time scales; her interests focus on oscillation criteria for certain linear and nonlinear second order dynamic equations. She is also interested in applications of time scales to biology, economics, and engineering.

1 Introduction

Oscillation theory on the real numbers and the integers has drawn increasing interest in recent years. Most of the results on the real numbers $\mathbb{R}$ have corresponding results on the integers $\mathbb{Z}$ and conversely since there is a close relationship between them. This connection, revealed by Hilger (1990), unifies continuous and discrete analysis by a new theory called time scale theory. A book on the subject of time scales by Bohner and Peterson (2001) summarises and organises much of the time scale calculus.

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For completeness, we recall the following concepts related to the notion to time scales. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers and, since oscillation of solutions is our primary concern, we make the blanket assumption that $\sup \mathbb{T} = \infty$. We assume throughout that $\mathbb{T}$ has the topology it inherits for the standard topology on $\mathbb{R}$. The *forward* and *backward jump operators* are defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in T : s < t\},$$

where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be *left-dense* if $t > \inf \mathbb{T}$ and $\rho(t) = t$, *right-dense* if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, *left-scattered* if $\rho(t) < t$, *right-scattered* if $\sigma(t) > t$, *dense* if $\rho(t) = t = \sigma(t)$, and *isolated* if $\rho(t) < t < \sigma(t)$. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be *right-dense continuous* provided g is continuous at right-dense points, and at left-dense points in $\mathbb{T}$, left-hand limits exist and are finite. The set of all right-dense continuous functions is denoted by $C_{rd}(\mathbb{T})$. The *graininess function* μ for a time scale $\mathbb{T}$ is defined by $\mu(t) := \sigma(t) - t$, and for any function $f : \mathbb{T} \to \mathbb{R}$ the notation $f^\sigma(t)$ denotes $f(\sigma(t))$.

The problem of obtaining conditions to ensure that all solutions of certain classes of second-order dynamic equations are oscillatory has been studied by several authors (see Agarwal et al., 2005; Akin et al., 2001; Bohner and Saker, 2004; Erbe, 2001; Erbe and Pererson, 2002; Erbe et al., 2002, 2003; Saker, 2004; Zhang and Shanliang, 2005). Some of these results have been for the nonlinear dynamic equation of the form

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0 \quad \text{for} \quad t \in [a, b]_\mathbb{T}, \tag{1.1.1}$$

where $p(t)$ and $q(t)$ are positive, real-valued, right-dense continuous functions defined on $[a, b]_\mathbb{T}$ and $\int_a^\infty \frac{\Delta t}{p(t)} < \infty$ and/or $\int_a^\infty \frac{\Delta t}{p(t)} = \infty$. Zhang and Shanliang (2005) consider (1.1.1) in the case $p(t) = 1$ and

$$y^{\Delta\Delta}(t) + q(t)f(y(t - \tau)) = 0,$$

where $\tau \in \mathbb{R}$, and established some oscillation results for both equations.

In this paper, we consider the second-order nonlinear functional dynamic equation

$$(p(t)y^\Delta)^\Delta + q(t)f(y(\tau(t))) = 0 \tag{1.1.2}$$

and the second-order nonlinear dynamic equation

$$(p(t)y^\Delta)^\Delta + q(t)f(y^\sigma(t)) = 0 \tag{1.1.3}$$

on a time scale $\mathbb{T}$. We shall assume the following conditions hold:

(H_1) $\quad p \in C_{rd}(\mathbb{T}, (0, \infty))$ satisfies $\displaystyle\int_{t_0}^\infty \frac{1}{p(t)} \Delta t = \infty, \quad t \in \mathbb{T};$

(H_2) $\quad q \in C_{rd}(\mathbb{T}, [0, \infty));$

(H_3) $\tau \in C_{rd}(\mathbb{T}, \mathbb{T})$ satisfies $\lim_{t\to\infty} \tau(t) = \infty$ and there exists $M > 0$ such that

$$|P(t) - P(\tau(t))| < M \quad \text{for all } t \in \mathbb{T},$$

where $P(t) = \int_{t_0}^{t} \frac{1}{p(s)} \Delta s$

(H_4) $f : \mathbb{R} \to \mathbb{R}$ is continuous, increasing, and $f(-u) = -f(u)$ for $u \in \mathbb{R}$.

By a solution of (1.1.2) we mean a nontrivial real-valued function y satisfying (1.1.2) for $t \geq t_0 \geq a \in \mathbb{T}$, where $a > 0$. A solution y of (1.1.2) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is nonoscillatory. Equation (1.1.2) is said to be oscillatory if all its solutions are oscillatory. Our attention is restricted to those solutions of $(p(t)y^{\Delta})^{\Delta}$ $+q(t)f(y(\tau(t))) = 0$ which exist on some half line $[t_y, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|y(t)| : t > t_0\} > 0$ for any $t_0 \geq t_y$.

We note that (1.1.2) in its general form includes several types of difference equations with delay arguments. In addition, different equations correspond to the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{Z}$, we have $y^{\Delta} = \Delta y$ and (1.1.2) becomes the delay difference equation

$$\Delta[p(t)(y(t+1) - y(t))] + q(t)f(y(\tau(t))) = 0,$$

where Δ denotes the forward difference operator. When $\mathbb{T} = \{q_0^k : k \in \mathbb{N}_0\}$ with $q_0 > 1$, (1.1.2) becomes the delay q_0-difference equation

$$y(q_0^2 t) - (q_0 + 1)y(q_0 t) + q_0 y(t) q_0 (q_0 - 1)^2 t^2 + q(t)f(y(\tau(t))) = 0.$$

In the next section, we establish a relationship between the oscillatory behaviour of (1.1.2) and (1.1.3). We present two lemmas necessary to prove our first main result. In the last section, we present oscillation criteria for the linear form of (1.1.3). We use the Riccati transformation to obtain these results and close with an example.

2 The oscillatory correlation of (1.1.2) and (1.1.3)

Throughout this section, we assume $\mathbb{T}$ is isolated. We begin with the following definition.

Definition 2.1: A nonempty closed subset K of a Banach space $\mathbb{X}$ is called a cone if it possess the following properties:

i if $\alpha \in \mathbb{R}^+$ and $x \in K$, then $\alpha x \in K$

ii if $x, y \in K$, then $x + y \in K$

iii if $x \in K - \{0\}$, then $-x \notin K$.

Let $\mathbb{X}$ be a Banach space and K be a cone with nonempty interior. Then we define a partial ordering $\leq$ on $\mathbb{X}$ by

$$x \leq y \quad \text{if and only if } y - x \in K.$$

We will use the following theorem (Erbe et al., 1995) in order to prove some of our results.

Theorem 2.2 (Knaster's fixed-point theorem)**:** *Let X be a partially ordered Banach space with ordering $\leq$. Let Ω be a subset of X with the following properties: The infimum of Ω belongs to Ω and every nonempty subset of Ω has a supremum which belongs to Ω. If $S : \Omega \to \Omega$ is an increasing mapping, then S has a fixed point in Ω.*

We continue with the following lemma.

Lemma 2.3: *A necessary and sufficient condition for equation* (1.1.3) *to be oscillatory is that the inequality*

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) \leq 0 \tag{2.2.1}$$

has no eventually positive solutions.

Proof: NECESSITY. Suppose that (1.1.3) is oscillatory, and without loss of generality, assume that (2.2.1) has an eventually positive solution y, namely, there exists $t_0 \in \mathbb{T}$ $(t_0 \geq a)$ such that $y(t) > 0$ for $t \geq t_0$. As $\sigma(t) \geq t$ for all t, $\sigma(t) \geq t_0$ for all $t \in [t_0, \infty)_{\mathbb{T}}$. Then $y^{\sigma}(t) > 0$ for $t \geq t_0$. Using this fact along with condition (H_4), we have $(p(t)y^{\Delta}(t))^{\Delta} \leq 0$ for $t \geq t_0$, and so $p(t)y^{\Delta}(t)$ decreases on $[t_0, \infty)_{\mathbb{T}}$.

We claim that $y^{\Delta}(t) > 0$ for all large t. If not, then for some $t_1 \in [t_0, \infty)_{\mathbb{T}}$, we have $y^{\Delta}(t_1) \leq 0$. It follows that $p(t)y^{\Delta}(t) \leq 0$, $t \in [t_1, \infty)$. Now, if $y^{\Delta}(t_2) < 0$ for some $t_2 \geq t_1$, then

$$\begin{aligned} y(t) - y(t_2) &= \int_{t_2}^{t} y^{\Delta}(s)\Delta s \\ &= \int_{t_2}^{t} \frac{p(s)y^{\Delta}(s)}{p(s)}\Delta s \\ &\leq p(t_2)y^{\Delta}(t_2)\int_{t_2}^{t} \frac{\Delta s}{p(s)} \\ &\to -\infty \quad \text{as } t \to \infty, \end{aligned}$$

which is a contradiction to our assumption that $y(t) > 0$ for $t \geq t_0$. Hence $y^{\Delta}(t) \equiv 0$ on $[t_1, \infty)$, and so $(p(t)y^{\Delta}(t))^{\Delta} \equiv 0$ and $q(t)f(y^{\sigma}(t)) > 0$, which is contradictory. Consequently, there exists $T \in \mathbb{T}$ $(T \geq t_0)$ such that

$$y(t) > 0, \quad y^{\Delta}(t) > 0, \quad \text{and} \quad (p(t)y^{\Delta}(t)t)^{\Delta} \leq 0$$

for all $t \geq T$. Integrating $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) \leq 0$ from t to s yields

$$p(s)y^{\Delta}(s) - p(t)y^{\Delta}(t) + \int_{t}^{s} q(u)f(y^{\sigma}(u))\Delta u \leq 0, \quad \text{for} \quad s, t \in \mathbb{T} \quad \text{and} \quad s \geq t,$$

i.e.,

$$p(t)y^{\Delta}(t) \geq p(s)y^{\Delta}(s) + \int_{t}^{s} q(u)f(y^{\sigma}(u))\Delta u. \tag{2.2.2}$$

Since $p(t)y^{\Delta}(t) > 0$ decreases for $t \geq T$, $\lim_{t\to\infty} p(t)y^{\Delta}(t) = k \geq 0$ exists. Letting $s \to \infty$ in (2.2.2) we obtain

$$y^{\Delta}(t) \geq \frac{1}{p(t)}\left(k + \int_t^{\infty} q(u)f(y^{\sigma}(u))\Delta u\right) \geq \frac{1}{p(t)}\int_t^{\infty} q(u)f(y^{\sigma}(u))\Delta u. \quad (2.2.3)$$

Since $\int_t^{\infty} q(u)f(y^{\sigma}(u))\Delta u$ exists and is continuous, integrating (2.2.3) from T to t yields

$$y(t) \geq y(T) + \int_T^t \frac{1}{p(s)}\int_s^{\infty} q(u)f(y^{\sigma}(u))\Delta u\Delta s, \quad t \geq T. \quad (2.2.4)$$

Let $\mathbb{X}$ be the set of all continuous functions on $[t_0,\infty)_{\mathbb{T}}$ satisfying $\lim_{t\to\infty} y(t) = 0$ with $\|\cdot\|$ defined by $\|y\| = \sup\{|y(t)| : t_0 \leq t < \infty\}$. Then $\mathbb{X}$ is a Banach space. Now, define the set

$$\Omega := \{\omega \in C([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+) : 0 \leq \omega(t) \leq 1 \text{ for } t \geq t_0\},$$

which is endowed with the usual pointwise ordering $\leq$:

$$\omega_1 \leq \omega_2 \Leftrightarrow \omega_1(t) \leq \omega_2(t) \quad \text{for} \quad t \geq t_0.$$

Using the fact $\mathbb{T}$ is isolated, one can show that any nonempty subset A of Ω has a supremum which belongs to Ω and $\inf\Omega \in \Omega$. Define a mapping S on Ω by

$$(S\omega)(t) = \begin{cases} 1, & \text{if } t \leq T, \\ \dfrac{1}{y(t)}\left(y(T) + \displaystyle\int_T^t \frac{1}{p(s)}\int_s^{\infty} q(u)f(y^{\sigma}(u)\omega^{\sigma}(u))\Delta u\Delta s\right), & \text{if } t \geq T. \end{cases}$$

We claim that $S\Omega \subset \Omega$ and S is nondecreasing. For any $\omega \in \Omega$, $(S\omega)(t)$ is certainly continuous and for $t \geq T$,

$$q(t)f(y^{\sigma}(t)\omega^{\sigma}(t)) \leq q(t)f(y^{\sigma}(t))$$

since $0 \leq \omega^{\sigma}(t) \leq 1$ and f is nondecreasing. This inequality along with (2.2.4) yield $0 \leq (S\omega)(t) \leq 1$ for $t \geq T$. Furthermore, if $\omega_1 \leq \omega_2$, $\omega_1, \omega_2 \in \Omega$, then, since f is nondecreasing, $f(y^{\sigma}(u)\omega_1(u)) \leq f(y^{\sigma}(u)\omega_2(u))$ and so $(S\omega_1)(t) \leq (S\omega_2)(t)$. Therefore, by Knaster's Fixed-Point Theorem, there is an $\tilde{\omega} \in \Omega$ such that $S\tilde{\omega} = \tilde{\omega}$. Hence,

$$\tilde{\omega}(t) = \frac{1}{y(t)}\left(y(T) + \int_T^t \frac{1}{p(u)}\int_u^{\infty} q(v)f(y^{\sigma}(v)\tilde{\omega}^{\sigma}(v))\Delta v\Delta u\right), \quad \text{for} \quad t \geq T.$$

Observe

$$\tilde{\omega}(t) \geq \frac{y(T)}{y(t)} > 0 \quad \text{for} \quad t \geq T.$$

Set $z(t) := \tilde{\omega}(t)y(t)$. Then $z(t) > 0$ is continuous and

$$z(t) = y(T) + \int_T^t \frac{1}{p(u)} \int_u^\infty q(v)f(z^\sigma(v))\Delta v \Delta u, \quad \text{for} \quad t \geq T.$$

As

$$z^\Delta(t) = \frac{1}{p(t)} \int_t^\infty q(u)f(z^\sigma(u))\Delta u \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta = -q(t)f(z^\sigma(t)),$$

$(p(t)z^\Delta(t))^\Delta + q(t)f(z^\sigma(t)) = 0$ has a positive solution, which is a contradiction to the assumption that all solutions of (1.1.3) are oscillatory.

SUFFICIENCY. Assume (2.2.1) has no eventually positive solutions. Then neither does (1.1.3), and so $(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0$ is oscillatory.

If y is an eventually negative solution of (1.1.3), then let $x = -y$. Then x is eventually positive and

$$(px^\Delta)^\Delta + qf(x^\sigma) = -(py^\Delta)^\Delta - qf(y^\sigma) = -[(px^\Delta)^\Delta + qf(x^\sigma)] = 0$$

for $t \geq T$ sufficiently large by Condition (H_4). Thus x is an eventually positive solution of (2.2.1), which is a contradiction. Hence, (1.1.3) is oscillatory. This completes the proof. □

Lemma 2.4: *Every solution of the second-order nonlinear functional dynamic equation* $(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) = 0$ *oscillates if and only if the inequality*

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) \leq 0 \tag{2.2.5}$$

has no eventually positive solutions.

The proof is similar to that of Lemma 2.3 and so we omit it. We continue with our first main result which is an extension of Theorem 2.1 of Zhang and Shanliang (2005).

Theorem 2.5: *Assume* $\frac{\mu(t)}{p(t)}$ *is bounded. Then the oscillation of the second-order nonlinear dynamic equation*

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0 \tag{1.1.3}$$

is equivalent to the oscillation of the second-order nonlinear functional dynamic equation

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) = 0 \tag{1.1.2}$$

where either $\tau(t) \leq t$ *for all* t *or* $\tau(t) \geq \sigma(t)$ *for all* t.

Proof: Since $\frac{\mu}{p}$ is bounded, there exists $N > 0$ such that $\frac{\mu(t)}{p(t)} \leq N$ for all t. Let $K := M + N$, where $M > 0$ is such that

$$|P(t) - P(\tau(t))| < M \quad \text{for all} \quad t \in \mathbb{T} \quad \text{where} \quad P(t) = \int_{t_0}^t \frac{1}{p(s)} \Delta s.$$

NECESSITY. The oscillation of (1.1.2) implies that of equation (1.1.3). Suppose that there is a nonoscillatory solution $y(t)$ of (1.1.3). We will only consider the case where there exists $t_0 \in \mathbb{T}$ such that $y(t) > 0$ for $t \geq t_0$, since the other case is similar.

From equation (1.1.3) and Conditions (H_1)–(H_4), there exists $t_1 \in \mathbb{T}$ $(t_1 \geq t_0)$ such that

$$y(t) > 0, \quad (py^\Delta)(t) > 0, \quad (py^\Delta)^\Delta(t) \leq 0, \quad y(\tau(t)) > 0, \quad t \geq t_1$$

as in the proof of Lemma 2.3. Hence, since $p(t)y^\Delta(t) > 0$ decreases for $t \geq t_1$, $\lim_{t\to\infty} p(t)y^\Delta(t) = L \geq 0$ exists. We will distinguish several cases.

(I) Assume $\tau(t) \leq t$ for all t. As y is increasing, $y^\sigma(t) \geq y(\tau(t))$. Furthermore, as f is increasing, we have

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) \leq (p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) = 0.$$

So $y(t)$ is an eventually positive solution of (2.2.5). By Lemma 2.4, equation (1.1.3) is nonoscillatory, which is a contradiction.

(II) Suppose $\tau(t) \geq \sigma(t)$ for all t.

(a) Assume $L > 0$. It follows that there exists $t_2 \in \mathbb{T}$ with $t_2 \geq t_1$ such that $p(t)y^\Delta(t) \leq L+1$ for all $t \geq t_2$. Since $\lim_{t\to\infty} \tau(t) = \infty$, there is a $t_3 \geq t_2$ such that $\tau(t) \geq t_2$ for $t \geq t_3$. Therefore, if $t \geq t_3$, we have

$$\begin{aligned}
y(\tau(t)) - y^\sigma(t) &= \int_{\sigma(t)}^{\tau(t)} \frac{p(s)y^\Delta(s)}{p(s)}\Delta s \\
&\leq (L+1)\int_{\sigma(t)}^{\tau(t)} \frac{\Delta s}{p(s)} \\
&= (L+1)[P(\tau(t)) - P(\sigma(t))] \\
&\leq (L+1)[|P(\tau(t)) - P(t)| + |P(t) - P(\sigma(t))|] \\
&= (L+1)\left[|P(\tau(t)) - P(t)| + \left|\int_{\sigma(t)}^{t} \frac{\Delta s}{p(s)}\right|\right] \\
&= (L+1)\left[|P(\tau(t)) - P(t)| + \frac{\mu(t)}{p(t)}\right] \\
&\leq (L+1)[M+N],
\end{aligned}$$

which leads to

$$y^\sigma(t) \geq y(\tau(t)) - (L+1)K, \quad t \geq t_3.$$

Let $z(t) := y(t) - (L+1)K$. Then for sufficiently large t, we have

$$z(t) > 0, \quad z(\tau(t)) \leq y^\sigma(t), \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta + q(t)f(z(\tau(t))) \leq 0.$$

This leads to a contradiction as in part (I) above.

(b) Assume $L = 0$. Since $y^\Delta(t) > 0$ and $y(t) > 0$, there is an $\epsilon_0 > 0$ and a $t_2 \geq t_1$ such that $y(t) > M\epsilon_0$ for all $t \geq t_2$. Corresponding to this ϵ_0, there exists $t_3 \geq t_1$ such that $p(t)y^\Delta(t) \leq \epsilon_0$ for all $t \geq t_3$. Now, if $t \geq T := \max\{t_2, t_3\}$, in the same manner as above we have

$$y^\sigma(t) \geq y(\tau(t)) - \epsilon_0 K \quad \text{for} \quad t \geq T.$$

Now set $z(t) := y(t) - \epsilon_0 K$. Then for sufficiently large t

$$z(t) > 0, \quad z(\tau(t)) \leq y^\sigma(t), \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta + q(t)f(z^\sigma(t)) \leq 0,$$

which again leads to a contradiction.

SUFFICIENCY. The oscillation of (1.1.3) implies that of (1.1.2). Suppose, to the contrary, that y is a nonoscillatory solution of (1.1.2) and without loss of generality, we assume there exists $t_1 \in \mathbb{T}$ such that

$$y(t) > 0, \quad p(t)y^\Delta(t) > 0, \quad \text{and} \quad (p(t)y^\Delta(t))^\Delta \leq 0, \quad t \geq t_1.$$

Since $p(t)y^\Delta(t) > 0$ is decreasing for $t \geq t_1$, $\lim_{t\to\infty} p(t)y^\Delta(t) = L \geq 0$ exists. We distinguish several cases.

(I) Assume $\sigma(t) \leq \tau(t)$ for all t. It follows that $y(\tau(t)) \geq y^\sigma(t) > 0$ as y is increasing. Consequently,

$$(p(t)y^\Delta(t))^\Delta + q(t)f(y^\sigma(t)) \leq (p(t)y^\Delta(t))^\Delta + q(t)f(y(\tau(t))) = 0,$$

and so (2.2.1) has an eventually positive solution. By Lemma 2.3, equation (1.1.3) has a nonoscillatory solution, which is a contradiction.

(II) Suppose next that $\tau(t) \leq \sigma(t)$ for all t.

(a) Assume $L > 0$. Then there exists $t_2 \in \mathbb{T}$ with $t_2 \geq t_1$ such that $p(t)y^\Delta(t) \leq L + 1$, for all $t \geq t_2$. Since $\lim_{t\to\infty} \tau(t) = \infty$, there is a $t_3 \geq t_2$ such that $\tau(t) \geq t_2$ for $t \geq t_3$. Therefore, if $t \geq t_3$, we have

$$y(\tau(t)) \geq y^\sigma(t) - (L+1)K, \quad t \geq t_3.$$

Let $z(t) = y(t) - (L+1)K$. Note that for all t large enough,

$$p(t)y^\Delta(t) \geq L.$$

By integrating both sides from t_0 to t we obtain

$$y(t) - y(t_0) \geq L \int_{t_0}^{t} \frac{1}{p(s)} \Delta s.$$

By letting $t \to \infty$, we see that $z(t) > 0$ for large enough t. Hence, for all sufficiently large t,

$$z(t) > 0, \quad z^\sigma(t) \leq y(\tau(t)), \quad \text{and} \quad (p(t)z^\Delta(t))^\Delta + q(t)f(z^\sigma(t)) \leq 0.$$

Hence, (2.2.1) has an eventually positive solution. By Lemma 2.3, we have that $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0$ is nonoscillatory, which is a contradiction.

(b) Assume $L = 0$. Since both $y^{\Delta}(t)$ and $y(t)$ are positive, there exists $\epsilon_0 > 0$ and $t_2 \geq t_1$ such that $y(t) > M\epsilon_0$ for all $t \geq t_2$. Corresponding to this ϵ_0, there exists $t_3 \geq t_1$ such that $p(t)y^{\Delta}(t) \leq \epsilon_0$ for all $t \geq t_3$. Now, if $t \geq T := \max\{t_2, t_3\}$, we have

$$y(\tau(t)) \geq y^{\sigma}(t) - \epsilon_0 K, \quad t \geq T.$$

Again, we set $z(t) := y(t) - \epsilon_0 K$. Then for sufficiently large t

$$z(t) > 0, \quad z^{\sigma}(t) \leq y(\tau(t)), \quad \text{and} \quad (p(t)z^{\Delta}(t))^{\Delta} + q(t)f(z^{\sigma}(t)) \leq 0.$$

Hence, 2.2.1 has an eventually positive solution. Again by Lemma 2.3, $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t)) = 0$ is nonoscillatory, which is a contradiction.

This completes the proof. □

Remark 2.6: Under the assumptions Theorem 2.5 we see that oscillatory behaviour of the more difficult functional equation can be established by considering the dynamic equation that only involves the forward jump operator σ.

As an example of Theorem 2.5, we have the following:

Example 2.7: Let $\mathbb{T} = \mathbb{N}$ and $\tau : \mathbb{T} \to \mathbb{T}$. Assume q, f, and τ satisfy conditions (H_2)–(H_4). If we let

$$p(t) = \frac{1}{\sqrt{t} + 1},$$

then condition (H_1) holds. Therefore, the oscillation of the two equations

$$\Delta(p(t)\Delta y(t)) + q(t)f(y(t+1)) = 0$$

and

$$\Delta(p(t)\Delta y(t)) + q(t)f(y(\tau(t))) = 0$$

is equivalent.

Remark 2.8: One can prove analogous results when considering

$$(p(t)y^{\Delta}(t))^{\Delta} + \sum_{i=1}^{n} q_i(t)f_i(y(\tau_i(t))) = 0$$

and

$$(p(t)y^{\Delta}(t))^{\Delta} + \sum_{i=1}^{n} q_i(t)f_i(y^{\sigma}(t)) = 0$$

and their corresponding inequalities.

We end this section with comparing $(p(t)y^{\Delta})^{\Delta} + q(t)f(y(\tau(t)))$ to

$$(p(t)y^{\Delta}(t))^{\Delta} + \tilde{q}(t)g(y(\tilde{\tau}(t))) = 0, \tag{2.2.6}$$

on a time scale $\mathbb{T}$ where $\tilde{q}, g,$ and $\tilde{\tau}$ satisfy conditions (H_1)–(H_4) and $\frac{\mu}{p}$ is bounded. From Theorem 2.5 we see that the oscillation of (2.2.6) is equivalent to that of

$$(p(t)y^{\Delta}(t))^{\Delta} + \tilde{q}(t)g(y^{\sigma}(t)) = 0. \tag{2.2.7}$$

We get the following result.

Theorem 2.9: *Assume $\frac{\mu}{p}$ is bounded on $\mathbb{T}$. Further assume that $\tilde{q}(t) \le q(t)$ for all large t and $|g(u)| \le |f(u)|$ for $|u| > 0$. Then, the oscillation of equation* (2.2.6) *implies that of equation* (1.1.2)*.*

Proof: Otherwise, without loss of generality, we assume that (1.1.2) has an eventually positive solution. From Theorem 2.5, equation (1.1.3) also has an eventually positive solution $y(t)$. Then

$$(p(t)y^{\Delta}(t))^{\Delta} + \tilde{q}(t)g(y^{\sigma}(t))) \le (p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y^{\sigma}(t))) = 0,$$

which implies (2.2.7) has an eventually positive solution. Therefore, equation (2.2.6) also has an eventually positive solution, which is a contradiction. □

3 Oscillation of a linear dynamic equation

In this section we give two theorems about the oscillatory behaviour of the second-order dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)y^{\sigma}(t) = 0 \tag{3.3.1}$$

on a time scale $\mathbb{T}$ where $\sup \mathbb{T} = \infty$, $p \in C_{rd}(\mathbb{T}, (0, \infty))$ and $q \in C_{rd}(\mathbb{T}, \mathbb{R})$. These are Theorems 3.2 and 3.7. We impose the following condition

$$\int_a^{\infty} \frac{1}{p(s)} \Delta s = \infty \quad \text{and} \quad \int_a^{\infty} q(s)\Delta s < \infty \quad \text{for some} \quad a \in \mathbb{T}. \tag{C$_1$}$$

To prove our main result, we need the following lemma.

Lemma 3.1 (Erbe et al., 2002)**:** *Assume*

$$\liminf_{t\to\infty} \int_T^t q(s)\Delta s \ge 0 \quad \textit{and} \not\equiv 0 \tag{C$_2$}$$

for all large T, and

$$\int_T^\infty \frac{1}{p(s)} \Delta s = \infty. \tag{C$_3$}$$

If y is a solution of (3.3.1) such that $y(t) > 0$, for $t \in [T, \infty)_{\mathbb{T}}$, then there exists $S \in [T, \infty)_{\mathbb{T}}$ such that $y^{\Delta}(t) > 0$ for $t \in [S, \infty)_T$.

Before we state Theorem 3.2, we need the following definitions.

$$\begin{aligned}
A_0(t) &= \int_t^\infty q(s) \Delta s, \\
A_1(t) &= A_0(t) + \int_t^\infty \frac{A_0^2(s)}{p(s) + \mu(s) A_0(s)} \Delta s, \\
&\vdots \\
A_n(t) &= A_0(t) + \int_t^\infty \frac{A_{n-1}^2(s)}{p(s) + \mu(s) A_{n-1}(s)} \Delta s,
\end{aligned}$$

if the integrals on the right-hand side exist.

The following is a generalisation of Theorem 3.1 of Zhang and Shanliang (2005).

Theorem 3.2: *Assume (C_1) and (C_2) hold, and one of the following two conditions holds:*

(i) *there exists some positive integer m such that A_n is well defined for $n = 0, 1, 2, \ldots, m-1$, and*

$$\lim_{t \to \infty} \int_a^t \frac{A_{m-1}^2(s)}{p(s) + \mu(s) A_{m-1}(s)} \Delta s = \infty.$$

(ii) *A_n is well defined for $n = 0, 1, 2, \ldots$, and there exists $t^* \in \mathbb{T}$ $(t^* \geq t_0)$ such that*

$$\lim_{n \to \infty} A_n(t^*) = \infty.$$

Then the second-order dynamic equation

$$(p(t) y^{\Delta}(t))^{\Delta} + q(t) y^{\sigma}(t) = 0 \tag{3.3.1}$$

is oscillatory.

Proof: If not, without loss of generality, we assume (3.3.1) has an eventually positive solution $y(t)$. From Lemma 3.1, we get that there exists $t_1 \in \mathbb{T}\,(t_1 \geq t_0)$ such that

$$y(t) > 0 \quad \text{and} \quad y^{\Delta}(t) > 0 \quad \text{for all} \quad t \geq t_1.$$

Define the function z by

$$z(t) = \frac{p(t)y^{\Delta}(t)}{y(t)} \quad \text{for} \quad t \geq t_1. \tag{3.3.2}$$

Then $z(t) > 0$ and

$$p(t) + \mu(t)z(t) = p(t) + \mu(t)\frac{p(t)y^{\Delta}(t)}{y(t)} = \frac{p(t)y(t) + p(t)\mu(t)y^{\Delta}(t)}{y(t)} > 0,$$

for $t \geq t_1$. From (3.3.2) we get that z is a solution of the Riccati equation

$$z^{\Delta}(t) = -q(t) - \frac{z^2(t)}{p(t) + \mu(t)z(t)}, \quad t \geq t_1. \tag{3.3.3}$$

Integrating both sides of (3.3.3) from t_1 to t we get

$$z(t) - z(t_1) + \int_{t_1}^{t} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\,\Delta s = -\int_{t_1}^{t} q(s)\,\Delta s, \quad t \geq t_1.$$

Then, as $z(t) > 0$,

$$\int_{t_1}^{t} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\Delta s \leq z(t_1) - \int_{t_1}^{t} q(s)\Delta s \leq z(t_1), \quad t \geq t_1.$$

Letting $t \to \infty$ we have that

$$\lim_{t\to\infty} \int_{t_1}^{t} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\Delta s < \infty.$$

Integrating (3.3.3) from t to s we obtain

$$\begin{aligned} z(t) &= z(s) + \int_t^s q(\tau)\Delta\tau + \int_t^s \frac{z^2(\tau)}{p(\tau) + \mu(\tau)z(\tau)}\Delta\tau \\ &> \int_t^s q(\tau)\Delta\tau + \int_t^s \frac{z^2(\tau)}{p(\tau) + \mu(\tau)z(\tau)}\Delta\tau \end{aligned}$$

for s, $t \in \mathbb{T}$ and $s \geq t \geq t_1$. Letting $s \to \infty$ we have

$$z(t) \geq \int_t^{\infty} q(s)\Delta s + \int_t^{\infty} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\Delta s, \quad t \geq t_1. \tag{3.3.4}$$

Assume Condition (i) holds and $m = 1$. From (3.3.4) we obtain that $z(t) \geq A_0(t)$ for all $t \geq t_1$.

Observe that $F(u) = \frac{u^2}{c_1 + c_2 u}$ is increasing for $u > 0$, where $c_1,\ c_2 \geq 0$ are constants. It follows that

$$\int_t^{\infty} \frac{A_0^2(s)}{p(s) + \mu(s)A_0(s)}\Delta s \leq \int_t^{\infty} \frac{z^2(s)}{p(s) + \mu(s)z(s)}\Delta s < \infty.$$

This contradicts (i). If $m > 1$, we have

$$z(t) \geq \int_t^\infty q(s)\Delta s + \int_t^\infty \frac{A_0^2(s)}{p(s) + \mu(s)A_0(s)}\Delta s = A_1(t), \quad \text{for } t \geq t_1.$$

Repeating the above procedure, we get that $z(t) \geq A_{m-1}(t)$ for all $t \geq t_1$, and

$$\int_t^\infty \frac{A_{m-1}^2(s)}{p(s) + \mu(s)A_{m-1}(s)}\Delta s \leq \int_t^\infty \frac{z^2(s)}{p(s) + \mu(s)z(s)}\Delta s < \infty,$$

which contradicts Condition (i).

Assume that Condition (ii) holds. Similar to the above proof, we obtain $A_n(t) \leq z(t)$ for $n = 0, 1, 2, \ldots$ Then, as $y(t) > 0$,

$$\lim_{n\to\infty} A_n(t^*) \leq z(t^*) < \infty,$$

which gives a contradiction to Condition (ii). The proof is complete. □

Remark 3.3: It is well known that the Leighton–Wintner condition

$$\int_a^\infty \frac{1}{p(t)}\Delta t = \int_a^\infty q(t)\Delta t = \infty$$

implies that every solution of (3.3.1) is oscillatory on $[a, \infty)_{\mathbb{T}}$.

Remark 3.4: If $\mathbb{T} = \mathbb{R}$ and $p(t) = 1$ for all t, then Theorem 3.2 is the same as Yan's result for second-order linear differential equations (Yan, 1987).

To prove the next result, we need the following lemmas:

Lemma 3.5 (Bohner and Peterson, 2001, Theorem 4.61)**:** *Assume $a \in \mathbb{T}$, $p > 0$, and let $\omega := \sup \mathbb{T}$. If $\omega < \infty$, then we assume $\rho(\omega) = \omega$. If $(py^\Delta)^\Delta(t) + q(t)y^\sigma(t) = 0$ has a positive solution on $[a, \omega)$, then there is a positive solution u, called a recessive solution at ω, such that for any second linearly independent solution v, called a dominant solution at ω,*

$$\lim_{t\to\omega^-} \frac{u(t)}{v(t)} = 0, \quad \int_a^\omega \frac{\Delta t}{p(t)u(t)u^\sigma(t)} = \infty, \quad \textit{and} \quad \int_b^\omega \frac{\Delta t}{p(t)v(t)v^\sigma(t)} < \infty,$$

where $b < \omega$ is sufficiently close. Furthermore

$$\frac{p(t)v^\Delta(t)}{v(t)} > \frac{p(t)u^\Delta(t)}{u(t)}$$

for $t < \omega$ sufficiently close.

Lemma 3.6 (Bohner and Peterson, 2001, Theorem 4.55)**:** *Assume z is a solution of the the Riccati equation*

$$Rz = 0, \quad \textit{where} \quad Rz(t) := z^\Delta(t) + q(t) + \frac{z^2(t)}{p(t) + \mu(t)z(t)}$$

on $[a,\sigma^2(b)]_{\mathbb{T}}$ *with* $p(t)+\mu(t)z(t)>0$ *on* $[a,\sigma^2(b)]_{\mathbb{T}}$*. Let* u *be a continuous function on* $[a,\sigma^2(b)]_{\mathbb{T}}$ *whose derivative is piecewise right-dense continuous with* $u(a)=u(\sigma^2(b))=0$*. Then we have for all* $t\in[a,\sigma^2(b)]_{\mathbb{T}}$*,*

$$\begin{aligned}(zu^2)^{\Delta}(t) &= p(t)[u^{\Delta}(t)]^2 - q(t)u^2(\sigma(t))\\ &\quad - \left\{\frac{z(t)u^{\sigma}(t)}{\sqrt{p(t)+\mu(t)z(t)}} - \sqrt{p(t)+\mu(t)z(t)}\,u^{\Delta}(t)\right\}^2.\end{aligned}$$

Using the previous lemmas, we have the following theorem which was proven for differential equations by Kelley and Peterson (2004).

Theorem 3.7: *Assume* $I=[a,\infty)_{\mathbb{T}}$*. If* $\int_a^{\infty}\frac{\Delta t}{p(t)}=\infty$ *and there is a* $t_0\ge a$ *and a* $u\in C^1_{rd}[t_0,\infty)$ *such that* $u(t)>0$ *on* $[t_0,\infty)_{\mathbb{T}}$ *and*

$$\int_{t_0}^{\infty}\{q(t)[u^{\sigma}(t)]^2 - p(t)[u^{\Delta}(t)]^2\}\Delta t=\infty,$$

then the second-order dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta}+q(t)y^{\sigma}(t)=0 \tag{3.3.1}$$

is oscillatory on I*.*

Proof: We prove this theorem by contradiction. So assume (3.3.1) is nonoscillatory on I. By Lemma 3.5, there is a dominant solution y at ∞ such that for $t_1\ge a$, sufficiently large,

$$\int_{t_1}^{\infty}\frac{\Delta t}{p(t)y(t)y^{\sigma}(t)}<\infty,$$

and we may assume $y(t)>0$ on $[t_1,\infty)_{\mathbb{T}}$. Let t_0 and u be as in the statement of this theorem. Let $T=\max\{t_0,t_1\}$; then let

$$z(t):=\frac{p(t)y^{\Delta}(t)}{y(t)},\quad t\ge T.$$

It follows that

$$p(t)+\mu(t)z(t)>0 \quad \text{for all } t\ge T.$$

Then by Lemma 3.6, we have for $t\ge T$

$$\begin{aligned}(zu^2)^{\Delta}(t) &= p(t)[u^{\Delta}(t)]^2 - q(t)u^2(\sigma(t))\\ &\quad - \left\{\frac{z(t)u(\sigma(t))}{\sqrt{p(t)+\mu(t)z(t)}} - \sqrt{p(t)+\mu(t)z(t)}\,u^{\Delta}(t)\right\}^2\\ &\le p(t)[u^{\Delta}(t)]^2 - q(t)u^2(\sigma(t)).\end{aligned}$$

Integrating from T to t, we obtain

$$z(t)u^2(t) \leq z(T)u^2(T) - \int_T^t \{q(t)u^2(\sigma(t)) - p(t)[u^\Delta(t)]^2\}\Delta t$$

which implies

$$\lim_{t\to\infty} z(t)u^2(t) = -\infty.$$

However, then there is a $T_1 \geq T$ such that for $t \geq T_1$

$$z(t) = \frac{p(t)y^\Delta(t)}{y(t)} < 0.$$

This implies that $y^\Delta(t) < 0$ for $t \geq T_1$, and hence y is decreasing on $[T_1, \infty)_{\mathbb{T}}$. However,

$$\begin{aligned}\int_{T_1}^\infty \frac{1}{p(s)}\Delta s &= y(T_1)y^\sigma(T_1)\int_{T_1}^\infty \frac{1}{p(s)y(T_1)y^\sigma(T_1)}\Delta s\\ &\leq y(T_1)y^\sigma(T_1)\int_{T_1}^\infty \frac{1}{p(s)y(s)y^\sigma(s)}\Delta s\\ &< \infty,\end{aligned}$$

which is a contradiction. □

We conclude with an example that shows how Theorem 3.7 can be used to obtain oscillation criteria.

Example 3.8: If $a > 0$ and

$$\int_a^\infty \sigma^\alpha(t)q(t)\Delta t = \infty,$$

where $0 < \alpha < 1$, then $y^{\Delta\Delta} + q(t)y^\sigma = 0$ is oscillatory on $[a, \infty)_{\mathbb{T}}$.

We will show that this follows from Theorem 3.7. In the Pötzsche Chain Rule (Bohner and Peterson, 2001, Theorem 1.90), let $g(t) = t$ and $f(t) = t^{\frac{\alpha}{2}}$, for $0 < \alpha < 1$. Then with $u(t) = (f\circ g)(t) = t^{\frac{\alpha}{2}}$, we have

$$\begin{aligned}u^\Delta(t) = (f\circ g)^\Delta(t) &= \left\{\int_0^1 \frac{\alpha}{2}[t + h\mu(t)\cdot 1]^{\frac{\alpha-2}{2}}dh\right\}\cdot 1\\ &= \frac{\alpha}{2}\int_0^1 (t + h\mu(t))^{\frac{\alpha-2}{2}}dh\\ &\leq \frac{\alpha}{2}\int_0^1 t^{\frac{\alpha-2}{2}}dh\\ &= \frac{\alpha}{2}t^{\frac{\alpha-2}{2}}\end{aligned}$$

since $\alpha - 2 < 0$. Therefore, it follows that $(u^{\Delta}(t))^2 \le \frac{\alpha^2}{4} t^{\alpha-2}$ for all t. Hence,

$$\begin{aligned}
&\int_a^{\infty} \{q(t)[u^{\sigma}(t)]^2 - p(t)[u^{\Delta}(t)]^2\}\Delta t \\
&\ge \int_a^{\infty} \left\{q(t)\sigma^{\alpha}(t) - \frac{\alpha^2}{4} t^{\alpha-2}\right\}\Delta t \\
&= \infty
\end{aligned}$$

since $0 < \alpha < 1$ implies

$$\int_a^{\infty} t^{\alpha-2}\Delta t < \infty.$$

Thus $y^{\Delta\Delta} + q(t)y^{\sigma} = 0$ is oscillatory on $[a, \infty)_{\mathbb{T}}$ by Theorem 3.7.

4 Conclusion and future directions

In this paper, we studied the oscillatory behaviour of the second-order functional dynamic equation

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) = 0$$

on an isolated time scale $\mathbb{T}$. We showed that the oscillation of the functional dynamic equation is equivalent to that of the dynamic equation

$$(p(t)y^{\Delta})^{\Delta} + q(t)f(y^{\sigma}(t)) = 0.$$

This was accomplished by establishing a relationship between the oscillatory solutions of the functional dynamic equation and the inequality $(p(t)y^{\Delta}(t))^{\Delta} + q(t)f(y(\tau(t))) \le 0$ and a relationship between oscillatory solutions of the dynamic equation and the inequality $(p(t)y^{\Delta})^{\Delta} + q(t)f(y^{\sigma}(t)) \le 0$. On any time scale $\mathbb{T}$, we considered the dynamic equation with $f(u) = u$ and established two sufficient conditions for oscillation using the Riccati transformation technique.

Possibilities for further exploration include considering the case $\int_{t_0}^{\infty} \frac{\Delta t}{p(t)} < \infty$ and a general time scale $\mathbb{T}$ as well as other generalisations of oscillations theorems from differential equations.

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