Complex Variables
Preliminary Exam
May 2014

Directions: Do all of the following eight problems. **Show all your work and justify your answers.** Each problem is worth 10 points.

Notation: $\mathbb{C}$ = the complex plane; $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ = the extended complex plane; $\mathcal{A}(G) = \{f : f \text{ is analytic on a region } G \subset \mathbb{C}\}$; $\mathcal{M}(G) = \{f : f \text{ is meromorphic on a region } G \subset \mathbb{C}_\infty\}$; $B(a; b) = \{z : |z - a| < b\}$; $\mathbb{D} = B(0; 1) = \{z : |z| < 1\}$; $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginary part of z, respectively.

1. Let $D = B(0; 1) \setminus \overline{B(3/4; 1/4)}$. Find a one-to-one conformal map f of D onto $\mathbb{D}$ such that $f(0) = 0$.

2. Find all $f \in \mathcal{A}(\mathbb{C})$ such that $|\arg f(z)| < \frac{\pi}{4}$ for all $z \in \mathbb{C}$.

3. Prove that if $f \in \mathcal{M}(\mathbb{C}_\infty)$, then f is a rational function.

4. Give an example of a function $f \in \mathcal{M}(\mathbb{C})$ such that f has a pole of order 5 at $z = 0$ and $\displaystyle\int_\gamma f(z)dz = 4$ where γ has parametrization $\gamma(t) = e^{it}, 0 \le t \le 2\pi$.

5. Let $f \in \mathcal{A}(\mathbb{D})$ satisfy $f(\mathbb{D}) \subset \mathbb{D}$ with $f(0) = 0$. Prove that $\displaystyle\sum_{n=0}^{\infty} (f(z))^n$ converges in $\mathcal{A}(\mathbb{D})$.

6. Let the sequence $\{f_n\} \subset \mathcal{A}(G)$ and $f \in \mathcal{A}(G)$ where G is a region. Suppose that $\{f_n\}$ converges to f in the topology of uniform convergence on compact subsets of G. Prove that $\{f_n^{(k)}\}$ converges to $f^{(k)}$ in the topology of uniform convergence on compact subsets of G, for each $k = 1, 2, 3, \cdots$

7. Locate and classify the isolated singularities (including any potential isolated singularity at the point at infinity) by type for each of the following functions. Additionally, for any singularity which is a pole, determine the order of the pole.

   (a) $\dfrac{\sin^2 z}{z^4}$ **(b)** $\sin\dfrac{1}{z} + \dfrac{1}{z^2(z-1)}$ **(c)** $\csc z - \dfrac{1}{z}$

8. Let $f \in \mathcal{A}(\mathbb{C})$. Suppose that the sequence $\{f(0), f'(0), f''(0), \cdots, f^{(n)}(0), \cdots\}$ is bounded. Show that $\{f(z), f'(z), f''(z), \cdots, f^{(n)}(z), \cdots\}$ is a normal family.