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An application from partial sums of e^z to a problem in several complex variables

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Abstract

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Let $B_r^n = \{z \in \mathbb{C}^n\colon |z| < r\}$, where $|\cdot|$ is the Euclidean norm, and for $X \subset \mathbb{C}^n$, let $\mathscr{H}X$ denote the closed convex hull of X in $\mathbb{C}^n$. In 1990, Graham showed that if f is a normalized holomorphic map from B_1^n into $\mathbb{C}^n$, and if f is either an open map or a polynomial map, then there is a sharp, uniform constant a, a given by $ae^{1+a} = 1$, such that $\mathscr{H}f(B_1^n) \supset B_a^n$. Graham posed the question to find, for normalized polynomial maps f of degree m, the best constant a_m so that $\mathscr{H}f(B_1^n) \supset B_{a_m}^n$. We answer this question and obtain, for each m, the sharp constant

$$a_m = a + \frac{a \ln m}{2(1+a)m} + \frac{a \ln\{\sqrt{2\pi}(1+a)/a\}}{(1+a)m} + o\left(\frac{1}{m}\right), \quad m \to \infty.$$

We also note that this solution extends an old result of Pólya and Szegő.

Keywords: Holomorphic maps in several complex variables; zeros of the partial sums of e^z; the Szegő curve.

In this short note, we show how a numerically motivated result on the zeros of the partial sums of e^z in classical one complex variable function theory can be used to answer a question arising in several complex variables. To put the problem in context, we introduce the following

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notations. With $\mathbb{C}^n := \{z = (z_1, z_2, \ldots, z_n)\colon z_k \in \mathbb{C}$ for $k = 1, 2, \ldots, n\}$, let $|z|^2 := \sum_{k=1}^{n} |z_k|^2$ and $B_r^n := \{z \in \mathbb{C}^n\colon |z| < r\}$, with $B_r := B_r^1$, $B := B_1^1$ and $B^n := B_1^n$. We consider the classes

$$\mathscr{S} := \{f\colon f \text{ is analytic and one-to-one on } B, \text{ with } f(0) = 0 \text{ and } f'(0) = 1\},$$

$$\mathscr{S}^* := \{f \in S\colon f(B) \text{ is starlike with respect to } z = 0\},$$

$$\mathscr{K} := \{f \in \mathscr{S}\colon f(B) \text{ is convex}\}.$$

Classical results (cf. [6]) obtained at the beginning of this century are as follows.

Distortion theorem:

$$\frac{(1 - |z|)^{p-1}}{(1 + |z|)^{p+1}} \leqslant |f'(z)| \leqslant \frac{(1 + |z|)^{p-1}}{(1 - |z|)^{p+1}}, \quad z \in B; \tag{1}$$

Growth theorem:

$$\frac{|z|}{(1 + |z|)^p} \leqslant |f(z)| \leqslant \frac{|z|}{(1 - |z|)^p}, \quad z \in B; \tag{2}$$

Koebe Covering theorem:

$$f(B) \supset B_{1/2^p}, \tag{3}$$

where $p = 2$, for all f in $\mathscr{S}$ and in $\mathscr{S}^*$. All three theorems are sharp when $p = 2$ for the function, known as the Koebe function, defined by

$$f(z) := \frac{z}{(1 - z)^p}. \tag{4}$$

Corresponding results hold for functions $f \in \mathscr{K}$ with the exponent $p = 1$ in (1)–(4).

In 1933, in the appendix of Montel's book on Univalent Function Theory, Cartan [5] posed the question: Can the Distortion, Growth and Koebe Covering theorems be extended to one-to-one functions f which are biholomorphic on B^n and normalized by $f(0) = 0$ and $Jf(0) = I$, where Jf denotes the Jacobian of f and I denotes the identity matrix? (The analogue of the Distortion theorem for f in (1) would give bounds for the modulus of the determinant of the Jacobian f.) Cartan also explicitly asked if there were extensions to the convex and starlike subclasses of maps from B^n into $\mathbb{C}^n$. In that appendix, he gave several examples of polynomial maps on $\mathbb{C}^n$. Since then, polynomial maps have been studied extensively in the literature of several complex variables.

That the three theorems cannot be directly extended to arbitrary biholomorphic maps from B^n into $\mathbb{C}^n$ has been known for some time (cf. [7,9]). As a simple counterexample, consider the map $F(z_1, z_2) := (z_1, z_2 e^{\alpha z_1})$ for $\alpha \in \mathbb{R}$, so that F is a normalized, one-to-one and biholomorphic mapping from B^2 into $\mathbb{C}^2$. It can be verified that for a suitable choice of α, each of the four inequalities in (1) and (2), as well as the inclusion of (3), *fails* for this map F.

To answer Cartan's questions about the convex and starlike maps, the following extensions of the Distortion, Growth and Covering theorems were obtained in [2].

Theorem A. *Let f be a normalized biholomorphic map from B^2 into $\mathbb{C}^2$ with $f(B^2)$ a convex domain. Then, there exists a positive constant c such that*

$$\frac{(1-|z|)^{c-3/2}}{(1+|z|)^{c+3/2}} \leqslant |\det Jf| \leqslant \frac{(1+|z|)^{c-3/2}}{(1-|z|)^{c+3/2}}, \quad z \in B^2,$$

where the constant c satisfies

$$\tfrac{3}{2} \leqslant c < 1.71\ldots . \tag{5}$$

In [2], it was conjectured that $c = \frac{3}{2}$. In [1], the next result was established.

Theorem B. *Let f be a normalized biholomorphic map from B^n into $\mathbb{C}^n$ with $f(B^n)$ a starlike domain. Then,*

$$\frac{|z|}{(1+|z|)^2} \leqslant |f(z)| \leqslant \frac{|z|}{(1-|z|)^2}, \quad z \in B^n,$$

and

$$f(B^n) \supset B^n_{1/4},$$

Furthermore, the above inequalities and inclusion are all sharp.

Although many of the classical results in $\mathbb{C}^1$ follow from each other, the situation in $\mathbb{C}^n$ for $n \geqslant 2$ is different, where new techniques of proof must be used, and the results derived appear to be independent of each other.

Theorems A and B were also obtained and improved later in [8,13].

A further result, answering Cartan's question, was obtained by Graham in an interesting paper [9], where he proved the following.

Theorem C. *Let f be a normalized holomorphic map from B^n into $\mathbb{C}^n$ with either f being an open map or f being a polynomial map. Let a be the unique positive constant satisfying $a\mathrm{e}^{1+a} = 1$, so that $a = 0.278\,46\ldots$. If $\mathscr{H}X$ denotes the closed convex hull of X, then $\mathscr{H}f(B^n) \supset B^n_a$.*

Note that if $f(B^n)$ is convex, then $f(B^n) \supset B^n_a$.

If we define the degree of a polynomial mapping $\mathbb{C}^n$ into $\mathbb{C}^n$ to be the maximal coordinate degree, then in [9], Graham posed the following question.

Question. If f is a polynomial map of degree m, m a fixed integer, can an $a = a_m$ be found such that

$$\mathscr{H}f(B^n) \supset B^n_{a_m}?$$

In this paper, we link together sources from several fields and generate a solution for Graham's question.

A major tool used by Graham is a set of ideas introduced in [14]. It is shown in [14] that for each positive integer m, there exists a positive constant a_m and a set $\{\alpha_j\}_{j=1}^m$, where $\alpha_j \in \mathbb{C}$ for $j = 1, 2, \ldots, m$, such that

$$\text{(i)} \qquad |\alpha_j| \leqslant \frac{1}{a_m},$$

$$\text{(ii)} \qquad \frac{1}{m}\sum_{j=1}^{m} \alpha_j = 1, \tag{6}$$

$$\text{(iii)} \qquad \sum_{j=1}^{m} \alpha_j^k = 0, \quad k = 2, 3, \ldots, m, \text{ if } m > 1.$$

Together, (ii) and (iii) imply that, for any holomorphic f mapping B into $\mathbb{C}^n$, the following representation formula is valid:

$$\frac{1}{m}\sum_{j=1}^{m} f(\alpha_j z) = f(0) + zf'(0) + \mathrm{O}(|z|^{m+1}), \tag{7}$$

for $|z| < a_m$. The idea of the proof of [14] for the existence of $\{\alpha_j\}_{j=1}^m$ was to observe that the α_j's satisfying (ii) and (iii) are the zeros of the monic polynomials p_m, defined by $p_m(z) := z^m P_m(-1/z)$, where $P_m(z) = \sum_{k=1}^m (mz)^k/k!$. If $s_m(z) := \sum_{j=0}^m z^j/j!$ denotes the familiar partial sum of e^z, then $P_m(z) = s_m(mz)$. As shown in [1], if f is a polynomial of degree m, then $\mathscr{H}f(B)$ contains the ball $f(0) + zf'(0)$, $|z| < a_m$. Thus, an answer can be given to Graham's question by determining explicit bounds for the zeros of $P_m(z)$.

In 1924, Szegő [12] defined a simple closed curve

$$D_\infty := \{z \in \mathbb{C}: |ze^{1-z}| = 1 \text{ and } |z| \leqslant 1\}, \tag{8}$$

lying in the closed disk $\bar{B}$. D_∞ has become known as the *Szegő curve* for the polynomials P_m. If $\{z_{k,m}\}_{k=1}^m$ denotes the zeros of P_m, it is known from the Eneström–Kakeya theorem (cf. [15, Chapter 4]) that $\{z_{k,m}\}_{k=1}^m \subset \bar{B}$ for each $m \geqslant 1$, and that $\{z_{k,m}\}_{k=1}^m \subset B$ for any $m > 1$. Clearly, the infinite set of all such zeros $\{z_{k,m}\}_{k=1,m=1}^{m,\infty}$ must possess at least one accumulation point in $\bar{B}$. In [12], Szegő showed that each accumulation point must lie on D_∞, and, conversely, that each point of D_∞ is an accumulation point of these zeros. Subsequently, it was shown in [3] that all of these zeros lie *outside* of D_∞. These facts are illustrated in Fig. 1.

Carpenter et al. [4] considered the problem of accurately estimating the zeros of P_m. They introduced the arc defined by

$$D_m := \left\{z \in \mathbb{C}: |ze^{1-z}|^m = \frac{m!\, e^m}{m^m}\left|\frac{1-z}{z}\right|, |z| \leqslant 1 \text{ and } |\arg z| \geqslant \cos^{-1}\left(\frac{m-2}{m}\right)\right\}, \tag{9}$$

for each $m = 1, 2, \ldots$.

A careful examination of the arc D_m in [4] showed that, if $C_\delta := \{z \in \mathbb{C}: |z-1| < \delta\}$ for any δ with $0 < \delta < 1$, then the zeros $\{z_{k,m}\}_{k=1}^m \backslash C_\delta$ are within $\mathrm{O}(1/m^2)$ of the arc D_m. If it could have been shown that all the zeros $\{z_{k,m}\}_{k=1}^m$ of P_m lie *outside* D_m (which would have been the

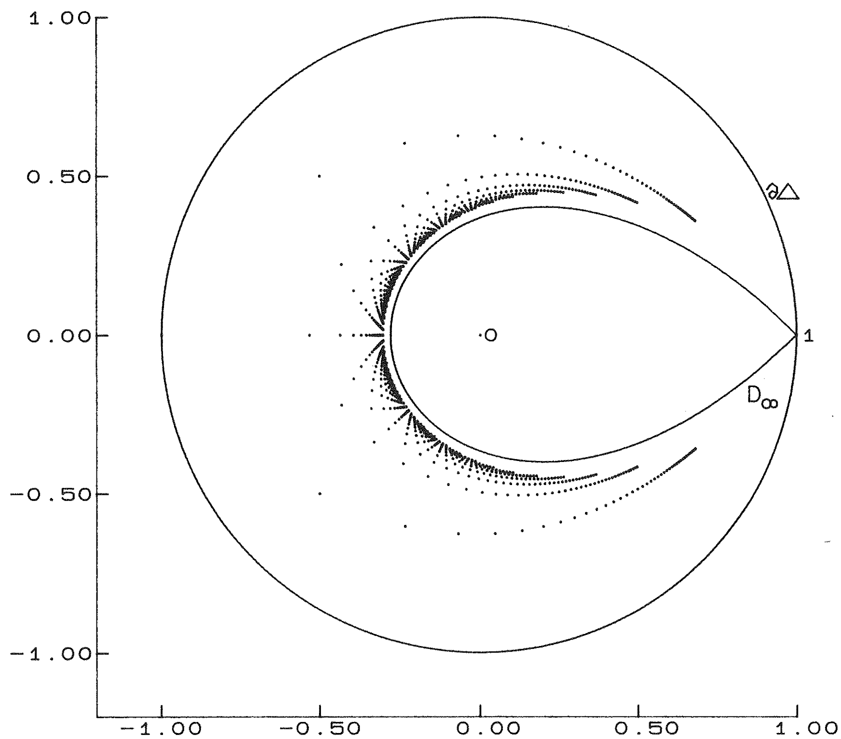

Fig. 1. Zeros of $\{P_n(z)\}_{n=1}^{40}$.

natural analog of Buckholtz's result [3] for D_∞), then an answer to Graham's question would have followed. Indeed, numerical computations did suggest this, noting that in [4] the zeros of P_{27} appeared, up to plotting accuracy, to lie *on* the arc D_{27} (see Fig. 2). However, in an attempt to obtain more precise information about the zeros of P_m relative to D_m, it was discovered that the zeros of P_m do *not* all lie outside D_m for every $m \geqslant 1$. Indeed, it was shown in [16] that there exists a positive integer m_0, such that at least one zero of P_m does not lie outside D_m for every $m > m_0$. A direct calculation indicates the rather surprising outcome that $m_0 = 96$. The size of m_0 necessitated great precision in calculating the zeros. Brent's MP package was used, with 120 significant digits, for these calculations.

Because of this result, it was natural to ask if a *modification* $\hat{D}_m$ of the arc D_m could be found for which all of the zeros of P_m would be outside of $\hat{D}_m$, for *each* $m \geqslant 1$. This was done recently in [16] where, for each $m \geqslant 1$, the arc $\hat{D}_m$ was defined by

$$\hat{D}_m := \left\{ z \in \mathbb{C}: |ze^{1-z}|^m = \frac{m!\, e^m}{m^m} \left| \frac{1 - \operatorname{Re} z}{z} \right|, |z| \leqslant 1 \text{ and } |\arg z| \geqslant \cos^{-1}\left(\frac{m-2}{m}\right) \right\}, \tag{10}$$

and it was shown that all the zeros of P_m *do* lie outside of $\hat{D}_m$ for each $m \geqslant 1$ (see Fig. 3). This is the relevant result needed to provide an answer to Graham's question. We include a brief outline of the proof of [16] that all the zeros of P_m lie outside of $\hat{D}_m$ for each $m \geqslant 1$.

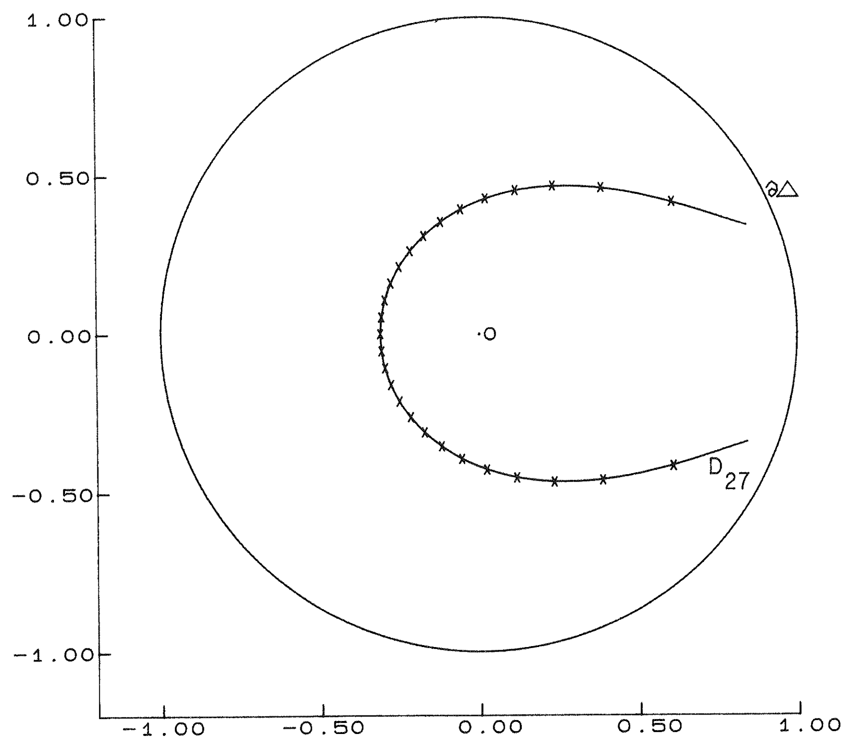

Fig. 2. D_{27} and the zeros of $P_{27}(z)$.

It is easy to verify by differentiation that

$$e^{-z}s_m(z)=1-\frac{1}{m!}\int_0^z \zeta^m e^{-\zeta}\,d\zeta, \tag{11}$$

and, on replacing z by mz and ζ by $m\zeta$ and recalling that $s_m(mz)=:P_m(z)$, we have

$$e^{-mz}P_m(z)=1-\frac{m^{m+1}}{m!}\int_0^z \zeta^m e^{-m\zeta}\,d\zeta=1-\frac{m^{m+1}}{m!}\,e^{-m}I_m(z), \tag{12}$$

where

$$I_m(z):=\int_0^z \left(\zeta e^{1-\zeta}\right)^m\,d\zeta.$$

Using integration along the radial path $\zeta=\rho e^{i\theta}$, $0\leqslant\rho\leqslant r$, for the integral $I_m(z)$, we obtain

$$|I_m(z)|\leqslant\int_0^r \left(\rho e^{1-\rho\cos\theta}\right)^m\,d\rho. \tag{13}$$

The integral on the right in (13) can be expressed as

$$\frac{1}{|\cos\theta|^{m+1}}\int_0^{r|\cos\theta|\exp\{1+r|\cos\theta|\}} v^{m-1}\frac{\mu(v)}{(1+\mu(v))}\,dv,\quad v:=\mu e^{1+\mu},$$

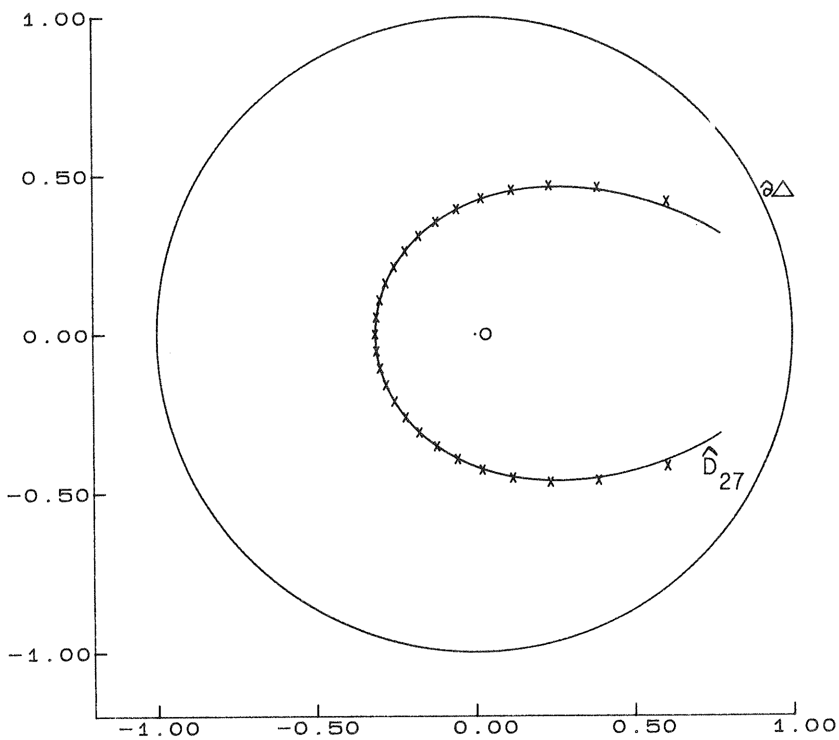

Fig. 3. $\hat{D}_{27}$ and the zeros of $P_{27}(z)$.

with careful use of the sign of cos θ. Since $\mu/(1+\mu)$ is strictly increasing, it follows that

$$|I_m(z)| < \frac{r\left(re^{1-r\cos\theta}\right)^m}{m(1-r\cos\theta)} = \frac{|z|\,|ze^{1-z}|^m}{m(1-\operatorname{Re} z)},\quad 0<r=|z|<1,\ m=1,2,\ldots .$$

Thus, if $z_{k,m}$ is a zero of P_m, then

$$\frac{m^{m+1}}{m!\,e^m} I_m(z_{k,m}) = 1$$

from (12), so that (13) gives

$$\frac{m^m |z_{k,m}|\,|z_{k,m}e^{1-z_{k,m}}|^m}{m!\,e^m(1-\operatorname{Re} z_{k,m})} > 1.$$

Therefore, all the zeros of P_m lie outside $\hat{D}_m$ for all $m \geqslant 1$, as claimed.

We now determine the point of $\hat{D}_m$ which is closest to the origin, i.e.,

$$r_m := \min\left\{|z|: z \in \hat{D}_m\right\},\quad m=1,2,\ldots . \tag{14}$$

Fixing any $m \geqslant 1$, let $z=\rho(\phi)e^{i\phi}$ be any point of $\hat{D}_m$. From (10),

$$\left(\rho(\phi)e^{1-\rho(\phi)\cos\phi}\right)^m = \frac{m!\,e^m(1-\rho(\phi)\cos\phi)}{m^m\rho(\phi)}, \tag{15}$$

since, by definition, $\rho(\phi)$ satisfies $\rho(\phi) \leqslant 1$, which implies that $1-\rho(\phi)\cos\phi \geqslant 0$. On differentiating (15) as a function of ϕ, a straightforward calculation gives that

$$\frac{d\rho(\phi)}{d\phi} = -\frac{\rho^2(\phi)\sin\phi[m(1-\rho(\phi)\cos\phi)-1]}{m(1-\rho(\phi)\cos\phi)^2+1}. \tag{16}$$

The definition in (10) implies that $\cos\phi \leqslant (m-2)/m$, showing that the quantity in brackets in (16) is at least unity for any $m \geqslant 1$. Thus, for $z=\rho(\phi)e^{i\phi}$ on $\hat{D}_m$ in the open upper half-plane, i.e., for $\cos^{-1}((m-2)/m) \leqslant \phi < \pi$, the derivative $d\rho(\phi)/d\phi$ in (16) is *negative*. This establishes that the point of $\hat{D}_m$, closest to the origin in the closed upper half-plane, is the intersection of the arc $\hat{D}_m$ with the negative real axis, and, as the arc $\hat{D}_m$ is, from (10), clearly symmetric about the real axis, the same is true for all of $\hat{D}_m$.

Returning to Graham's question, an a_m, affirmatively answering Graham's question, is defined by $a_m := r_m$ where

$$\left(r_m e^{1+r_m}\right)^m = \frac{m!\,e^m}{m^m}\left(\frac{1+r_m}{r_m}\right), \quad m=1, 2, \ldots . \tag{17}$$

We remark that if a is the unique positive constant satisfying $ae^{1+a}=1$, so that $a=0.27846\ldots$, then it can be shown from (17) that

$$r_m = a + \frac{a\ln m}{2(1+a)m} + \frac{a\ln\{\sqrt{2\pi}(1+a)/a\}}{(1+a)m} + o\left(\frac{1}{m}\right), \quad m\to\infty. \tag{18}$$

We finally show that (18) is *sharp* in the following sense. It is well known that the partial sum $s_n(z)$ of e^z has exactly one negative real zero if n is an odd positive integer, and that it has no real zeros if n is an even positive integer. If $\{z_k(n)\}_{k=1}^n$ denotes the set of zeros of $s_n(z)$ and if we write $z_k(n) := r_k(n)e^{i\theta_k(n)}$ where $|z_k(z)| = r_k(n)$ and where $0<|\theta_k(n)| \leqslant \pi$, then from (11) we have

$$n! = e^{i(n+1)\theta_k(n)}\int_0^{r_k(n)} u^n e^{-u\cos\theta_k(n)} e^{-iu\sin\theta_k(n)}\,du, \quad n=1, 2, \ldots . \tag{19}$$

On taking moduli in (19),

$$n! \leqslant \int_0^{r_k(n)} u^n e^{-u\cos\theta_k(n)}\,du, \quad n=1, 2, \ldots,$$

so that

$$n! < \int_0^{r_k(n)} u^n e^u\,du, \quad \text{for any } \theta_k(n) \text{ with } 0<|\theta_k(n)|<\pi. \tag{20}$$

For n odd, say $n := 2l+1$, let $\tilde{r}_{2l+1} > 0$ be such that $s_{2l+1}(-\tilde{r}_{2l+1}) = 0$. From (19) and (20), we have

$$(2l+1)! = \int_0^{\tilde{r}_{2l+1}} u^{2l+1}e^u\,du < \int_0^{r_k(2l+1)} u^{2l+1}e^u\,du,$$

which evidently implies that

$$\tilde{r}_{2l+1} < r_k(2l+1), \tag{21}$$

for any zero $z_k(2l+1)$ with $|\theta_k(2l+1)| \neq \pi$. This shows that, for *each* odd positive integer m, the unique negative real zero of $P_m(z) = s_m(mz)$ is the *closest* zero of $P_m(z)$ to the origin. In other words, the very *best* choice of a_m, for every $m = 2l+1$ odd, in Graham's problem is just $\tilde{r}_{2l+1}$. Now, in [11, II. Abschnitt, Exercise 215], it is shown that $\tilde{r}_{2l+1}$ satisfies (18), with m replaced by $2l+1$. In this sense, the expression in (18), which holds for all integers m, is sharp and extends, to the case of even integers, the result of [11].

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