Complex Analysis

Preliminary Examination

August 2003

Answer all questions completely. Calculators may not be used. Notation: $\mathbb{D} = \{z : |z| < 1\}$; for G a region in $\mathbb{C}$, let $H(G) = \{f : f \text{ is analytic on } G\}$.

1. Let $z_1, z_2, \ldots, z_n$ be n points in $\mathbb{C}$ and for $z \in \mathbb{C}$ let $d(z, z_k)$ denote the distance between z and z_k. If z is confined to the closure of a bounded domain Ω, show that $\prod_{k=1}^{n} d(z, z_k)$ attains its maximum on the boundary of Ω.

2. Let $f(z)$ be a one-to-one analytic map from $\mathbb{D}$ into $\mathbb{D}$. Suppose $f(\frac{1}{2}) = 0$ and $f(0) = -\frac{1}{2}$. Find $f(-\frac{1}{2})$ and justify your answer.

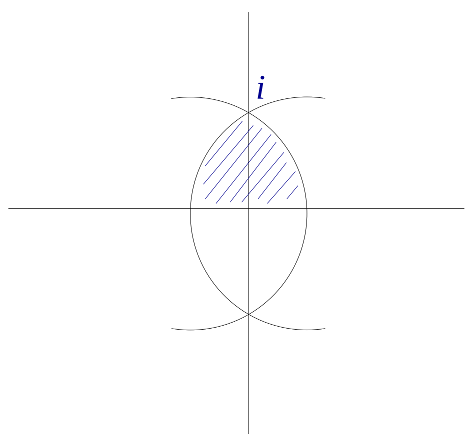
3. Let $G = \{z : 1 < |z| < 2\}$. Suppose f is analytic in G and
$$\lim_{|z| \to 1,\, z \in G} f(z) = 0.$$
Prove f is identically zero on G.

4. Let $w_1, w_2, \ldots, w_n$ be n points in $\mathbb{D}$ and let
$$f(z) = \prod_{j=1}^{n} \frac{(z - w_j)}{(1 - \overline{w_j} z)}.$$
Prove that f maps $\mathbb{D}$ onto $\mathbb{D}$ exactly n times (according to multiplicity). HINT: Use the argument principle.

5. Let $\mathbb{C}_\infty$ denote the extended complex plane (Riemann sphere) and let D_1 and D_2 be disjoint closed circular discs in $\mathbb{C}$. Prove that $\mathbb{C}_\infty \setminus \{D_1 \cup D_2\}$ is conformally equivalent to an annulus.

6. Let f be an entire function such that $f(z + 1) = f(z + i) = f(z)$ for all z. Show that f is constant.

7. Let Ω be the shaded region bounded by the x-axis and the circular arcs pictured below:



Assume the circular arcs meet perpendicularly at i and $-i$. Find a conformal map of Ω onto $\mathbb{D}$ sending $i/2$ to 0.

8. Let G be a region in $\mathbb{C}$. Suppose $f_n \to f$ in $H(G)$. Show that for each $k > 0$, the derivatives $f_n^{(k)}$ converge to $f^{(k)}$ in $H(G)$.