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OSCILLATION OF A SECOND-ORDER LINEAR DELAY DYNAMIC EQUATION

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This paper is dedicated to Professor Lynn Erbe

ABSTRACT. In this paper, we consider the second order linear dynamic equation

$$\left(p(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y\left(\tau(t)\right) = 0$$

on a time scale $\mathbb{T}$. Our goal is to establish some new oscillation results for this equation. Here we assume that $\tau(t) \leq t$ and $\tau : \mathbb{T} \to \mathbb{T}$. We apply results from the theory of lower and upper solutions for related dynamic equations along with some additional estimates on positive solutions.

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1. INTRODUCTION

In 1988 the theory of time scales was introduced by Stefan Hilger in his Ph.D. Thesis in order to unify continuous and discrete analysis (see [13]). Not only does this unify the theories of differential equations and difference equations, but it also extends these classical situations to cases "in between"– e.g., to the so-called q-difference equations. Moreover, the theory can be applied to other different types of time scales. Since its introduction, many authors have expounded on various aspects of this new theory, and we refer specifically to the paper by Agarwal et al. [2] and the references cited therein. A book on the subject of time scales by Bohner and Peterson [5] summarizes and organizes much of time scale calculus.

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on a time scale (i.e., a closed subset of the real line $\mathbb{R}$). This has lead to many attempts to harmonize the oscillation theory for the continuous and the discrete cases, to include them in one comprehensive theory, and to extend the results to more general time scales. We refer the reader to the papers [1], [6], [10], [14], [15], and the references cited therein.

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Since we are interested in the oscillatory behavior of solutions near infinity, we assume throughout this paper that our time scale is unbounded above. We also assume $t_0 \in \mathbb{T}$, and for convenience, $t_0 > 0$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by

$$[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}.$$

Our main interest is to consider the second-order linear dynamic equation

$$\left(p(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y\left(\tau(t)\right) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}. \tag{1.1}$$

We shall assume the following conditions hold:
(H_1) $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty))$ satisfies $\displaystyle\int_{t_0}^{\infty} \frac{1}{p(t)} \Delta t = \infty$;
(H_2) $q \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [0, \infty))$;
(H_3) $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{T})$ satisfies $\lim\limits_{t\to\infty} \tau(t) = \infty$ and $0 < \tau(t) \le t$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.

Our attention is restricted to those solutions $y(t)$ of (1.1) which exist on some half-line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|y(t)| : t > T\} > 0$ for any $T \ge t_y$. A solution $y(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

We note that (1.1) in its general form includes several types of differential and difference equations with delay arguments. In addition, different equations correspond to the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have $y^{\Delta} = y'$, and so (1.1) becomes the delay differential equation

$$\left(p(t)y'(t)\right)' + q(t)y(\tau(t)) = 0.$$

In the case $\mathbb{T} = \mathbb{Z}$, $y^{\Delta} = \Delta y$ and (1.1) becomes the second-order delay difference equation

$$\Delta\left(p(t)\Delta y(t)\right) + q(t)y(\tau(t)) = 0$$

where Δ denotes the forward difference operator.

In this paper we intend to use the method of upper and lower solutions and the Riccati transformation technique to establish some sufficient conditions which ensure that every solution of (1.1) oscillates. These results extend some earlier ones given in Higgins [11]. We illustrate our results with examples.

2. PRELIMINARY RESULTS

In this section we state fundamental results needed to prove our main results. We begin with the following lemma.

Lemma 2.1 ([7, Lemma 2]). *Let $y(t)$ be a solution of*

$$\left(p(t)y^{\Delta}(t)\right)^{\Delta} + \sum_{i=1}^{n} q_i(t)y(\tau_i(t)) = 0$$

which satisfies

$$y(t) > 0, \quad y^\Delta(t) > 0, \quad \text{and} \quad \big(p(t)y^\Delta(t)\big)^\Delta \le 0$$

for all $t \ge \tau_i(t) \ge T \ge t_0$. *Then for each* $1 \le i \le n$ *we have*

$$y(\tau_i(t)) > \eta_i(t,T)y^\sigma(t), \quad t \ge \tau_i(t) > T$$

where

$$H(t,a) = \int_a^t \frac{1}{p(s)}\,\Delta s \quad \text{and} \quad \eta_i(t,a) = \frac{H(\tau_i(t),a)}{H(\sigma(t),a)},\ 1 \le i \le n.$$

Lemma 2.2. *For any* $0 < k < 1$, *there exists* $T_k \ge T \ge t_0$ *such that* $\eta(t,T) \ge k\eta(t,t_0)$ *for* $t \ge T_k$.

Proof. Consider the quotient $\dfrac{\eta(t,T)}{\eta(t,t_0)}$. It follows that

$$\begin{aligned}
\lim_{t\to\infty}\frac{\eta(t,T)}{\eta(t,t_0)} &= \lim_{t\to\infty}\frac{\int_T^{\tau(t)}\frac{\Delta s}{p(s)}}{\int_T^{\sigma(t)}\frac{\Delta s}{p(s)}}\cdot\frac{\int_{t_0}^{\sigma(t)}\frac{\Delta s}{p(s)}}{\int_{t_0}^{\tau(t)}\frac{\Delta s}{p(s)}} \\
&= \lim_{t\to\infty}\frac{\int_T^{\tau(t)}\frac{\Delta s}{p(s)}}{\int_{t_0}^{T}\frac{\Delta s}{p(s)}+\int_T^{\tau(t)}\frac{\Delta s}{p(s)}}\cdot\frac{\int_{t_0}^{T}\frac{\Delta s}{p(s)}+\int_T^{\sigma(t)}\frac{\Delta s}{p(s)}}{\int_T^{\sigma(t)}\frac{\Delta s}{p(s)}} \\
&= \lim_{t\to\infty}\frac{1}{\dfrac{\int_{t_0}^{T}\frac{\Delta s}{p(s)}}{\int_T^{\tau(t)}\frac{\Delta s}{p(s)}}+1}\left(\frac{\int_{t_0}^{T}\frac{\Delta s}{p(s)}}{\int_T^{\sigma(t)}\frac{\Delta s}{p(s)}}+1\right) \\
&= \left(\frac{1}{0+1}\right)(0+1) \\
&= 1
\end{aligned}$$

Hence for any $0 < k < 1$, there exists $T_k \ge T \ge t_0$ such that $\eta(t,T) \ge k\eta(t,t_0)$ for $t \ge T_k$. □

We shall also need the following lemma which is often referred to as the Riccati substitution technique.

Lemma 2.3 ([7, Lemma 1])**.** *The linear equation*

$$Lx \equiv \big(p(t)x^\Delta(t)\big)^\Delta + q(t)x^\sigma = 0$$

is nonoscillatory if and only if there is a function z *satisfying the Riccati dynamic inequality*

$$z^\Delta + q(t) + \frac{z^2}{p(t)+\mu(t)z} \le 0 \tag{2.1}$$

with $p(t)+\mu(t)z(t) > 0$ *for large* t.

In order to prove our main results, we need a method of studying separated boundary value problems (SBVPs). Namely, we will define functions called upper and lower solutions that, not only imply the existence of a solution of a SBVP, but also provide bounds on the location of the solution. Consider the SBVP

$$-(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = f(t, x^{\sigma}), \quad t \in [a, b]^{\kappa^2} \tag{2.2}$$

$$x(a) = A, \qquad x(b) = B \tag{2.3}$$

where the functions $f \in C([a, b]^{\kappa^2} \times \mathbb{R}, \mathbb{R})$ and $p, q \in C_{rd}([a, b]^{\kappa^2})$ are such that $p(t) > 0$ and $q(t) \geq 0$ on $[a, b]^{\kappa^2}$. Recall that $\mathbb{T}^{\kappa^n}$ means to remove the last n points of $\mathbb{T}$ provided they are left-scattered. We define the set

$$\mathbb{D}_1 := \{x \in \mathbb{X} : x^{\Delta} \text{ is continuous and } px^{\Delta} \text{ is delta differentiable on } [a, b]^{\kappa} \text{ and } (px^{\Delta})^{\Delta} \text{ is rd-continuous on } [a, b]^{\kappa^2}\},$$

where the Banach space $\mathbb{X} = C([a, b])$ is equipped with the norm $\|\cdot\|$ defined by

$$\|x\| := \max_{t \in [a,b]_{\mathbb{T}}} |x(t)| \quad \text{for all} \quad x \in \mathbb{X}.$$

A function x is called a solution of the equation $-(p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma} = 0$ on $[a, b]^{\kappa^2}$ if $x \in \mathbb{D}_1$ and the equation $-(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0$ holds for all $t \in [a, b]^{\kappa^2}$. Next we define for any $u, v \in \mathbb{D}_1$ the sector $[u, v]_1$ by

$$[u, v]_1 := \{w \in \mathbb{D}_1 : u \leq w \leq v\}.$$

Definition 2.4 ([4, Definition 6.1])**.** We call $\alpha \in \mathbb{D}_1$ a lower solution of the SBVP (2.2)-(2.3) on $[a, b]$ provided

$$-(p\alpha^{\Delta})^{\Delta}(t) + q(t)\alpha^{\sigma}(t) \leq f(t, \alpha^{\sigma}(t)) \quad \text{for all} \quad t \in [a, b]^{\kappa^2}$$

and

$$\alpha(a) \leq A, \quad \alpha(b) \leq B.$$

Similarly, $\beta \in \mathbb{D}_1$ is called an upper solution of the SBVP (2.2)-(2.3) on $[a, b]$ provided

$$-(p\beta^{\Delta})^{\Delta}(t) + q(t)\beta^{\sigma}(t) \geq f(t, \beta^{\sigma}(t)) \quad \text{for all} \quad t \in [a, b]^{\kappa^2}$$

and

$$\beta(a) \geq A, \quad \beta(b) \geq B.$$

Theorem 2.5 ([4, Theorem 6.5])**.** *Assume that there exist a lower solution α and an upper solution β of the SBVP* (2.2)–(2.3) *such that*

$$\alpha(t) \leq \beta(t) \quad \textit{for all} \quad t \in [a, b].$$

Then the SBVP (2.2)–(2.3) *has a solution $x \in [\alpha, \beta]_1$ on $[a, b]$.*

The following is an extension of the previous theorem to $[a, \infty)_{\mathbb{T}}$.

Theorem 2.6 ([12, Theorem 1.5]). *Assume that there exists a lower solution α and an upper solution β of (2.2) with $\alpha(t) \le \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. Then*

$$-(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = f(t, x^{\sigma}) \tag{2.4}$$

has a solution x with $x(a) = A$ and $x \in [\alpha, \beta]_1$ on $[a,\infty)_{\mathbb{T}}$.

3. MAIN RESULTS

In this section we give six results concerning the oscillatory behavior of

$$\left(p(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y\left(\tau(t)\right) = 0 \tag{1.1}$$

on the time scale $[t_0,\infty)_{\mathbb{T}}$ where $\sup \mathbb{T} = \infty$ and conditions (H_1)–(H_3) hold. These are Theorems 3.1, 3.6, and 3.10 and Corollaries 3.3, 3.11, and 3.12. Define

$$\gamma := \liminf_{t\to\infty} \frac{\sigma(t)}{\tau(t)}\eta(t,t_0) \quad \text{ and } \quad Q(t) := \gamma\frac{\tau(t)}{\sigma(t)}q(t),\ t \ge t_0.$$

Theorem 3.1. *Assume that the equation*

$$\left(p(t)y^{\Delta}(t)\right)^{\Delta} + \lambda Q(t)y^{\sigma}(t) = 0 \tag{3.1}$$

is oscillatory on $[t_0,\infty)_{\mathbb{T}}$ for some $0 < \lambda < 1$. Then all solutions of (1.1) are oscillatory.

Proof. If not, we may assume that $u(t)$ is a nonoscillatory solution of (1.1) with $u(t) > 0$ on $[T,\infty)_{\mathbb{T}}$ for some $T \ge t_0$. As $\tau(t) \to \infty$ as $t \to \infty$, we may also assume $u(\tau(t)) > 0$ for $t \ge T$. Integrating twice from T to t gives

$$u(t) \le u(T) + p(T)u^{\Delta}(T)\int_T^t \frac{1}{p(s)}\,\Delta s, \quad t \ge T.$$

We claim that $u^{\Delta}(t) > 0$ for all large t. If not, for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$, we have $u^{\Delta}(t_1) \le 0$. It follows that $p(t)u^{\Delta}(t) \le 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Now, if $u^{\Delta}(t_2) < 0$ for some $t_2 \ge t_1$, then

$$u(t) - u(t_2) = \int_{t_2}^t \frac{p(s)u^{\Delta}(s)}{p(s)}\,\Delta s \le p(t_2)u^{\Delta}(t_2)\int_{t_2}^t \frac{\Delta s}{p(s)} \to -\infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction to our assumption that $u(t) > 0$ for $t \ge t_0$. Hence it follows that $u^{\Delta}(t) \equiv 0$ on $[t_1,\infty)_{\mathbb{T}}$, and so $\left(p(t)u^{\Delta}(t)\right)^{\Delta} \equiv 0$ and $q(t)u(\tau(t)) > 0$, which is contradictory. Consequently,

$$u(t) > 0, \quad p(t)u^{\Delta}(t) > 0, \quad \left(p(t)u^{\Delta}(t)\right)^{\Delta} \le 0, \quad u(\tau(t)) > 0 \quad \text{on} \quad [T,\infty)_{\mathbb{T}}.$$

For any $0 < k < 1$, there is a $T_k \ge T$ such that

$$u(\tau(t)) \ge \eta(t,T)u^{\sigma}(t) \ge k\eta(t,t_0)u^{\sigma}(t), \quad t \ge T_k$$

by Lemma 2.1 and Lemma 2.2. By definition of γ, we may also assume that T_k is such that

$$\frac{\sigma(t)}{\tau(t)}\eta(t,t_0) \geq k\gamma, \quad t \geq T_k.$$

Hence

$$u(\tau(t)) \geq k^2\gamma\frac{\tau(t)}{\sigma(t)}u^\sigma(t), \quad t \geq T_k.$$

Using this inequality in (1.1) yields

$$\big(p(t)u^\Delta(t)\big)^\Delta + k^2Q(t)u^\sigma(t) \leq 0, \quad t \geq T_k. \tag{3.2}$$

Now if we put $z(t) = \dfrac{p(t)u^\Delta(t)}{u(t)}$, then

$$z^\Delta(t) = \frac{\big(p(t)u^\Delta(t)\big)^\Delta}{u^\sigma(t)} - \frac{p(t)(u^\Delta(t))^2}{u(t)u^\sigma(t)}$$

and by means of (3.2), we have the Riccati dynamic inequality

$$z^\Delta + k^2Q(t) + \frac{z^2}{p(t)+\mu(t)z} = \frac{1}{u^\sigma}\left[\big(pu^\Delta\big)^\Delta + k^2Q(t)u^\sigma\right] \leq 0.$$

Hence, by Lemma 2.3, $\big(pu^\Delta\big)^\Delta + k^2Q(t)u^\sigma = 0$ is nonoscillatory. Choosing $0 < k < 1$ such that $\lambda < k^2 < 1$, we have $k^2Q(t) > \lambda Q(t)$. By the Sturm–Picone comparison theorem [8, Lemma 6], we have $\big(pu^\Delta\big)^\Delta + \lambda Q(t)u^\sigma = 0$ is nonoscillatory. This contradiction proves the theorem. □

The next result gives an integral condition for oscillation of (3.1).

Theorem 3.2. *If*

$$\int_{t_0}^\infty q(t)\frac{\tau(t)}{\sigma(t)}\,\Delta t = \infty, \tag{3.3}$$

then every solution of (3.1) *is oscillatory on* $[t_0,\infty)_\mathbb{T}$.

Proof. Suppose that there exists an eventually positive solution y of (3.1). Then there exists $T \in [t_0,\infty)_\mathbb{T}$ such that

$$y(t) > 0, \quad y^\Delta(t) > 0, \quad \big(p(t)y^\Delta(t)\big)^\Delta \leq 0 \quad \text{for all} \quad t \geq T \geq t_0.$$

It follows that y is nondecreasing on $[T,\infty)_\mathbb{T}$, and so there is an $\alpha > 0$ such that $y(t) > \alpha$ for all large t. Let $Y(t) = p(t)y^\Delta(t)$. Then

$$\begin{aligned} Y(t) &= Y(T) + \int_T^t Y^\Delta(s)\,\Delta s \\ &= Y(T) - \lambda\gamma\int_T^t \frac{\tau(s)}{\sigma(s)}q(s)y^\sigma(s)\,\Delta s \\ &\leq Y(T) - \lambda\gamma\alpha\int_T^t \frac{\tau(s)}{\sigma(s)}q(s)\,\Delta s \\ &\to -\infty \text{ as } t \to \infty. \end{aligned}$$

This means that $y^\Delta(t) < 0$ for all large t, and this is a contradiction to the fact that both $y(t)$ and $y^\Delta(t)$ are positive for large t. Thus every solution of

$$\big(p(t)y^\Delta(t)\big)^\Delta + \lambda Q(t)y^\sigma(t) = 0 \tag{3.1}$$

is oscillatory. □

We now state and prove a corollary of Theorem 3.1.

Corollary 3.3. *All solutions of the linear second-order dynamic equation*

$$\big(p(t)y^\Delta(t)\big)^\Delta + q(t)y(\tau(t)) = 0 \tag{1.1}$$

are oscillatory in case (3.3) *holds.*

Proof. Let $u(t)$ be a nonoscillatory solution of (1.1) with $u(t) > 0$ and $u(\tau(t)) > 0$ for $t \geq T$ for some $T \in [t_0, \infty)_{\mathbb{T}}$. Since $\big(p(t)u^\Delta(t)\big)^\Delta \leq 0$ for all t, we have $p(t)u^\Delta(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. As in the proof of Theorem 3.1, for any $0 < k < 1$ there exists a $T_k \geq T$ such that

$$\big(p(t)u^\Delta(t)\big)^\Delta + k^2Q(t)u^\sigma(t) \leq 0$$

for $t \geq T \geq T_k$. Let $\alpha(t) = u(T)$ and $\beta(t) = u(t)$. Then, in Definition 2.4 with $q(t) = \tilde{q}(t) = -k^2Q(t)$ and $f(t, x^\sigma) \equiv 0$, we have

$$-\left(p\alpha^\Delta\right)^\Delta(t) + \tilde{q}(t)\alpha^\sigma(t) = -k^2Q(t)u(T) \leq 0$$

and

$$-\left(p\beta^\Delta\right)^\Delta(t) + \tilde{q}(t)\beta^\sigma(t) = -\big(p(t)u^\Delta(t)\big)^\Delta - k^2Q(t)u^\sigma(t) \geq 0 \quad \text{with} \quad \lambda = k^2.$$

So α, β are lower and upper solutions, respectively, of

$$\big(p(t)y^\Delta\big)^\Delta + \lambda Q(t)y^\sigma(t) = 0. \tag{3.1}$$

As u is increasing, $\alpha(t) \leq \beta(t)$ on $[T_k, \infty)_{\mathbb{T}}$. Then by Theorem 2.6, there is a solution $y(t)$ of (3.1) satisfying $u(T) \leq y(t) \leq u(t)$ on $[T_k, \infty)_{\mathbb{T}}$. Consequently, y is a nonoscillatory solution of (3.1). This is a contradiction to Theorem 3.2 and proves the theorem. □

The following example is illustrative.

Example 3.4. Consider the delay dynamic equation

$$\big(ty^\Delta(t)\big)^\Delta + \frac{\sigma(t)}{t\tau(t)}y(\tau(t)) = 0 \quad \text{for } t \geq t_0. \tag{3.4}$$

It follows that $p(t) = t$ satisfies condition (H_1) and condition (3.3) becomes

$$\lim_{t\to\infty}\int_{t_0}^{t} q(s)\frac{\tau(s)}{\sigma(s)}\,\Delta s = \lim_{t\to\infty}\int_{t_0}^{t}\frac{\sigma(s)}{s\tau(s)}\frac{\tau(s)}{\sigma(s)}\,\Delta s = \infty.$$

Therefore, by Corollary 3.3, every solution of (3.4) oscillates on $[t_0, \infty)_{\mathbb{T}}$.

In order to prove our next result, we need the following lemma.

Lemma 3.5 ([9, Lemma 1]). *Assume that*

$$\int_{t_0}^{\infty} \frac{\Delta t}{p(t)} = \infty,$$

and

$$\int_{t_0}^{\infty} \tau(t)q(t)\,\Delta t = \infty, \tag{3.5}$$

and assume that (1.1) *has a positive solution* y *on* $[t_0,\infty)_{\mathbb{T}}$. *Then there exists a* $T \in [t_0,\infty)_{\mathbb{T}}$, *sufficiently large, so that*

1. $y^{\Delta}(t) > 0,\; y(t) > ty^{\Delta}(t)$ *for* $t \in [T,\infty)_{\mathbb{T}}$;
2. y *is strictly increasing and* $y(t)/t$ *is strictly decreasing on* $[T,\infty)_{\mathbb{T}}$.

Theorem 3.6. *Assume* (3.5) *holds. If*

$$\lim_{t\to\infty}\left(\frac{t}{p(t)}\int_t^{\infty} q(s)\frac{\tau(s)}{s}\,\Delta s\right) = \infty, \tag{3.6}$$

then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. To the contrary, suppose y is a nonoscillatory solution of (1.1). Then there exists $T \in \mathbb{T}$ such that

$$y(t) \geq y(\tau(t)) > 0, \quad y^{\Delta}(t) \geq 0, \quad \text{and} \quad \big(p(t)y^{\Delta}(t)\big)^{\Delta} \leq 0$$

for all $t \geq T \geq t_0$. It follows that for $s \geq t \geq T$, we have

$$\int_t^s q(r)y(\tau(r))\,\Delta r = -\int_t^s \big(p(r)y^{\Delta}(r)\big)^{\Delta}\,\Delta r \leq p(r)y^{\Delta}(r).$$

From Lemma 3.5, it follows that for $t \in \mathbb{T}$ sufficiently large

$$\begin{aligned} y(t) &\geq ty^{\Delta}(t) \\ &\geq \frac{t}{p(t)}\int_t^{\infty} q(s)y(\tau(s))\,\Delta s \\ &\geq \frac{t}{p(t)}\int_t^{\infty} q(s)\frac{\tau(s)}{s}y(s)\,\Delta s \\ &\geq \frac{t}{p(t)}y(t)\int_t^{\infty} q(s)\frac{\tau(s)}{s}\,\Delta s. \end{aligned}$$

So for$t \in \mathbb{T}$ sufficiently large

$$1 \geq \frac{t}{p(t)}\int_t^{\infty} q(s)\frac{\tau(s)}{s}\,\Delta s,$$

a contradiction to (3.6). This completes the proof. □

The following examples illustrate Corollary 3.3 and Theorem 3.6.

Example 3.7. Let $h > 0$ and $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$. When $p(t) = t$, (1.1) becomes

$$h[y(t+h) - y(t)] + \sigma(t)[y(t+2h) - 2y(t+h) + y(t)] + h^2 q(t) y(\tau(t)) = 0. \tag{3.7}$$

Condition (3.3) reduces to

$$\lim_{n\to\infty} h \sum_{k=t_0/h}^{n} \frac{\tau(hk)}{t+h} q(hk) = \infty.$$

So, according to Corollary 3.3, every solution of (3.7) oscillates on $[t_0, \infty)_{\mathbb{T}}$. Furthermore, (3.5) and (3.6) become

$$\lim_{n\to\infty} h \sum_{k=t_0/h}^{n} \tau(hk) q(hk) = \infty \quad \text{and} \quad \lim_{m\to\infty} \left\{ \lim_{n\to\infty} \sum_{k=m}^{n} q(hk) \frac{\tau(hk)}{k} \right\} = \infty,$$

respectively. Thus, by Theorem 3.6, every solution of (3.7) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Example 3.8. Let $\tilde{q} > 1$ and $\mathbb{T} = \tilde{q}^{\mathbb{N}_0} = \{\tilde{q}^k : k \in \mathbb{N}_0\}$. When $p(t) = t$, (1.1) becomes

$$\tilde{q}\,(\tilde{q}-1)\,t[y(\tilde{q}t) - y(t)] + \tilde{q}t[y(\tilde{q}^2 t) - (\tilde{q}+1)\,y(\tilde{q}t) + \tilde{q}y(t)] + \tilde{q}\,(\tilde{q}-1)\,t^2 q(t) y(\tau(t)) = 0. \tag{3.8}$$

Condition (3.3) becomes

$$\frac{\tilde{q}-1}{\tilde{q}} \lim_{n\to\infty} \sum_{t=t_0}^{\tilde{q}^{n-1}} q(t)\tau(t) = \infty.$$

Then, by Corollary 3.3, every solution of (3.8) oscillates on $[t_0, \infty)_{\mathbb{T}}$. Furthermore, (3.5) and (3.6) reduce to

$$(\tilde{q}-1) \lim_{n\to\infty} \sum_{t=t_0}^{\tilde{q}^{n-1}} t q(t)\tau(t) = \infty \quad \text{and} \quad (\tilde{q}-1) \lim_{t\to\infty} \left\{ \lim_{n\to\infty} \sum_{s=t}^{\tilde{q}^{n-1}} q(s)\tau(s) \right\} = \infty,$$

respectively. Consequently, by Theorem 3.6, every solution of (3.8) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

To prove our next result Theorem 3.10 concerning the oscillatory behavior of

$$\left(p(t)y^{\Delta}\right)^{\Delta} + q(t)y(\tau(t)) = 0, \tag{1.1}$$

we need the following auxiliary result which was proven in [3].

Lemma 3.9. *If z and y are differentiable on a time scale $\mathbb{T}$ with $y(t) \neq 0$ for all $t \in \mathbb{T}$, then*

$$y^{\Delta} \left(\frac{z^2}{y}\right)^{\Delta} = \left(z^{\Delta}\right)^2 - y y^{\sigma} \left[\left(\frac{z}{y}\right)^{\Delta}\right]^2.$$

Theorem 3.10. *Assume*

$$\int_{t_0}^{\infty} \tau(t)q(t)\,\Delta t = \infty \tag{3.5}$$

holds. If there exists a differentiable function z such that

$$\lim_{t\to\infty} \int_{t_0}^{t} \left[\frac{(z^{\sigma}(s))^2}{\sigma(s)}\tau(s)q(s) - p(s)(z^{\Delta}(s))^2\right] \Delta s = \infty, \tag{3.9}$$

then every solution of (1.1) *is oscillatory on* $[t_0,\infty)_{\mathbb{T}}$.

Proof. As before, we suppose y is an eventually positive solution of (1.1). So there exists $T \in \mathbb{T}$ such that

$$y(t) \geq y(\tau(t)) > 0, \quad y^{\Delta}(t) \geq 0, \quad \text{and} \quad \left(p(t)y^{\Delta}(t)\right)^{\Delta} \leq 0$$

for all $t \geq T \geq t_0$. Using the Riccati substitution

$$w = \frac{z^2\left(py^{\Delta}\right)}{y},$$

by the quotient rule [5, Theorem 1.20], (1.1), and Lemma 3.5, we have

$$\begin{aligned}
-w^{\Delta} &= \frac{z^2p\left(y^{\Delta}\right)^2 - \left[(z^2)^{\Delta}\left(py^{\Delta}\right) + (z^{\sigma})^2\left(py^{\Delta}\right)^{\Delta}\right]y}{yy^{\sigma}} \\
&= \frac{py^{\Delta}\left[z^2y^{\Delta} - (z^2)^{\Delta}\,y\right]}{yy^{\sigma}} + \frac{(z^{\sigma})^2\,q(y\circ\tau)}{y^{\sigma}} \\
&= \frac{(z^{\sigma})^2\,q(y\circ\tau)}{y^{\sigma}} - py^{\Delta}\left[\frac{z^2}{y}\right]^{\Delta} \\
&= \frac{(z^{\sigma})^2\,q(y\circ\tau)}{y^{\sigma}} + p\left(yy^{\sigma}\left[\left(\frac{z}{y}\right)^{\Delta}\right]^2 - \left(z^{\Delta}\right)^2\right) \\
&\geq \frac{(z^{\sigma})^2\,q\tau}{\sigma} - p\left(z^{\Delta}\right)^2.
\end{aligned}$$

Then for sufficiently large $t \geq T_1 \geq t_0$,

$$\begin{aligned}
w(T_1) &\geq w(T_1) - w(T) \\
&= -\int_{T_1}^{t} w^{\Delta}(s)\,\Delta s \\
&\geq \int_{T_1}^{t}\left(\frac{(z^{\sigma}(s))^2}{\sigma(s)}q(s)\tau(s) - p(s)\left(z^{\Delta}(s)\right)^2\right)\Delta s \\
&\to \infty \text{ as } t\to\infty.
\end{aligned}$$

This contradiction completes the proof. □

Choosing $z(t) = t$ in Theorem 3.10, we have

Corollary 3.11. *Assume* (3.5)*holds. If*

$$\int_{t_0}^{\infty} [q(t)\tau(t)\sigma(t) - p(t)]\ \Delta t = \infty,$$

then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

If we let $z(t) = \sqrt{t}$ in Theorem 3.10 and recall $\sigma(t) \geq t$ for all $t \in \mathbb{T}$, we have

Corollary 3.12. *Assume* (3.5) *holds. If*

$$\int_{t_0}^{\infty} \left(q(t)\tau(t) - \frac{p(t)}{4t} \right) \Delta t = \infty, \tag{3.10}$$

then every solution of (1.1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

The following example illustrates Corollary 3.12.

Example 3.13. Consider the delay dynamic equation

$$\left(ty^{\Delta}(t)\right)^{\Delta} + \frac{\sigma(t)}{\tau(t)} y(\tau(t)) = 0 \tag{3.11}$$

for $t \geq t_0 > 0$. Then $p(t) = t$ satisfies condition (H_1) and condition (3.5) becomes

$$\lim_{t\to\infty} \int_{t_0}^{t} \tau(s)q(s)\,\Delta s = \lim_{t\to\infty} \int_{t_0}^{t} \sigma(s)\,\Delta s = \infty.$$

In this case (3.10) is satisfied since

$$\lim_{t\to\infty} \int_{t_0}^{t} \left(\tau(s)q(s) - \frac{p(s)}{4s} \right) \Delta s = \lim_{t\to\infty} \int_{t_0}^{t} \left(\sigma(s) - \frac{1}{4} \right) \Delta s = \infty.$$

Thus, by Corollary 3.12, every solution of (3.11) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

4. CONCLUSION AND FUTURE RESULTS

In this paper we have obtained sufficient conditions for the oscillatory behavior of

$$\left(p(t)y^{\Delta}(t)\right)^{\Delta} + q(t)y\left(\tau(t)\right).$$

This was done by comparing nonoscillatory solutions of the delay dynamic equation with the solutions of a corresponding linear dynamic equation and then using known properties of the linear equation to obtain a desired contradiction.

Possibilities for further exploration include considering the case when $\int^{\infty} p(t)\ \Delta t = \infty$, and replacing the delay $\tau(t)$ with the advance $\xi : \mathbb{T} \to \mathbb{T}$ where $\sigma(t) \leq \xi(t)$ and $\lim_{t\to\infty} \xi(t) = \infty$ in both cases $\int^{\infty} \frac{\Delta t}{p(t)} = \infty$ and $\int^{\infty} p(t)\ \Delta t = \infty$.

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