A SURVEY OF APPLICATIONS OF THE JULIA VARIATION

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Abstract.
This is an introductory survey on applications of the Julia variation to problems in geometric function theory. A short exposition is given which develops a method for treating extremal problems over classes F of analytic functions on the unit disk D for which appropriate subsets F_n can be constructed so that (i) $F = \overline{\bigcup_n F_n}$ and (ii) for each $f \in F_n$ a geometric constraint will hold that $\partial f(D)$ will have at most n "sides". Applications of this method which have been made to problems in the literature are reviewed, e.g., Netanyahu's problems about the distortion theorems for starlike and convex functions constrained to contain a fixed disk; Goodman's problems about omitted values for classes of univalent functions; integral means estimates for derivatives of convex functions; maximization problems for functionals on linear fractional transforms of convex and starlike functions.

Key words. Julia variation, variational methods, geometric function theory

AMS subject classifications. 30C

This is a survey paper on the applications of a variational method introduced by J. Krzyz in [29]. It is based on Julia's modification of Hadamard's variation of the Green's function. In the early 1900's, in order to investigate the behavior of Green's function for slight deformations on the boundary of its domain, Hadamard developed his variational formulas [23]. The validity of the formulas relied on the domain having (at least) a continuously differentiable or smooth boundary. In the 1930's Julia, modifying Hadamard's procedure, derived a new variational formula in terms of the Riemann mapping function for the domain. This also required the boundary of the domain to be smooth. This requirement proved to be too restrictive to solve most extremal problems since the corresponding extremal domains have non-smooth, typically piece-wise smooth, boundaries. In particular, Schiffer suggested in [39] that the requirement of a smooth boundary was too restrictive for extremal problems for the general class of univalent functions; he then proceeded to develop his method of interior variations to circumvent the issue.

The first author was introduced to the Hadamard and Julia variational formulas during his graduate studies at the University of Maryland by J. Hummel. Hummel had used results on the Julia variation, obtained in [38], to create in [27] a general variational method for starlike functions. He gave a discussion of the Hadamard and Julia formulas in his lecture notes in [26]. We note also that Julia's formula was used by M.S. Robertson in [37] to develop a variational method for analytic functions on the unit disk with positive real part.

Aside from Schiffer's and Hummel's references, the Hadamard and Julia variational formulas remained generally dormant until Krzyz in [29] applied the Julia variational formula to convex polygons. A rigorous proof that the formula is valid when the boundary contains corners was provided by Barnard and Lewis in [10]. It was found later that an independent proof had been given earlier by Warshawski in [44]. Consequently, the initial condition of smoothness on the boundary of the domain could be replaced by a piece-wise smooth condition. This allowed for the method to be applied to a fairly large class of functions; in particular, to any class of functions whose image domains could be approximated by domains bounded by a finite number

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of arcs with explicit geometric conditions. The extremal domains with their piece-wise smooth boundaries are then among the approximating domains with a small number of sides. Since the approach of using dense subclasses does not usually give uniqueness of the extremal functions other methods are needed to obtain uniqueness.

For completeness, we will give a brief outline of the method. Let $\mathbb{C}$ denote the complex plane and let

$$\begin{aligned} D_r &= \{z : |z| < r\}, & D &= \{z : |z| < 1\} = D_1, \\ C_r &= \{z : |z| = r\}, & \gamma &= \{z : |z| = 1\} = C_1, \\ \mathcal{A} &= \{f : f \text{ is analytic on } D,\ f(z) = z + ...\ \}, \\ S &= \{f : f \in \mathcal{A}, f \text{ is 1-to-1 on } D\} \\ S_X &= \{f : f \in \mathcal{A}, f(D) \text{ is geometrically characterized by } X\}. \end{aligned}$$

Two specific subclasses of S to which we will make frequent reference are S^* and K, the subclasses of starlike functions (with respect to the origin) and convex functions, respectively. Recall that a function $f \in S$ is starlike if and only if

$$\text{Re } \{\frac{zf'(z)}{f(z)}\} > 0$$

for $z \in D$. Also, a function $f \in S$ is convex if and only if

$$\text{Re } \{1 + \frac{zf''(z)}{f'(z)}\} > 0$$

for $z \in D$.

Let E be a domain in $\mathbb{C}$. Let r_0 be the *mapping radius* or *inner radius* of the domain E at z_0. It is determined by the limit

$$\lim_{z \to z_0} g(z, z_0) + \log|z - z_0| = \log r_0$$

where $g(z, z_0)$ is the Green's function of E at z_0. Alternately, the mapping radius is given as the first order coefficient of the univalent mapping function $f : D \to E$ such that $f(0) = z_0$, i.e., let $f(z) = a_0 + a_1 z + a_2 z^2 + ...$, then $r_0 = |a_1|$. Both the mapping radius and the Green's function depend on the domain (and z_0), and as was shown in [25], the mapping radius is monotonically, set theoretically and continuously dependent on the size of the domain.

Let X_0 be a specific geometric characterization. We will suppose that there exist compact subclasses

$$S_n = \{f \in S_{X_0} : \partial f(D) \text{ has at most } n \text{ smooth "sides"}\}$$

such that for each $f \in S_n, \partial f(D)$ is a bounded Jordan curve and

$$\overline{\bigcup_n S_n} = S_{X_0}.$$

The definition of "sides" will depend on the geometric characterization of X_0.

For example, for the class of convex functions K the subclass K_n could be all of the functions for which $f(D)$ is a bounded convex polygon with at most n sides.

Let X_0 be a specific geometric characterization. Suppose L is a continuous linear functional, $L : \mathcal{A} \to \mathbb{C}$, and for the class S_{X_0} we want to find $\sup_{f \in S_{X_0}} \text{Re } L(f)$. Standard normal family arguments give the existence of an extremal function whenever S_{X_0} is a normal family.

The following proposition will now be proven.

Proposition 1. *Let $z = re^{i\theta}, |z| = r < 1$, and let $T(\zeta) = L\left(zf'(z)\frac{1+\zeta z}{1-\zeta z}\right)$, where $\zeta = e^{i\phi}$. If $\text{Re } T(\zeta) = c$ (for any constant c) has at most m solutions on $|\zeta| = 1$, then for $n \geq m$*

$$\sup_{f \in S_{X_0}} \text{Re } L(f) = \sup_n \max_{f \in S_n} \text{Re } L(f) = \max_{f \in S_m} \text{Re } L(f).$$

Proof. The compactness of the subfamilies S_n along with the density of $\bigcup_n S_n$ in S_{X_0} insures that, for $n \geq m$,

$$\sup_{f \in S_{X_0}} \text{Re } L(f) = \sup_{n \geq m} \max_{f \in S_n} \text{Re } L(f).$$

We will show that $\max_{f \in S_n} \text{Re } L(f) = \max_{f \in S_m} \text{Re } L(f)$ for $n \geq m$.

Let n be a fixed integer greater than m. Consider any function f in S_n where $\partial f(D)$ has strictly more than m "sides". By the assumption that $\partial f(D)$ is a (bounded) Jordan curve (where $f(D)$ is a Jordan domain), we have, by applying Caratheodory's Extension Theorem [19], that f has a continuous extension to $\bar{D} = D \bigcup \gamma$.

Let $\Omega = f(D)$ and $\Gamma = f(\gamma) = \partial\Omega$. Then Γ has $m+1$ or more "sides." Call these sides, $\Gamma_1, \Gamma_2, ...$ and call the preimages $\gamma_1, \gamma_2, ...$, where $f(\gamma_j) = \Gamma_j$.

We will locally vary Γ by moving one "side" in the following manner (see Figure 1). Pick a "side" of Γ, say Γ_j. For $w \in \Gamma_j, w = f(z)$ for some $z \in \gamma_j$. Let $n(w) = \frac{zf'(z)}{|zf'(z)|}$ and let $p(w)$ be the distance along the unit normal from w to a point w^* on a new "side" Γ_j^*. The distance moved is controlled by $\epsilon p(w)$, which is positive along the outward pointing normal and negative along the inward pointing normal. That is, it is positive if the "side" is moved out and negative if the "side" is moved in. (The variational distances at the corners are shown to be $o(\epsilon)$ in [10].) Form a new polygonal curve Γ_ϵ containing the "side" Γ_j^*. When the new curve Γ_ϵ can be chosen so that it still has n or more "sides," the variation is allowable. Details can be found in Barnard [6].

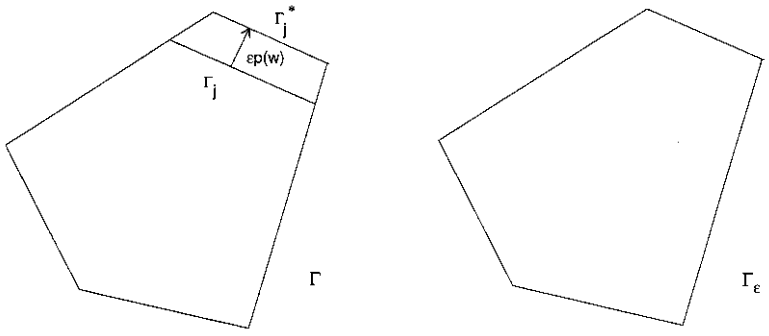

Figure 1

We note here that our initial definition of "sides" will require a geometric condition which allows the "sides" to be moved both in and out in the manner described above.

For a given $z \in D$ ($z = re^{i\theta}$), the varied function f_ϵ which maps D onto the varied domain Ω_ϵ for which $\partial\Omega_\epsilon = \Gamma_\epsilon$ becomes by the Julia Variational Formula [6],

$$f_\epsilon(z) = f(z) + \epsilon z f'(z) \int_\Gamma \frac{1+\zeta z}{1-\zeta z} d\Psi + o(\epsilon)$$

where $\zeta = e^{i\phi}, o(\epsilon)$ is uniform for z in compact subsets of D and, for $w = f(\zeta)$,

$$d\Psi = \frac{p(w)n(w)}{i[\zeta f'(\zeta)]^2} dw.$$

A change in variable

$$d\Psi(\phi) = \frac{p(w)}{|f'(\zeta)|} d\phi$$

insures that $d\Psi(\phi)$ is real and gives

$$f_\epsilon(z) = f(z) + \epsilon z f'(z) \int_\gamma \frac{1+\zeta z}{1-\zeta z} d\Psi(\phi) + o(\epsilon).$$

We note for future reference that the change in mapping radius from $f(D)$ to $f_\epsilon(D)$ is given by

$$\epsilon \int_\gamma d\Psi(\phi) + o(\epsilon).$$

If the varied side is Γ_j, the integral need only be taken over γ_j since the value of $d\Psi(\phi)$ is defined to be zero over the rest of γ by construction.

If two "sides" of Γ are locally varied, say Γ_k was moved out and Γ_l was moved in, the varied function becomes

$$f_\epsilon(z) = f(z) + \epsilon \left[z f'(z) \int_{\gamma_k} \frac{1+\zeta z}{1-\zeta z} d\Psi - z f'(z) \int_{\gamma_l} \frac{1+\zeta z}{1-\zeta z} d\Psi \right] + o(\epsilon).$$

Using the linearity of L, we have

$$\begin{aligned} \text{Re } L(f_\epsilon) = \text{Re } L(f) &+ \epsilon \int_{\gamma_k} \text{Re } L\left(z f'(z) \frac{1+\zeta z}{1-\zeta z} \right) d\Psi \\ &- \epsilon \int_{\gamma_l} \text{Re } L\left(z f'(z) \frac{1+\zeta z}{1-\zeta z} d\Psi \right) + o(\epsilon). \end{aligned}$$

We have then

$$\begin{aligned} \frac{\partial}{\partial \epsilon} \text{Re } L(f_\epsilon)|_{\epsilon=0} = &\int_{\gamma_k} \text{Re } L\left(z f'(z) \frac{1+\zeta z}{1-\zeta z} \right) d\Psi \\ &- \int_{\gamma_l} \text{Re } L\left(z f'(z) \frac{1+\zeta z}{1-\zeta z} d\Psi \right). \end{aligned}$$

Let $T(\zeta) = L(z f'(z) \frac{1+\zeta z}{1-\zeta z})$. Then,

$$\frac{\partial}{\partial \epsilon} \text{Re } L(f_\epsilon)|_{\epsilon=0} = \int_{\gamma_k} \text{Re } T(\zeta) d\Psi - \int_{\gamma_l} \text{Re } T(\zeta) d\Psi. \tag{0.1}$$

The function $T(\zeta)$ maps $|\zeta| = 1$ onto a curve β. On this curve β, there are $m+1$ arcs resulting from $T(\gamma_1), T(\gamma_2), ..., T(\gamma_{m+1})$ (see Figure 2). The Mean Value Theorem for integrals allows us to conclude that on each arc γ_j there exists a ζ_j such that

$$\int_{\gamma_j} \text{Re } T(\zeta) d\Psi(\phi) = \text{Re } T(\zeta_j) \int_{\gamma_j} d\Psi(\phi). \tag{0.2}$$

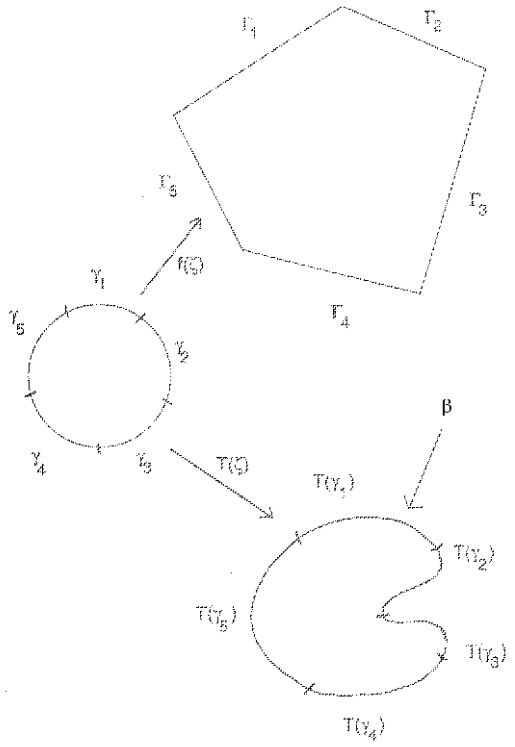

Figure 2

This implies that for γ_k and γ_l there are ζ_k and ζ_l satisfying (0.2) which makes (0.1) become

$$\frac{\partial}{\partial \epsilon} \text{Re } L(f_\epsilon)|_{\epsilon=0} = \text{Re } T(\zeta_k) \int_{\gamma_k} d\Psi - \text{Re } T(\zeta_l) \int_{\gamma_l} d\Psi.$$

By hypothesis , Re $T(\zeta) = c$ (for any constant c) has at most m solutions. Thus, the most such points, $T(\zeta_j)$, any vertical line (Re $T(\zeta) = c$) could intersect is m. Since $\partial f(D)$ has strictly more than m sides, there must exist two points $T(\zeta_k)$ and $T(\zeta_l)$ in β such that the real part of one is greater than the real part of the other, say

$$\text{Re } T(\zeta_k) > \text{Re } T(\zeta_l).$$

Thus, if the two sides that are initially varied are Γ_k and Γ_l, then the partial in (0.1) can be chosen so that

$$\frac{\partial}{\partial \epsilon} \text{Re } L(f_\epsilon)|_{\epsilon=0} > \text{Re } T(\zeta_l) \left[\int_{\gamma_k} d\Psi(\phi) - \int_{\gamma_l} d\Psi(\phi) \right].$$

As noted earlier, $\int_{\gamma_k} d\Psi(\phi) - \int_{\gamma_l} d\Psi(\phi)$ represents the change in the mapping radius. From continuity Γ can be varied so that the resulting function f_ϵ is again in S_n, i.e., if $\Psi(\phi)$ is chosen on γ_k and γ_l so that the resulting mapping radius of f_ϵ is 1, then $\int_{\gamma_k} d\Psi(\phi) - \int_{\gamma_l} d\Psi(\phi) = 0$. Hence,

$$\frac{\partial}{\partial \epsilon}\text{Re } L(f_\epsilon)|_{\epsilon=0} > [\text{Re } T(\zeta_l)](0) = 0.$$

Recall that

$$\begin{aligned}\frac{\partial}{\partial \epsilon}\text{Re } L(f_\epsilon)|_{\epsilon=0} = &\int_{\gamma_k} \text{Re } L\left(zf'(z)\frac{1+\zeta z}{1-\zeta z}\right) d\Psi \\ &- \int_{\gamma_l} \text{Re } L\left(zf'(z)\frac{1+\zeta z}{1-\zeta z} d\Psi\right).\end{aligned}$$

We have

$$\text{Re } L(f_\epsilon) = \text{Re } L(f) + \epsilon\left[\frac{\partial}{\partial \epsilon}\text{Re } L(f_\epsilon)|_{\epsilon=0}\right] + o(\epsilon).$$

Thus, for any function $f \in S_n$ such that $\partial f(D)$ has strictly more than m "sides," each of which can be moved both in and out, a function can be found which increases the real part of the functional L. Therefore, we have that the function $f \in S_n$ which maximizes the real part of the functional L is actually in S_m. Thus, for $n \geq m$,

$$\max_{f \in S_n} \text{Re } L(f) = \max_{f \in S_m} \text{Re } L(f).$$

This proves Proposition 1.

Remark. In [8] it is shown that by combining this method with Loewner theory on slit domains that this method may be applied to domains where $\partial f(D)$ is not a Jordan curve (where $f(D)$ is a slit domain) with slight variations. Hence, this variational method involving Proposition 1 may be used to solve extremal problems over a much broader class of functions.

It is this basic idea that the authors have been able to apply to a variety of extremal problems. The first arose in Barnard's dissertation and answered a question which arose out of Netanyahu's work in [36]. Netanyahu had asked to determine the extremal functions for the standard distortion theorems for the class of starlike or convex functions whose image domains contain a fixed disk centered at the origin. Netanyahu had done this [36] for the class $NS(d)$ of univalent functions in S whose image domains contain a fixed disk of radius d centered at the origin, but his methods did not appear to work for the corresponding classes $NS^*(d)$ or $NK(d)$. The extremal functions for the classes $NS^*(d)$ and $NK(d)$ were found by Barnard in [6], however, a more general result containing resolutions for Netanyahu's questions as special cases was given in [10] by Barnard and Lewis.

To obtain the more general result, we first answered M. Reade's question as to determining a geometric characterization for the Mocanu class, $MS^*(\alpha)$, of α-starlike functions. $MS^*(\alpha)$ was a continuously parametrized family of functions, $0 \leq \alpha \leq \infty$, defined analytically, which contained the class $S^* = MS^*(0)$ and the class $K = MS^*(1)$. In particular, Mocanu had defined in [35] the class $MS^*(\alpha)$ as those functions in $\mathcal{A}$ for which

$$\text{Re } \left\{(1-\alpha)\left(\frac{zf'(z)}{f(z)}\right) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > 0.$$

We determined in [10] that by defining an α-arc for $\alpha > 0$, loosely speaking, as a translate of an arc of the α^{th} power of the line $x = 1$ (see [10] for details), then a domain $f(D)$ is the image of an α-starlike function f if and only if any two points in $f(D)$ can be connected by an α-arc lying in $f(D)$. (Note $\alpha = 1$ gives 1-arcs as straight line segments [hence, convexity], while the limiting case of $\alpha = 0$ gives starlikeness with respect to the origin.) By using as a dense subclass those domains bounded by a finite number of α-arcs as "sides", we proved the following result.

For a given $d, 0 \leq d < 1$ and $1 < \rho \leq \infty$, let $MS^*(\alpha, d, \rho)$ be those functions in $MS^*(\alpha)$ for which $d \leq |f(z)/z| \leq \rho$. Let $F = F(\cdot, \alpha, d, \rho)$ be the function in $MS^*(\alpha, d, M)$ whose image is the symmetric domain bounded by an arc $\gamma_d \subset C_d$, which is symmetric about $-d$, an arc $\gamma_\rho \subset C_\rho$, which is symmetric about ρ and two α-arcs connecting the endpoints of γ_d and γ_ρ. We proved the following main theorem and corollary.

Theorem 1. *Let α, d and ρ be fixed nonnegative numbers satisfying $0 \leq \alpha \leq \infty$, $0 \leq d < 1$ and $1 < \rho \leq \infty$. Then,*

(A) The function $g(z) = \log[F(z)/z], z \in D$ is univalent and convex in the direction of the imaginary axis.

(B) If $f \in MS^*(\alpha, d, \rho)$, then $\log[f(z)/z]$ is subordinate to g.

Corollary 1. *Let α, d and ρ be as in Theorem 1. Let Φ be a given nonconstant entire function. If $f \in MS^*(\alpha, d, \rho)$, then*

(A) For given $z \in D \backslash \{0\}$

$$\text{Re } \left\{ \Phi \left[\log \frac{f(z)}{z} \right] \right\} \leq \max_{0 < \theta \leq 2\pi} \text{Re } \left\{ \Phi \left[\log \frac{F(z)}{z} \right] \right\},$$

(B) For given $r, 0 < r < 1$, and $\lambda > 0$,

$$\int_{-\pi}^{\pi} |f(re^{i\theta})|^\lambda d\theta \leq \int_{-\pi}^{\pi} |F(re^{i\theta})|^\lambda d\theta,$$

(C) For a given positive integer $N \geq 2$,

$$\sum_{k=2}^{N} |a_k|^2 \leq \sum_{k=2}^{N} |A_k|^2,$$

where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $F(z) = z + \sum_{k=2}^{\infty} A_k z^k, z \in D$.

Another application of this variational method was to show in [4] that any $f \in S$ for which $\partial f(D)$ has an analytic slit which contains a point $f(e^{i\theta_1}) = f(e^{i\theta_2})$ where the opposing normals exist and are of unequal modulus, i.e., $|f'(e^{i\theta_1})| \neq |f'(e^{i\theta_2})|$, can always be varied locally on the boundary to produce a domain of strictly larger mapping radius. For consequences of this result, see [4].

General applications of the variational method were used in a series of papers on the omitted values problem of Goodman. The problem of omitted values was first posed by Goodman [20] in 1949, restated by MacGregor [33] in his survey article in 1972, then reposed in a more general setting by Brannan [3] in 1977. It also appears in Bernardi's survey article [17] and has appeared in several open problem sets since then, including [7], [18], [21] and [34].

For a function $f \in S$, let $A(f)$ denote the Lebesgue measure of the set $D\backslash f(D)$ and for $0 < r < 1$ let $L(f, r)$ denote the Lebesgue measure of the set $\{D\backslash f(D)\} \bigcap C_r$. Two explicit problems posed by Goodman and by Brannan were to determine

$$A = \sup_{f \in S} A(f)$$

and

$$L(r) = \sup_{f \in S} L(f, r). \tag{0.3}$$

Goodman [20] showed that $0.22\pi < A < 0.50\pi$. The lower bound which he obtained was generated by a domain of the type shown in Figure 3.

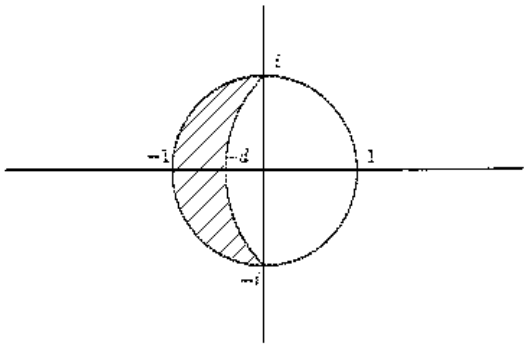

Figure 3

Later, Goodman and Reich [22] gave an improved upper bound of 0.38π for A. Lewis had shown in [31] by applying some deep results of Alt and Caffarelli [2] in partial differential equations for free boundary problems that the extremal domains had to have piecewise analytic boundaries and that our methods developed in [10] for these omitted value problems could be used to give a geometric description for the boundaries of the extremal domains as follows. There is an f_0 in S with $A = A(f_0)$ such that $f_0(D)$ is circularly symmetric with respect to the positive real axis, i.e., it has the property that for $0 < r < 1$,

$$\frac{\partial}{\partial\theta}|f_0(re^{i\theta})| \leq 0 \text{ and } \frac{\partial}{\partial\theta}|f_0(re^{-i\theta})| \leq 0, \text{ for } 0 < \theta < \pi$$

(cf. Hayman [25]). Moreover, the $\partial f_0(D)$ consists of the negative real axis up to -1, and arc σ of the unit circle that is symmetric about -1 and an arc λ lying in D, except for its endpoints. The arc λ is symmetric about the reals, connects to the endpoints of σ and has monotonically decreasing modulus in the closure of the upper half disc. These results follow by standard symmetrization methods. Much deeper methods are needed to show (as in [5] and [31]) that f_0 has a piecewise analytic extension to λ with f_0' continuous on $f_0^{-1}(\lambda)$ and $|f_0'(f_0^{-1}(w))| \equiv c < 1$ for all $w \in \lambda \bigcap \{D\backslash(-1, 1)\}$. Using these properties of f_0 it was shown by the authors in [11] that by "rounding the corners" of certain gearlike domains a close approximation for the extremal function could be obtained. This gives the best known lower bound of

$$0.24\pi < A.$$

The upper bound is conceptually harder since it requires an estimate on the omitted area for each function in S. Indeed, it appears difficult to use the geometric description of f_0 to calculate A directly. However, an indirect proof was used by Barnard and Lewis [9] to obtain the best known upper bound of

$$A < 0.31\pi.$$

Open Problem. *Show that f_0 is unique and determine A explicitly.*

For the class S^* of functions in S whose images are starlike with respect to the origin, the problem of determining the corresponding

$$A^* = \sup_{f \in S^*} A(f)$$

has been completely solved by Lewis in [31]. The extremal function $f_1 \in S^*$ defined by

$$A^* = A(f_1) \approx 0.235\pi$$

is unique (up to rotation). The extremal domain $\partial f(D)$ is a circularly symmetric domain whose boundary $\partial f_1(D)$ has two radial rays projecting into D with their endpoints connected by an arc λ_1 that is symmetric about the reals and has $|f_1'(\zeta)| \equiv c_1$ for all $\zeta \in f_1^{-1}(\lambda_1)$.

The problem of determining $L(r)$ in (0.3) was solved by Jenkins in [28] where he proved that for a fixed $r, 1/4 \leq r < 1$,

$$L(r) = 2\pi \arccos(8\sqrt{r} - 8r - 1).$$

The extremal domain in this case is the circularly symmetric domain (unique up to rotation) having as its boundary the negative reals up to $-r$ and a single arc of C_r symmetric about the point $-r$.

The corresponding problem for starlike functions of determining

$$L^*(r) = \sup_{f \in S^*} L(f, r)$$

was solved by Lewandowski in [30] and by Stankiewicz in [40]. The extremal domain in that case is the circularly symmetric domain (unique up to rotation) having as its boundary two radial rays and the single arc of C_r connecting their endpoints. An explicit formula for the mapping function in this case was first given by Suffridge in [41].

However, for the class K of functions in S whose images are convex domains the corresponding problems of determining

$$A_K(r) = \sup_{f \in K} A(f, r)$$

and

$$L_K(r) = \sup_{f \in K} L(f, r),$$

where $A(f, r)$ denotes the Lebesgue measure of $D_r \backslash f(D)$, presents some interesting difficulties. One particular difficulty is that the basic tool of circular symmetrization

used in the solution of each of the previous determinations is no longer available. The example of starting with the convex domain bounded by a square shows that convexity is not alway preserved under circular symmetrization. However, Steiner symmetrization (cf. Hayman [25]) can still be used in certain cases such as sectors. Another difficulty is the introduction of distinctly different extremal domains for different ranges of r. Since every function in K covers a disk of radius $1/2$ (cf. Duren [19]) r needs only to be considered in the interval $(1/2, 1)$. Waniurski has obtained some partial results in [43]. He defined r_1 and r_2 to be the unique solutions to certain transcendental equations where $r_1 \approx 0.594$ and $r_2 \approx 0.673$. If $F_{\pi/2}$ is the map of D onto the half plane $\{w : \text{Re } w > -1/2\}$ and F_α maps D onto the sector

$$\left\{w : \left|arg\left(w + \frac{\pi}{4\alpha}\right)\right| < \alpha\right\}$$

whose vertex, $v = -\pi/4\alpha$, is located inside D, then

$$A_K(r) = A(F_{\pi/2}, r) \text{ for } 1/2 < r < r_1,$$
$$L_K(r) = L(F_{\pi/2}, r) \text{ for } 1/2 < r < r_1,$$

and

$$L_K(r) = L(F_\alpha, r) \text{ for } r_1 < r < r_2.$$

Barnard had announced in his survey talk on open problems in complex analysis at the 1985 *Symposium on the Occasion of the Proof of the Bieberbach Conjecture* the following conjecture.

Conjecture 1. *The extremal domains for determining $A_K(r)$ and $L_K(r)$ will be half planes, symmetric sectors and domains bounded by single arcs of C_r along with tangent lines to the endpoints of these arcs; the different domains will depend on different ranges of r in $(1/2, 1)$.*

This conjecture was also made later, independently, by Waniurski at the end of his paper in [43] in 1987.

In trying to determine the extremal domains for $A_K(r)$ and $L_K(r)$ via our variational method developed in [10] the following problem was investigated. For $F \subset S$ and $1/4 \le d \le 1$ let $F_d = \{f \in F : \min_{z\in D} |f(z)/z| = d\}$. Then, determine the sharp constant $A = A(F_d)$ such that for any $f \in F_d$

$$I_{-1}(f') = \lim_{r\to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\frac{1}{f'(re^{i\theta})}\right| d\theta \le \frac{A}{d}. \tag{0.4}$$

For $0 \le \alpha < 1$ let $S^*(\alpha)$ denote the subclass of S of starlike functions of order α, i.e., a function $f \in S^*(\alpha)$ if and only if f satisfies the condition $\text{Re } zf'(z)/f(z) > \alpha$ for $z \in D$. It is well known that $K \subset S^*(1/2)$. It follows fairly easily from subordination theory that $A(S^*(1/2)) \le 4/\pi$. Furthermore, this estimate is sharp for $S^*(1/2)$ since the functions $f_n(z) = z/(1 - z^n)^{1/n}$ belong to $S^*(1/2)$ for each $n > 0$. However, this estimate is not sharp for the class K of convex functions which is a proper subset of $S^*(1/2)$. Considerable numerical evidence suggested to the authors to make the following conjecture.

Conjecture 2. *For each $d, 1/2 \le d \le 1, A = A(K_d) = 1$ in (0.4) with equality holding for all domains which are bounded by regular polygons centered at the origin.*

This conjecture was announced in March 1985 at the *Symposium on the Occasion of the Proof of the Bieberbach Conjecture* at Purdue University. It also appeared as Conjecture 8 in the first author's "Open Problems and Conjectures in Complex Analysis" in [7]. It was thought, by many function theorists, that the conjecture would be easily settled, given the vast literature on convex functions and the large research base for determining integral mean estimates, see [19].

An initial difficulty was the non-applicability of Baernstein's circular symmetrization methods, since convexity, unlike univalence and starlikeness, is not preserved under circular symmetrization as noted above. Although Steiner symmetrization does preserve convexity, see [42], it did not appear to be helpful for the problem and, indeed, we found that extremal domains need possess no standard symmetry.

A confusing issue, which also arises, is that the integral means of the standard approximating functions f_n in K defined by

$$f_n'(z) = \prod_{k=1}^{n}(1 - ze^{i\theta_k})^{-2\alpha_k}, 0 < \alpha_k \leq 1, \sum_{k=1}^{n}\alpha_k = 1$$

decrease when the arbitrarily distributed θ_k are replaced by uniformly distributed $t_k = k\pi/n$, as was shown in [45]. The conjecture suggests that multiplication by the minimum modulus d must overcome this decrease.

We make the following definition.

Definition. Let Γ be a curve in $\mathbb{C}$ such that the left- and right-hand tangents to the curve Γ exist at each point on Γ. The curve Γ will be said to *circumscribe a circle* C if the left- and right-hand tangents to the curve Γ at each point on Γ lie on tangent lines to the circle C.

We will employ the following notation.

Notation. Let $f \in S$ and suppose that λ is a subarc of γ on which f is smooth. For $z = e^{i\theta} \in \lambda$ let $d_\theta = < f(z), \frac{zf'(z)}{|zf'(z)|} >$, i.e, d_θ is the directed length of the projection of $f(z)$ onto the outward unit normal to the $\partial f(D)$ at $f(z)$.

In 1993 in [12], we proved the following theorem which verified Conjecture 2.

Theorem 2. *Let $f \in K$, $d = \min_\theta |f(e^{i\theta})|$ and $d^* = \sup_\theta d_\theta$. Then,*

$$\frac{1}{d^*} \leq \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{|f'(e^{i\theta})|}d\theta \leq \frac{1}{d}$$

with equality holding if the $\partial f(D)$ circumscribes C_d.

Our original proof which was quite lengthy used as a major step the following idea which has independent interest. For any domain Λ which contains the origin, let the notation $m.r.(\Lambda)$ denote the inner radius of Λ at the origin. Given a d and a convex domain $\Omega = f(D)$ where $f \in K_d$, one can always construct two varied domains Ω^* and Ω^{**} which satisfy the containment relationship

$$\Omega^* \subseteq \Omega^{**}, \tag{0.5}$$

as follows.

The domain Ω^{**} is constructed by multiplication by $1 + \epsilon$ for ϵ sufficiently small and positive. This produces a radial enlargement $(1+\epsilon)\Omega = \Omega^{**}$ of Ω. The domain Ω^*

is constructed by moving each smooth boundary point $w = f(e^{i\theta})$ a distance ϵd in the direction of the outward pointing normal. An adjustment is made at the non-smooth boundary points to produce the domain Ω^* so that boundary of Ω^* has the same geometric restrictions as the boundary of Ω, e.g., same number of sides. Subordination then shows that (0.5) implies that $m.r.(\Omega^*) \leq m.r.(\Omega^{**})$. The variational formulas can then be applied to yield that

$$\Delta m.r.(\Omega^*, \Omega) = \frac{\epsilon d}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{f'(e^{i\theta})} \right| d\theta + o(\epsilon) \leq \epsilon = \Delta m.r.(\Omega^{**}, \Omega)$$

which give the result. Similar arguments are used for d^*.

We obtained, arising out of the proof, the rather unexpected sufficient condition for equality to occur in (2) for the classes K_d. However, because the proof used a scheme to approximate convex functions by polygonally convex functions, we did not obtain a necessary condition for equality.

We devised in [13] a new, simpler proof for the conjecture which extends Theorem 2. The proof relaxes the convexity requirement and validates the necessity of the sufficient condition.

Theorem 3. *Let $f \in S^*(\alpha)$ for some $0 \leq \alpha < 1$. Suppose f is smooth on $X \subset \partial D$ where X is a countable union of pairwise disjoint subarcs of γ such that the complement of X in γ has measure zero. Let $d_* = \inf_{\theta \in X} d_\theta$, $d^* = \sup_{\theta \in X} d_\theta$. Then,*

$$\frac{1}{d^*} \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|f'(e^{i\theta})|} d\theta \leq \frac{1}{d_*}$$

with equality holding if and only if $\partial f(D)$ circumscribes C_{d_*}.

Another application of the method originated from a question of Clunie and Sheil-Small. If $f \in S$ and $w \notin f(D)$, then the function

$$\hat{f} = f/(1 - f/w) \tag{0.6}$$

belongs again to S. The transformation $f \to \hat{f}$ is an important tool in the study of geometric function theory. If F is a subset of S, let

$$\hat{F} = \{\hat{f} : f \in F, w \in \mathbb{C}^* \backslash f(D)\}.$$

Here, $\mathbb{C}^* = \mathbb{C} \bigcup \{\infty\}$. Since we admit $w = \infty$, it is clear that $F \subset \hat{F} \subset S$.

If F is compact in the topology of local uniform convergence, then so is $\hat{F}$. If F is rotationally invariant, that is, if $f_\alpha(z) = e^{-i\alpha} f(e^{i\alpha} z)$ belongs to F whenever f does, then $\hat{F}$ is also rotationally invariant. It is an interesting question to ask which properties of F are inherited by $\hat{F}$. Since $\hat{S} = S$, this question is trivial for S.

In [14], [15] and [16] Barnard and Schober considered transforms of the class K of convex mappings (and the class S^* of starlike mappings). Simple examples show that $\hat{K}$ is strictly larger than K. Since the coefficients of functions in K are uniformly bounded [by one (in modulus)], Clunie and Sheil-Small had asked whether the coefficients of functions in $\hat{K}$ have a uniform bound. The affirmative solution of this problem was given by Hall [24].

Open Question. *Find the best uniform coefficient bound as well as the individual coefficient bounds for functions in $\hat{K}$.*

In [14] the variational procedure developed in [10] is applied to a class of extremal problems for $\hat{K}$. If $\lambda : \hat{K} \to \mathbb{R}$ is a continuous functional that satisfies certain admissibility criteria, it was shown that the problem

$$\max_{\hat{K}} \lambda$$

has a relatively elementary extremal function $\hat{f}_1$. More specifically, it was shown that $\hat{f}_1$ is either a half-plane mapping $\hat{f}_1(z) = z/(1 - e^{i\alpha}z)$ or is generated through (0.6) by a parallel strip mapping $f_1 \in K$.

The class of functionals considered in [14] contain the second-coefficient functional $\lambda(\hat{f}) = \text{Re } a_2$ and the functionals $\lambda(\hat{f}) = \text{Re } \Phi(\log \hat{f}(z)/z)$ where Φ is entire and z is fixed. The latter functionals include the problems of maximum and minimum modulus ($\Phi(w) = \pm w$). In general, the extremal strip domains $f_1(D)$ need not be symmetric about the origin. This adds a nontrivial and interesting character to the problem.

A sharp estimate for the second coefficient of functions in $\hat{K}$ is given explicitly in the following result. Surprisingly, the answer is not an obvious one.

Theorem 4. *If $\hat{f}(z) = z + a_2z^2 + ...$ belongs to $\hat{K}$, then*

$$|a_2| \leq \frac{2}{x_0} \sin x_0 - \cos x_0 \approx 1.3270$$

where $x_0 \approx 2.0816$ is the unique solution of the equation

$$\cot x = \frac{1}{x} - \frac{1}{2}x$$

in the interval $(0, \pi)$. Equality occurs for the functions $e^{-i\alpha}\hat{f}_1(e^{i\alpha}z), \alpha \in \mathbb{R}$, where $\hat{f}_1(z) = f_1(z)/[1 - f_1(z)/f_1(1)]$ and f_1 is the vertical strip mapping defined by

$$f_1(z) = \frac{1}{2i \sin x_0} \log \frac{1 + e^{ix_0}z}{1 - e^{ix_0}z}. \tag{0.7}$$

We make the following conjecture.

Conjecture 3. *The extremal functions for maximizing $|a_n|$ over $\hat{K}$ are the vertical strip mappings defined by (0.7) where a different x_0 is needed for each n.*

In [15] the Koebe disk, radius of convexity and sharp estimates for the coefficient functional$|ta_3 + a_2^2|$ were found for functions in the class $\hat{K}$. Also, in [1] R.M. Ali found sharp upper and lower bounds for $|f(z)|$ for $\hat{f} \in \hat{K}$.

A key lemma in the proofs of the above was the determination of the geometric characterization of the "sides" to be varied in the images of the functions in a dense subclass of the transformed convex domains. The variations of the functions in K_n whose image domains are convex polygons with at most n sides did not reduce the problem for $\hat{K}$ to examining domains with a sufficiently small enough number of sides. We observed that functions $\hat{f}$ in $\hat{K_n}$ map D onto curvilinear polygons with at most n sides and with interior angles at π. Furthermore, if $\hat{f} = f/(1 - f/w)$, then the sides of the $\partial\hat{f}(D)$ all lie on circles or lines through the point $w_1 = -w$ (see Figure 4). In fact, these two properties characterize functions in $\widehat{K_n}$. That is, if $g \in S$ and the $\partial g(D)$ is a curvilinear n-gon with interior angles at most π and if the sides of the

$\partial g(D)$ all lie on circles or lines through a point $-w \notin g(D)$, then $f = g/(1 - g/w)$ belongs to K_n and so $\hat{f} = g$ belongs to $\widehat{K_n}$.

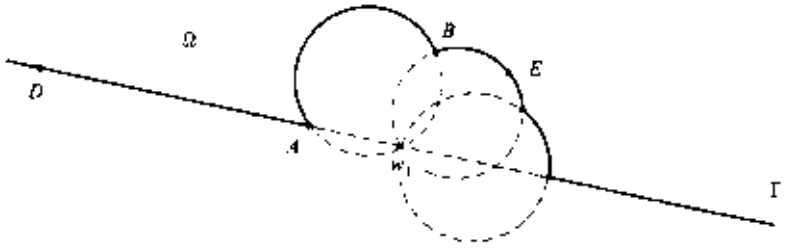

Figure 4

These observations thus produced the domains whose boundaries were characterized in a geometric fashion so that our methods could be applied.

In a more recent application of the Julia variational method J. MA [32] has proved, in his dissertation under Ruscheweyh, the following results.

Let points $z_j \in D$ with $z_j \neq 0, j = 1, 2, ..., m$ and positive integers $l_j, j = 0, 1, 2, ..., m$ be given. Define $N = l_0 + l_1 + l_2 + ... + l_m$. For $g \in \mathcal{A}$ consider the complex N vector $v(g)$ with components

$$a_k \ (k = 2, 3, ..., l_0 + 1), \qquad g^{(k)}(z_j) \ (k = 0, 1, ..., l_j - 1; j = 1, 2, ..., m).$$

Denote by $V_N(K)$ the set $\{v(g) : g \in K\}$.

Let $\Lambda(v)$ be a real-valued function that is differentiable on some open neighborhood of $V_N(K)$. Define a continuous functional Ψ on K by

$$\Psi(g) := \Lambda(v(g)) = \Lambda(a_2, a_3, ..., a_{l_0+1}, g(z_1), g'(z_1), ..., g^{(l_1-1)}(z_1), ..., g(z_m), g'(z_m), ..., g^{(l_m-1)}(z_m))$$

A functional of this form will be called a functional of finite degree and the number N is call the degree of the functional.

Let us denote

$$\Lambda_v := \left(\frac{\partial \Lambda}{\partial w_1}, ..., \frac{\partial \Lambda}{\partial w_N} \right),$$

where

$$\frac{\partial \Lambda}{\partial w_k} = \frac{1}{2} \left(\frac{\partial \Lambda}{\partial x_k} - i \frac{\partial \Lambda}{\partial y_k} \right), \qquad k = 1, 2, 3, ..., N.$$

The main results are the following theorems.

Theorem 5. *Let Ψ be a functional of degree N on the class K and for $f_0 \in K$ let*

$$\Psi(f_0) = \max(\min)\{\Psi(g) : g \in K\}.$$

Suppose that $\Lambda_v(v(f_0)) \neq 0$. Then,

$$f_0(z) = \int_0^z \frac{d\eta}{\prod_{k=1}^{N} (1 - z_k \eta)^{2\lambda_k}}, \tag{0.8}$$

where $|x_k| = 1, \lambda_k >= 0, k = 1, ..., N$ *and* $\sum_{k=1}^{N} \lambda_k = 1$.

Theorem 6. *Let* $H(w_1, w_2, ..., w_N)$ *be an analytic function on a neighborhood of the set* $V_N(K)$. *If for* $f_0 \in K$,

$$\text{Re } \{H(v(f_0))\} = \max(\min)\{\Psi(g) : g \in K\}$$

then f_0 *has the form (0.8).*

We note that extremal functions with the form (0.8) map D onto convex polygons with at most N sides.

REFERENCES

[1] Ali, R. M. "A distortion theorem for the class of Möbius transformations of convex mappings." *Rocky Mountain J. Math.* **19** (1989) pp. 1083-1094.

[2] Alt, H., and Caffarelli, L. "Existence and regularity for a minimum problem with free boundary." *J. Reine Angew. Math.* **325** (1981) pp. 105-144.

[3] Anderson, J.M., Barth, K.E., and Brannan, D.A. "Research problems in complex analysis." *Bull. London Math Soc.* **9** (1977) pp. 129-162.

[4] Barnard, R.W. "Criteria for local variations for slit mappings." *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.* **35/36** (1982/3) pp. 19-23.

[5] Barnard, R.W. "The omitted area problem for univalent functions." *Contemporary Math.* **38** (1985) pp. 53-60.

[6] Barnard, R.W. "On bounded univalent functions whose ranges contain a fixed disk." *Trans. Amer. Math. Soc.* **225** (1977) pp 1-28.

[7] Barnard, R.W. "Open problems and conjectures in complex analysis." in *Computational Methods and Function Theory*, Lecture Notes in Math. No. 1435 (Springer-Verlag, 1990), 1-26.

[8] Barnard, R.W. "A variational technique for bounded starlike functions." *Canadian Math. J.* **27** No. 2, (1975) pp. 337-347.

[9] Barnard, R.W., and Lewis, J.L. "On the omitted area problem." *Mich. Math. J.* **34** (1987) pp. 13-22.

[10] Barnard, R.W., and Lewis, J.L. "Subordination theorems for some classes of starlike functions." *Pacific J. Math.* **56** (1975) pp. 333-366.

[11] Barnard, R.W., and Pearce, K. "Rounding corners of gearlike domains and the omitted area problem." *J. Comput. Appl. Math.* **14** (1986) pp. 217-226.

[12] Barnard, R.W. and Pearce, K. "Sharp bounds on the H_p means of the derivative of a convex function for $p = -1$", *Complex Variables, Theory and Applications* **21** (1993), 149-158.

[13] Barnard, R.W., and Pearce, K. "A short proof of a conjecture on the integral means of the derivative of a convex function" *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.* **48** (1994) pp. 1-5.

[14] Barnard, R.W., and Schober, G., "Möbius transformations for convex mappings," *Complex Variables, Theory and Applications* **3** (1984) pp. 45-54.

[15] Barnard, R.W., and Schober, G., "Möbius transformations for convex mappings II," *Complex Variables, Theory and Applications* **7** (1986) pp. 205-214.

[16] Barnard, R.W., and Schober, G., "Möbius transformations of starlike mappings." *Complex Variables, Theory and Applications* **15** (1990) pp. 211-221.

[17] Bernardi, S.D. "A survey of the development of the theory of schlicht functions." *Duke Math. J.* **19** (1952) pp. 263-287.

[18] Brannan, D.A., and Clunie, J.G. (eds.). *Aspects of contemporary complex analysis.* (Durham, 1979), Academic Press, London, (1980).

[19] Duren, P. *Univalent Functions.* Spring-Verlag, Vol. 259, New York, (1980).

[20] Goodman, A.W. "Note on regions omitted by univalent functions." *Bull. Amer. Math. Soc.* **55** (1949) pp. 363-369.

[21] Goodman, A.W. *Univalent Functions.* Mariner (1982).

[22] Goodman, A.W., and Reich, E. "On the regions omitted by univalent functions II." *Canad. J. Math.* **7** (1955) pp. 83-88.

[23] Hadamard, J."Leçons sur le calcul des variations." Paris, 1910. FM pp. 41-432.

[24] Hall, R. R. "On a conjecture of Clunie and Sheil-Small." *Bull, London Math. Soc.* **12** (1980) pp. 25-28.

[25] Hayman, W.K. *Multivalent functions.* Cambridge Tracts in Math. and Math. Phys., No. 48, Cambridge University Press, Cambridge, MR21 #7301 (1958).

[26] Hummel, J. *Lectures on variational methods in the theory of univalent functions.* No. 8, University of Maryland, 1972.

[27] Hummel, J. "A variational method for starlike functions." *Proc. Amer. Math. Soc.* **9** (1958) pp. 82-87.

[28] Jenkins, J.A. "On values omitted by univalent functions." *Amer. J. Math.* **2** (1953) pp. 406-408.

[29] Krzyz, J. "Distortion theorems for bounded convex domains II." *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.* **14** (1960) pp 1028-1030.

[30] Lewandowski, Z. "On circular symmetrization of starshaped domains." *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.* **17** (1963) pp. 35-38.

[31] Lewis, J.L. "On the minimum area problem." *Indiana Univ. Math. J.* **34** (1985) pp. 631-661.

[32] Ma, J. "Julia variations for extremal problems of convex univalent functions." Ph.D. Dissertation.

[33] MacGregor, T.H. "Geometric problems in complex analysis." *Amer. Math. Monthly.* **79** (1972) pp. 447-468.
[34] Miller, S. (ed.). *Complex analysis.* (Brockport, NY, 1976), Lecture Notes in Pure and Appl. Math., 36, Dekker, New York, (1978).
[35] Mocanu, P.T. "Une propriété de convexité geénéralisée la théorie de la représentation conforme." *Mathematica (Cluj)* **11**(34) (1969) pp. 127-133.
[36] Netanyahu, E. "On univalent functions in the unit disk whose image contains a given disk." *J. D'analyse Math.* **23** (1970) pp. 305-322.
[37] Robertson, M.S. "Extremal problems for analytic functions with positive real part and applications." *Trans. Amer. Math. Soc.* **106** (1963) pp. 236-253.
[38] Schaeffer, A.C., Schiffer, M., and Spencer, D.C. "The coefficient regions of schlicht functions." *Duke Math J.* **16** (1949) pp. 493-527.
[39] Schiffer, M. "Some recent developments in the theory of conformal mapping." Appendix in *Dirichlet's Principle*, R. Courant. Interscience , New York (1967).
[40] Stankiewicz, J. "On a family of starlike functions." *Ann. Univ. Mariae Curie-Sklodowska, Sect. A.* 22-24, (1968-70) pp. 175-181.
[41] Suffridge, T. "A coefficient problem for a class of univalent functions." *Mich. Math. J.* **16** (1969) pp 33-42.
[42] Valentine, F.A. *Convex Sets*, McGraw-Hill, New York, 1964.
[43] Waniurski, J. "On values omitted by convex univalent mappings." *Complex Variables, Theory and Appl.* **8** (1987) pp. 173-180.
[44] Warshawski, S.E. "On Hadamard's variation formula for Green's function." *J. Math. Mech.* **9** (1960) pp. 497-512.
[45] Zheng, J. "Some extremal problems involving n points on the unit circle", Dissertation, Washington University (St. Louis), 1991.