Complex Variables
Preliminary Exam
May 2006

Directions: Do all of the following ten problems. **Show all your work and justify your answers.** Each problem is worth 10 points. $\Re(z)$ and $\Im(z)$ denote the real part of z and the imaginar part of z, respectively.

1. Use the Cauchy-Riemann equations to determine the domain of analyticity of the following complex-valued functions:
(**a**) $f(x+iy) = x^2+2iy$ (**b**) $g(x+iy) = e^y \cos x - ie^y \sin x$.
On the domains of analyticity find the derivatives $f'(z)$ and $g'(z)$, where $z = x+iy$.

2. Let $f(z) = u(x,y) + iv(x,y)$, where $z = x+iy$, be analytic on a domain D and let ∇u and ∇v denote the gradients of u and v.
(a) Prove that $|\nabla u| = |\nabla v| = |f'|$.
(b) Prove that if $|\nabla u|$ is constant on D then $f(z)$ is a linear function.

3. **(a)** State Cauchy's Theorem and the Cauchy Integral Formula.
(b) Proof the Cauchy Estimates for the derivatives of an analytic function, i.e., prove that

$$|f^{(m)}(z_0)| \leq \frac{m!}{\rho^m} M, \quad m = 1, 2, \ldots$$

for every function $f(z)$ analytic for $|z - z_0| \leq \rho$ and such that $|f(z)| \leq M$ for $|z - z_0| = \rho$.

4. Suppose $f(z)$ is entire and has a pole at $z = \infty$. Show that f is a polynomial.

5. **(a)** State the Ratio Test and the Root Test for convergence of power series.
(b) Find the radius of convergence of the following power series:

(1) $\displaystyle\sum_{k=0}^{\infty} 3^k z^k$ **(2)** $\displaystyle\sum_{n=1}^{\infty} n^n (z-2)^n$ **(3)** $\displaystyle\sum_{p \text{ prime}} z^p = z^2+z^3+z^5+z^7+\cdots$.

6. Locate and classify all singularities (including the singularity at $z = \infty$) of:
(**a**) $\dfrac{z}{(1-z^2)^2}$ (**b**) $\dfrac{z}{\sin z}$ (**c**) $z^2 \cos(1/z)$

7. Use the Residue Calculus to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{1+\cos^2\theta}.$$

8. Let $\mathcal{F}$ be the set of analytic functions $f(z)$ in $\mathbb{D}$ such that $f(0) = 0$ and $|\Im(f(z))| < \pi$ for $z \in \mathbb{D}$. Use the Schwarz lemma to find

$$\max_{f \in \mathcal{F}} |f'(0)|.$$

9. Find a conformal mapping $w = f(z)$ from the right half-plane $\{z : \Re(z) > 0\}$ onto the domain $D = \{w : |w| > 1, |\text{Arg}\,(w)| < \pi/4\}$ such that $f(1) = 2$.

10. Show that

$$\frac{\pi}{\cos(\pi z)} = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)}{z^2 - (n-1/2)^2}.$$