Sums of Exponentials with Polynomial Coefficients

by

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1 Introduction

In [1], Ammar, Dayawansa and Martin considered the problem of matching data points $\alpha_1, \alpha_2, \ldots$ to a curve of the form

$$y(t) = p_1(t)e^{\lambda_1 t} + p_2(t)e^{\lambda_2 t} + \cdots + p_s(t)e^{\lambda_s t} \tag{1}$$

where $p_1(t), p_2(t), \ldots, p_s(t)$ are polynomials in t. The data is assumed to be generated by sampling a physical process over equally space time increments whose dynamics are described by curves of the form (1). They solved this interpolation problem by a method that is similar to the classical method of Prońy [4].

In this paper we pursue the solution to this exponential interpolation problem by a continuation method described in [2]. There, in [2], from $y(t)$, a smooth mapping $F : \mathbf{R}^n \to \mathbf{R}^n$ ($\mathbf{R}^n$ n-dimensional euclidean space) is constructed. Then, from the zeros of F, the coefficients of the polynomials and the factors $y_1, y_2, \ldots, y_s$ are recovered.

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More precisely, if $p_1(t) = a_0 + a_1 t + \cdots + a_k t^k, p_2(t) = b_0 + b_1 t + \cdots + b_l t^l, \ldots, p_s(t) = c_0 + c_1 t + \cdots + c_m t^m$ with $k \leq l \leq \ldots \leq m$, and the physical process is sampled at time intervals $\Delta t, 2\Delta t, \ldots$ giving rise to the assumed values

$$\begin{array}{lcl} p_1(\Delta t)e^{\lambda_1 \Delta t} + \cdots + p_s(\Delta t)e^{\lambda_s \Delta t} & = & \alpha_1 \\ p_1(2\Delta t)e^{\lambda_1 2\Delta t} + \cdots + p_s(2\Delta t)e^{\lambda_s 2\Delta t} & = & \alpha_2 \\ \vdots & & \vdots \\ p_1(n\Delta t)e^{\lambda_1 n\Delta t} + \cdots + p_s(n\Delta t)e^{\lambda_s n\Delta t} & = & \alpha_n, \end{array} \tag{2}$$

where $n = (k+2) + (l+2) + \cdots + (m+2)$.

The coordinate functions of $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ are defined as follows: first assume $\Delta t = 1$ and define $x = e^{\lambda_1}, y = e^{\lambda_2}, \ldots, z = e^{\lambda_s}$. Next set $X = (a_0, a_1, \ldots, a_l, x;\ b_0, b_1, \ldots, b_l, y;\ \ldots, c_0, c_1, \ldots, c_n, z)$. Then the coordinate functions $F_1, F_2, \ldots, F_n$ of F are defined as

$$\begin{array}{lcl} F_1(X) & = & p_1(1)x + p_2(1)y + \cdots + p_s(1)z - \alpha_1 \\ F_2(X) & = & p_1(2)x^2 + p_2(2)y^2 + \cdots + p_s(2)z^2 - \alpha_2, \\ \vdots & & \vdots \\ F_n(X) & = & p_1(n)x^n + p_2(n)y^n + \cdots + p_s(n)z^n - \alpha_n. \end{array} \tag{3}$$

Therefore, finding the zero of the polynomial system (3)- or, more precisely, finding the zeros which are the permutations in the components of a fundamental root (see section 3 of [2] for a clarification of this point)- enables us to recover the polynomial coefficients of $p_1(t), p_2(t), \ldots, p_s(t)$ and the exponential factors $\lambda_1, \lambda_2, \ldots, \lambda_s$, and so $y(t)$ matches the data points.

2 Preliminaries

In the papers [2] and [3] the continuation method that we wish to exploit in this paper was described. However, for the sake of completeness we give some basic definitions and proofs of two theorems needed to justify our computational work.

First, some definitions:

Definition 2.1 *If $F : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is at least continuously differentiable, then $JF(x)$, its Jacobian at x, is a continuous real valued function. If* $q(t) =$

$(q_1(t), \ldots, q_n(t))$ is a path in $\mathbb{R}^n$, it is said to be lifted by the mapping F to a covering path $r(t) = (r_1(t), \ldots, r_n(t))$ in $\mathbb{R}^n$ if $F(r(t)) = q(t)$ for all $0 \leq t \leq 1$.

The following theorem gives conditions under which paths of the form $(1-t)F(a)$ for some a in an open set W of $\mathbb{R}^n$ may be lifted to a covering path lying in W. It is Theorem 2 of [3] or 2.1 of [2].

Theorem 2.1 *Let C be a bounded convex domain of $\mathbb{R}^n$ and $F : C \to \mathbb{R}^n$ be at least continuously differentiable. If $JF(x) \neq 0$ for x in C_1, then for each zero x^* of F in C there is an open subset W of C such that for each a in W the path $(1-t)F(a)$ may be lifted by F to a covering path $x(t)$in W such that $x^* = x(1)$.*

The open subset W may be described as follows: Let ∂C be the boundary of C and U the nonempty component of $\mathbb{R}^n - F(\partial C)$ which contains the origin. Define the open subset $St(0)$ of U as follows: y in U is in $St(0)$ if $(1 \dot{-} t)y,\ 0 \leq t \leq 1$, lies wholly in U. W is the connected component of $F^{-1}(St(0))$ which contains the zero x^*.

In our case of interest, due to the deterministic processes described and studied, we can see that because of uniqueness there is but one root and so one set W is determined by the theorem.

In practice we do not explicitly find a covering path $x(t)$, but rather we observe that a covering path $x(t)$ is a trajectory of the differential equation

$$F'(x)\frac{dx}{dt} = -F(a) \tag{1}$$

with initial condition $x(0) = a$. We then follow this solution to $t = 1$.

It is clear that in order to use this method knowledge of where $JF(x)$ is zero is necessary. To that end we calculate the Jacobian of the system in (3), but first we prove a lemma.

Lemma 2.1 *Let $A, B, \ldots, C$ be the transpose of the n-vectors $(1, x, x^2, \ldots, x^{n-1})$, $(1, y, y^2, \ldots, y^{n-1})$, $\ldots, (1, z, z^2, \ldots, z^{n-1})$ respectively, where $n = (k+1) + (l+1) + \ldots + (m+1)$, $k \leq l \leq \ldots \leq m$. Let $\Delta = \Delta(x, y, \ldots, z)$ be the determinant of the $n \times n$ matrix whose column presentation is*

$$[A, A', \ldots, A^{(k)};\ B, B', \ldots, B^{(l)};\ C, C', \ldots, C^{(m)}]$$

and the primes denote differentiation with respect to the variable $x, y, \ldots, z$. Then, modulo a constant

$$\Delta = (x-y)^p (x-z)^q \ldots (y-z)^r$$

where

$$p = (k+1)(l+1), q = (k+1)(m+1), \ldots, r = (l+1)(m+1).$$

Proof: Recall that the derivative of an $n \times n$ determinant Δ is a sum of n determinants where each summand is an $n \times n$ determinant obtained from Δ by leaving fixed all columns of Δ save one which is differentiated entry by entry. Evidently,

$$\frac{\partial \Delta}{\partial x} = \det\, [A, \ldots, A^{(k-1)}, A^{(k+1)}; \ldots]$$

consists of one term because of the identity of two columns in the remaining terms. Observe that $\frac{\partial \Delta}{\partial x} |_{x=y} = 0$ because of the identity of at least two columns. Again,

$$\frac{\partial^2 \Delta}{\partial x^2} = \det\, [A, \ldots, A^{(k-2)}, A^{(k)}, A^{(k+1)}; \ldots] + \det\, [A, \ldots, A^{(k-1)}, A^{(k+2)}; \ldots]$$

As before,

$$\frac{\partial^2 \Delta}{\partial x^2} |_{x=y} = 0$$

For what minimum value of p is $\frac{\partial^p \Delta}{\partial x^p} \neq 0$ when $x = y$? Clearly the end position must be at least

$$[A^{(l+1)}, A^{(l+2)}, \ldots, A^{(l+m+1)};\ B, \ldots, B^{(l)}; \ldots].$$

This means the expression $(x-y)^p$, $p = (k+1)(l+1)$, must appear as a factor of Δ.

A similar situation must hold for the remaining variables. That is, $(x-z)$ appears as a factor at least $q = (k+1)(m+1)$ times. So x must appear as a power in Δ at least $(k+1)(l+1) + \cdots + (k+1)(m+1)[(l+1) + \cdots + (m+1)] = (k+1)(n-(k+1))$ times. Note that this is the highest power that x can appear in the expansion of Δ. For $x^{n-1} \cdot x^{n-3} \cdots x^{n-(2k+1)}$ is the product of the first $(k+1)$ terms appearing along the skew diagonal. The exponent on x is $(k+1)n-(k+1)^2=(k+1)(n-(k+1))$. Hence Δ divided by $(x-y)^p \ldots (x-z)^q$ is independent of x. A similar statement results from considering $(y-z)$ and the remaining differences in the variables. Hence the lemma follows.

Theorem 2.2 *Let $F: \mathbb{R}^n \to \mathbb{R}^n$ have as its coordinate functions the system of equations given in (3). Then, apart from a constant, $JF(x)$ consists entirely of factors of the form $a_k, b_l, \ldots c_m$, powers of the variables $x, y, \ldots, z$ and powers in all differences of expressions $x-y, x-z, \ldots, y-z$. Note that the value of $JF(x)$ is independent of all but the leading coefficients of the polynomials $p_1(t), p_2(t), \ldots, p_s(t)$.*

Proof: The matrix representation for F' is an $n \times n$ matrix where $n = (k+2)+(l+2)+\cdots+(m+2)$. The presentation below gives the first $k+2$ columns, it is those columns which arise from the summand $p_1(t)e^{\lambda_1 t} = (a_0 + a_1 t + \ldots + a_k t^k)x^t, x = e^{\lambda_1}$, which makes up $y(t)$.

The remaining columns are similar and can easily be filled in by the reader.

$$\begin{bmatrix} x & x & \ldots & x & p_1(1) \\ x^2 & 2x^2 & \ldots & 2^k x^2 & 2p_1(2)x \\ \vdots & \vdots & & \vdots & \vdots \\ x^j & jx^j & \ldots & j^k x^j & jp_1(j)x^{j-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x^n & nx^n & \ldots & n^k x^n & np_1(n)x^{n-1} \end{bmatrix}.$$

We evaluate this determinant by a sequence of column operations, the first of which is to factor x from columns 1 through $k+1$. Then multiply columns 2 through $k+1$ by $a_0, a_1, \ldots, a_{k-1}$, respectively. Add the result and subtract it from column $k+2$. There then results the block form

$$\begin{bmatrix} 1 & 1 & \dots & 1 & a_k \\ x & 2x & \dots & 2^k x & 2^{k+1} a_k x \\ \vdots & \vdots & & \vdots & \vdots \\ x^{j-1} & jx^{j-1} & \dots & j^k x^{j-1} & j^{k+1} a_k x^{j-1} \\ \vdots & \vdots & & \vdots & \vdots \\ x^{n-1} & nx^{n-1} & \dots & n^k x^{n-1} & n^{k+1} a_k x^{n-1} \end{bmatrix}.$$

Next factor a_k from column $k+2$ and by a sequence of successive column subtractions beginning with $k+1$ from $k+2$, k from $k+1, \dots, 1$ from 2 and subsequent factorings of x, the resulting form of the determinant appears as

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ x & 1 & \dots & 2^{k+1} - 2^k \\ \vdots & \vdots & & \vdots \\ x^{j-1} & (j-1)x^{j-2} & \dots & (j^{k+1} - j^k)x^{j-2} \\ \vdots & \vdots & & \vdots \\ x^{n-1} & (n-1)x^{n-2} & \dots & (n^{k+1} - n^k)x^{n-2} \end{bmatrix}.$$

By further successive column operations and subsequent factorizations of the variable x the block form appears as

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ x & 1 & \dots & 0 \\ x^2 & 2x & \dots & 2 \\ \vdots & \vdots & & \vdots \\ x^j & jx^{j-1} & \dots & j(j-1)x^{j-2} \\ \vdots & \vdots & & \vdots \\ x^{n-1} & (n-1)x^{n-2} & \dots & (n-1)(n-2)x^{n-3} \end{bmatrix}.$$

Treating the other variables similarly in kind, it is clear that we can reduce the Jacobian to the determinant of Δ of the previous lemma modulo powers of $x, y, \dots, z$ and the leading coefficients of the polynomials $p_1(t), p_2(t), \dots, p_s(t)$. The theorem follows.

References

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