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A PRODUCT THEOREM FOR $\mathfrak{F}_p$ CLASSES AND AN APPLICATION

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ABSTRACT. For $\operatorname{Re} p > 0$ let $\mathfrak{F}_p = \{f \mid f(z) = \int_{|x|=1}(1 - xz)^{-p}\, d\mu(x), |z| < 1, \mu$ a probability measure on $|x| = 1\}$ and let $\mathfrak{F}_p \cdot \mathfrak{F}_q = \{fg \mid f \in \mathfrak{F}_p, g \in \mathfrak{F}_q\}$. Brickman, Hallenbeck, MacGregor and Wilken proved a product theorem for the $\mathfrak{F}_p$ classes; they showed that if $p > 0$, $q > 0$, then $\mathfrak{F}_p \cdot \mathfrak{F}_q \subset \mathfrak{F}_{p+q}$. We give an (essentially complete) converse for the result of Brickman et al., i.e., we show that if $\mathfrak{F}_p \cdot \mathfrak{F}_q \subset \mathfrak{F}_{p+q}$, then $p > 0$, $q > 0$ or else $p = q = 1 + it$ for some t real. As an immediate consequence we disprove a conjecture about the extreme points of the closed convex hulls of the classes $\operatorname{Sp}(\gamma)$, $0 < |\gamma| < \pi/2$, of γ-spirallike univalent functions, i.e., writing $m = 1 + e^{-2i\gamma}$, we show $\{z/(1 - xz)^m \mid |x| = 1\} \subsetneq \mathcal{E}\mathcal{H}\operatorname{Sp}(\gamma)$, $0 < |\gamma| < \pi/2$.

Introduction. Let $\mathcal{A}$ be the class of functions analytic on the open unit disk $\mathfrak{D} = \{z \mid |z| < 1\}$. Then $\mathcal{A}$ is a locally convex linear topological space with the topology of uniform convergence on compacta of $\mathfrak{D}$. For $\mathfrak{B} \subset \mathcal{A}$ let $\mathcal{H}\mathfrak{B}$ denote the closed convex hull of $\mathfrak{B}$ and let $\mathcal{E}\mathcal{H}\mathfrak{B}$ denote the set of extreme points of $\mathcal{H}\mathfrak{B}$.

Let $\mathcal{S}$ be the usual subclass of $\mathcal{A}$ of univalent functions f normalized by $f(0) = 0$ and $f'(0) = 1$. Let St be the subclass of $\mathcal{S}$ of starlike functions (with respect to the origin) and let $\operatorname{Sp}(\gamma)$, $-\pi/2 < \gamma < \pi/2$, be the subclasses of $\mathcal{S}$ of γ-spirallike functions introduced by L. Špaček [6]. It is well known that the classes $\operatorname{Sp}(\gamma)$, $-\pi/2 < \gamma < \pi/2$, include the class St as a special case, namely $\operatorname{Sp}(0) = \mathrm{St}$.

For $\operatorname{Re} p > 0$ let $\mathfrak{F}_p$ be the subclasses of $\mathcal{A}$ of functions f given by

$$f(z) = \int_{|x|=1} \frac{1}{(1 - xz)^p}\, d\mu(x), \qquad z \in \mathfrak{D},$$

for some probability measure μ on $\{x \mid |x| = 1\}$. It is easily seen that each $\mathfrak{F}_p$ class is closed and convex. Further,

$$\mathcal{E}\mathfrak{F}_p = \left\{ \frac{1}{(1 - xz)^p} \,\Big|\, |x| = 1 \right\}.$$

For $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$ let the product $\mathfrak{F}_p \cdot \mathfrak{F}_q$ be given by

$$\mathfrak{F}_p \cdot \mathfrak{F}_q = \{ fg \mid f \in \mathfrak{F}_p, g \in \mathfrak{F}_q \}.$$

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In an early paper [2] tying extreme point theory and univalent function theory together, L. Brickman, T. H. MacGregor and D. R. Wilken identified the extreme points of the closed convex hull of St as

$$\mathcal{E}\mathcal{H}\mathrm{St} = \left\{ \frac{z}{(1-xz)^2} \,\middle|\, |x| = 1 \right\}. \tag{1}$$

Subsequent to their results T. H. MacGregor [4] suggested a conjecture for the extreme points of the closed convex hulls of the classes $\mathrm{Sp}(\gamma)$, $-\pi/2 < \gamma < \pi/2$; namely,

$$\mathcal{E}\mathcal{H}\mathrm{Sp}(\gamma) = \left\{ \frac{z}{(1-xz)^{1+\exp(-2i\gamma)}} \,\middle|\, |x| = 1 \right\}, \qquad -\frac{\pi}{2} < \gamma < \frac{\pi}{2}. \tag{2}$$

In a sequel [1] to the above paper of Brickman et al., a second proof of (1) was given which used the following product theorem.

THEOREM A. *Let $p > 0$, $q > 0$. Then $\mathcal{F}_p \cdot \mathcal{F}_q \subset \mathcal{F}_{p+q}$.*

Because the proof of Theorem A in [1] depended strongly on the hypothesis that $p > 0$, $q > 0$, an alternate proof was looked for which would allow this hypothesis to be weakened. For it was clear that if Theorem A could be sufficiently strengthened then the second proof of (1) could be adapted to prove the conjecture (2) suggested by MacGregor.

We will now show that no general extension of Theorem A holds; i.e., we have the following essentially complete converse of Theorem A.

THEOREM 1. *Let $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$. If $\mathcal{F}_p \cdot \mathcal{F}_q \subset \mathcal{F}_{p+q}$, then $p > 0$, $q > 0$ or $p = q = 1 + it$ for some $t \in R$.*

As an immediate consequence we have the following corollary which nullifies the conjecture (2) about the extreme points of the closed convex hulls of the classes $\mathrm{Sp}(\gamma)$, $0 < |\gamma| < \pi/2$.

COROLLARY 1. *Let $m = 1 + e^{-2i\gamma}$. Then*

$$\left\{ \frac{z}{(1-xz)^m} \,\middle|\, |x| = 1 \right\} \subsetneq \mathcal{E}\mathcal{H}\mathrm{Sp}(\gamma), \qquad 0 < |\gamma| < \frac{\pi}{2}.$$

Technical background. It is well known that a function $f \in \mathcal{A}$ is an element of $\mathrm{Sp}(\gamma)$, $-\pi/2 < \gamma < \pi/2$, if and only if,

$$f(z) = z \exp \int_{|x|=1} -(1 + e^{-2i\gamma}) \log(1 - xz)\, d\mu(x), \qquad z \in \mathcal{D}, \tag{3}$$

for some probability measure μ on $\{x \mid |x| = 1\}$.

We will use the following partial converse of Theorem A obtained by P. C. Cochrane [3] to simplify part of our proof.

THEOREM B. *Suppose $\mathcal{F}_p \cdot \mathcal{F}_q \subset \mathcal{F}_{p+q}$. Then $p/q > 0$.*

For completeness we include a proof of Theorem B: Suppose

$$\mathcal{F}_p \cdot \mathcal{F}_q \subset \mathcal{F}_{p+q}. \tag{4}$$

As observed in [2], a necessary (and sufficient) condition for (4) is

$$(1-xz)^{-p}(1-yz)^{-q} \in \mathcal{F}_{p+q}, \qquad |x|=|y|=1,$$

i.e.,

$$(5) \qquad (1-xz)^{-p}(1-yz)^{-q} = \int_{|u|=1} \frac{1}{(1-uz)^{p+q}} d\mu(u), \qquad |x|=|y|=1,$$

for some probability measure μ on $\{u \mid |u|=1\}$. Let μ_1 denote the first moment of μ. If we expand both sides of (5) and then equate the first coefficients, we obtain

$$(6) \qquad px+qy=(p+q)\mu_1, \qquad |x|=|y|=1.$$

Together (6) and the moment condition $|\mu_1| \leq 1$ imply that p and q must be collinear, i.e., $p/q>0$.

We will also use several results from hypergeometric function theory, which we recall now for reference. We refer the reader to L. J. Slater [5].

The hypergeometric function $F(a,b;c;\cdot)$, $c \neq 0,-1,-2,\ldots$, has the power series expansion about the origin

$$(7) \qquad F(a,b;c;z)=\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} z^n$$

where

$$(d)_n = \begin{cases} 1, & n=0, \\ d(d+1)\cdots(d+n-1), & n>0. \end{cases}$$

It is easily seen that the series in (7) converges absolutely and locally uniformly on $\mathcal{D}$.

If the parameters b and c satisfy the inequality $\operatorname{Re} c > \operatorname{Re} b > 0$, then $F(a,b;c;\cdot)$ has an integral representation

$$(8) \qquad F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.$$

It can be shown that (8) extends $F(a,b;c;\cdot)$ analytically from $\mathcal{D}$ to $\mathcal{C} \setminus [1,\infty)$.

If the parameters a, b, and c are restricted so that $\operatorname{Re} c > \operatorname{Re}(a+b)$, it can be shown that $F(a,b;c;z)$ converges as $z \to 1$, $\operatorname{Re} z<1$. In fact, in a thesis presented to the Royal Society at Göttingen in 1812 Gauss showed that

$$(9) \qquad \lim_{\substack{z \to 1 \\ \operatorname{Re} z<1}} F(a,b;c;z)=F(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

for $\operatorname{Re} c > \operatorname{Re}(a+b)$. We note in (9) that $F(a,b;c;1) \neq 0$.

We recall Euler's identity which states

$$(10) \qquad F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z)$$

for $c \neq 0,-1,-2,\ldots$.

Finally, we note the simple identity $F(1,m;1;z)=(1-z)^{-m}$, $z \in \mathcal{D}$.

We will also need the following lemma, which follows by analytic continuation from a lemma used in [1] to prove Theorem A.

LEMMA 1. *Let* $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$. *Then for* $|x| = |y| = 1$

$$(1 - xz)^{-p}(1 - yz)^{-q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 t^{p-1}(1-t)^{q-1}[1 - (tx + (1-t)y)z]^{-(p+q)}\, dt, \qquad z \in \mathfrak{D}.$$

Main results.

THEOREM 1. *Let* $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$. *If* $\mathfrak{F}_p \cdot \mathfrak{F}_q \subset \mathfrak{F}_{p+q}$, *then* $p > 0$, $q > 0$ *or* $p = q = 1 + it$ *for some* $t \in R$.

PROOF. We will break the proof into two parts. We will first show that if $\operatorname{Re} p \neq 1$ and $\operatorname{Re} q \neq 1$, then $\mathfrak{F}_p \cdot \mathfrak{F}_q \subset \mathfrak{F}_{p+q}$ implies that $p > 0$, $q > 0$. In the second part we drop the restrictions $\operatorname{Re} p \neq 1$ and $\operatorname{Re} q \neq 1$. We will there show that together the first part and Theorem B imply that if $\mathfrak{F}_p \cdot \mathfrak{F}_q \subset \mathfrak{F}_{p+q}$, then $p > 0$, $q > 0$ or $p = q = 1 + it$ for some $t \in R$.

Part I. $\operatorname{Re} p \neq 1$, $\operatorname{Re} q \neq 1$.

We may assume $\operatorname{Re} p \leqslant \operatorname{Re} q$. Suppose

$$\mathfrak{F}_p \cdot \mathfrak{F}_q \subset \mathfrak{F}_{p+q}. \tag{11}$$

As observed in [2], a necessary (and sufficient) condition for (11) is

$$(1 - xz)^{-p}(1 - yz)^{-q} \in \mathfrak{F}_{p+q}, \qquad |x| = |y| = 1,$$

i.e.,

$$(1 - xz)^{-p}(1 - yz)^{-q} = \int_{|u|=1} \frac{1}{(1 - uz)^{p+q}}\, d\mu(u), \qquad |x| = |y| = 1, \tag{12}$$

for some probability measure μ on $\{u \mid |u| = 1\}$. If we use Lemma 1 to represent $(1 - xz)^{-p}(1 - yz)^{-q}$ and then convolve (Hadamard convolution) both sides of (12) with the function $F(1, 1; p + q; \cdot)$, we obtain

$$\frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 t^{p-1}(1-t)^{q-1}[1 - (tx + (1-t)y)z]^{-1}\, dt = \int_{|u|=1} \frac{1}{1 - uz}\, d\mu(u), \qquad z \in \mathfrak{D},$$

which can be alternately expressed as, using (8),

$$\frac{1}{1 - yz} F\left(1, p; p + q; \frac{(x - y)z}{1 - yz}\right) = \int_{|u|=1} \frac{1}{1 - uz}\, d\mu(u), \qquad z \in \mathfrak{D}. \tag{13}$$

Since the roles of $(1 - xz)^{-p}$ and $(1 - yz)^{-q}$ as first factor or second factor in (12) may be interchanged, it also follows that

$$\frac{1}{1 - xz} F\left(1, q; p + q; \frac{(y - x)z}{1 - xz}\right) = \int_{|u|=1} \frac{1}{1 - uz}\, d\mu(u), \qquad z \in \mathfrak{D}. \tag{14}$$

From (13) and (14) we can deduce

$$\operatorname{Re} \frac{1}{1 - yz} F\left(1, p; p + q; \frac{(x - y)z}{1 - yz}\right) > \frac{1}{2}, \qquad z \in \mathfrak{D}, \tag{15}$$

$$\text{(16)} \qquad \operatorname{Re} \frac{1}{1-xz} F\left(1, q; p+q; \frac{(y-x)z}{1-xz}\right) > \frac{1}{2}, \qquad z \in \mathfrak{D}.$$

The proof of Part I now breaks into three cases reflecting the restriction $\operatorname{Re} p \neq 1$, $\operatorname{Re} q \neq 1$.

Case (i). $1 < \operatorname{Re} p \leqslant \operatorname{Re} q$.

Let $x \neq y$. Since $\operatorname{Re}(p+q) > \operatorname{Re}(p+1)$, the hypergeometric function $F(1, p; p+q; w)$ converges at $w = 1$. Let $z \to \bar{x}$ radially in (15); then

$$(x-y)z/(1-yz) \to 1$$

and we conclude that

$$\text{(17)} \qquad \operatorname{Re} \frac{1}{1-y\bar{x}} F(1, p; p+q; 1) \geqslant \frac{1}{2}, \qquad x \neq y.$$

Similarly, from (16) we can obtain

$$\text{(18)} \qquad \operatorname{Re} \frac{1}{1-x\bar{y}} F(1, q; p+q; 1) \geqslant \frac{1}{2}, \qquad x \neq y.$$

Since (17) and (18) hold for all $x \neq y$, it follows that $F(1, p; p+q; 1)$ and $F(1, q; p+q; 1)$ are real and, in fact, that

$$\text{(19)} \qquad F(1, p; p+q; 1) \geqslant 1, \qquad F(1, q; p+q; 1) \geqslant 1.$$

Gauss's theorem (9) implies

$$\text{(20)} \quad F(1, p; p+q; 1) = \frac{\Gamma(p+q)\Gamma(q-1)}{\Gamma(p+q-1)\Gamma(q)} = \frac{p+q-1}{q-1} = 1 + \frac{p}{q-1}.$$

Similarly

$$\text{(21)} \qquad F(1, q; p+q; 1) = 1 + q/(p-1).$$

Together (19), (20) and (21) imply that $p/(q-1)$ and $q/(p-1)$ are real (and positive) and then that $p > 0$, $q > 0$.

Case (ii). $0 < \operatorname{Re} p \leqslant \operatorname{Re} q < 1$.

Let $x \neq y$. Since $\operatorname{Re}(p+q) < \operatorname{Re}(q+1)$, the hypergeometric function $F(1, q; p+q; w)$ diverges at $w = 1$. Indeed, applying Euler's identity (10) to $F(1, q; p+q; w)$ yields

$$\text{(22)} \qquad F(1, q; p+q; w) = (1-w)^{p-1} F(p+q-1, p; p+q; w).$$

Gauss's theorem (9) implies $F(p+q-1, p; p+q; w)$ converges at $w = 1$ and $F(p+q-1, p; p+q; 1) \neq 0$. Thus, (22) implies $F(1, q; p+q; w)$ diverges as $w \to 1$, $\operatorname{Re} w < 1$. In fact, if we let $F(p+q-1, p; p+q; 1) = A \neq 0$, then

$$F(1, q; p+q; w) \sim A/(1-w)^{1-p}$$

for w sufficiently close to 1, $|w| < 1$. Thus,

$$\text{(23)} \qquad \begin{aligned} &\operatorname{Re} \frac{1}{1-xz} F\left(1, q; p+q; \frac{(y-x)z}{1-xz}\right) \\ &\quad \sim \operatorname{Re} \frac{1}{1-xz} \frac{A}{[1-(y-x)z/(1-xz)]^{1-p}} \end{aligned}$$

for z sufficiently close to $\bar{y}$, $|z|<1$. If we choose $x=-1$ and $y=1$, then the right-hand side of (23) becomes

$$\text{(24)} \qquad \operatorname{Re} \frac{A}{1+z}\left(\frac{1+z}{1-z}\right)^{1-p}.$$

If, now, we let $z \to 1$ radially, then (23) and (24) show that (16) will be contradicted unless $p > 0$.

Similarly, we can show that (15) will fail to hold unless $q > 0$.

Case (iii). $0 < \operatorname{Re} p < 1 < \operatorname{Re} q$.

Let $x \neq y$. Since $\operatorname{Re}(p+q) > \operatorname{Re}(p+1)$, the hypergeometric function $F(1, p; p+q; w)$ converges at $w=1$. We can conclude from (15), as in Case (i), that $p/(q-1) > 0$. On the other hand, since $\operatorname{Re}(p+q) < \operatorname{Re}(q+1)$, the hypergeometric function $F(1, q; p+q; w)$ diverges at $w=1$. Thus, we can conclude from (16), as in Case (ii), that $p > 0$. It follows then that $q > 0$ also.

Part II. $\operatorname{Re} p > 0$, $\operatorname{Re} q > 0$.

We may assume $\operatorname{Re} p \leqslant \operatorname{Re} q$. Suppose $\mathfrak{F}_p \cdot \mathfrak{F}_q \subset \mathfrak{F}_{p+q}$. Part I implies we only need to consider three cases.

Case (i). $\operatorname{Re} p = 1$, $\operatorname{Re} q > 1$.

As in Case (i) of Part I, we can conclude $p/(q-1) > 0$. Theorem B implies $p/q > 0$. It follows that $p > 0$, $q > 0$.

Case (ii). $0 < \operatorname{Re} p < 1$, $\operatorname{Re} q = 1$.

As in Case (ii) of Part I, we can conclude $p > 0$. Again, Theorem B implies $p/q > 0$. It follows then $q > 0$.

Case (iii). $\operatorname{Re} p = \operatorname{Re} q = 1$.

Theorem B implies $p/q > 0$. Since $\operatorname{Re} p = \operatorname{Re} q = 1$, we must have $p = q = 1 + it$ for some $t \in R$.

COROLLARY 1. *Let* $m = 1 + e^{-2i\gamma}$. *Then,*

$$\text{(25)} \qquad \left\{\frac{z}{(1-xz)^m} \,\Big|\, |x| = 1\right\} \underset{\neq}{\subset} \mathcal{E}\mathcal{H}\operatorname{Sp}(\gamma), \qquad 0 < |\gamma| < \frac{\pi}{2}.$$

PROOF. It is easily seen from (3), the representation formula for $\operatorname{Sp}(\gamma)$, $-\pi/2 < \gamma < \pi/2$, that each function $f_x(z) = z/(1-xz)^m$, $|x| = 1$, uniquely maximizes the functional $\operatorname{Re} J_x$ over $\operatorname{Sp}(\gamma)$, where $J_x g = \overline{mx}\, g''(0)$, $|x| = 1$. Hence, each f_x, $|x| = 1$, is necessarily an extreme point of $\mathcal{H}\operatorname{Sp}(\gamma)$, $-\pi/2 < \gamma < \pi/2$, and inclusion holds in (25).

Theorem 1 implies for $0 < |\gamma| < \pi/2$ and $0 < t < 1$ that $\mathfrak{F}_{tm} \cdot \mathfrak{F}_{(1-t)m} \not\subset \mathfrak{F}_m$. In particular, the proof of Theorem 1 shows that for $0 < |\gamma| < \pi/2$ and $0 < t < 1$ there exist $|x| = |y| = 1$, $x \neq y$, such that

$$(1-xz)^{-tm}(1-yz)^{-(1-t)m} \notin \mathfrak{F}_m$$

which implies

$$\frac{z}{(1-xz)^{tm}(1-yz)^{(1-t)m}} \notin z\mathfrak{F}_m = \mathcal{H}\left\{\frac{z}{(1-uz)^m} \,\Big|\, |u| = 1\right\}.$$

It is easily seen, however, from the representation formula (3) for Sp(γ), that for $-\pi/2 < \gamma < \pi/2$ and $0 < t < 1$ and for all x and y, $|x| = |y| = 1$, that

$$\frac{z}{(1-xz)^{tm}(1-yz)^{(1-t)m}} \in \mathrm{Sp}(\gamma).$$

Thus, the inclusion (25) must be proper whenever $0 < |\gamma| < \pi/2$.

REMARKS. (1) Recently D. Moak at Texas Tech University has shown that if $t \in R$ and $\mathcal{F}_{1+it} \cdot \mathcal{F}_{1+it} \subset \mathcal{F}_{2+it}$, then $t = 0$. Thus, a full converse of Theorem A has been shown to hold.

(2) We have as yet not been able to exhibit any extreme points of $\mathcal{H}\mathrm{Sp}(\gamma)$, $0 < |\gamma| < \pi/2$, other than the functions $f_x(z) = z/(1-xz)^{1+e^{-2i\gamma}}$, $|x| = 1$. Variational methods suggest, of course, some natural candidates.

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