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NEW SUPPORT POINTS OF $\mathcal{S}$ AND EXTREME POINTS OF $\mathcal{HS}$

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ABSTRACT. Let $\mathcal{S}$ be the usual class of univalent analytic functions f on $\{z||z| < 1\}$ normalized by $f(z) = z + a_2z^2 + \cdots$. We prove that the functions

$$f_{x,y}(z) = \frac{z - \frac{1}{2}(x+y)z^2}{(1-yz)^2}, \qquad |x| = |y| = 1, x \neq y,$$

which are support points of $\mathcal{C}$, the subclass of $\mathcal{S}$ of close-to-convex functions, and extreme points of $\mathcal{HC}$, are support points of $\mathcal{S}$ and extreme points of $\mathcal{HS}$ whenever $0 < |\arg(-x/y)| < \pi/4$. We observe that the known bound of $\pi/4$ for the acute angle between the omitted arc of a support point of $\mathcal{S}$ and the radius vector is achieved by the functions $f_{x,y}$ with $|\arg(-x/y)| = \pi/4$.

Introduction. Let $\mathcal{A}$ be the set of analytic functions on the open unit disk. With the usual topology of uniform convergence on compacta $\mathcal{A}$ is a locally convex linear topological space. Suppose $\mathcal{B} \subset \mathcal{A}$. A function b in $\mathcal{B}$ is called a support point of $\mathcal{B}$ if b maximizes Re J over $\mathcal{B}$ for some continuous linear functional J on $\mathcal{A}$ such that Re J is not constant on $\mathcal{B}$. Let $\mathcal{HB}$ denote the closed convex hull of $\mathcal{B}$. A function b in $\mathcal{HB}$ is called an extreme point of $\mathcal{HB}$ if $b = tb_1 + (1-t)b_2$ implies $b = b_1 = b_2$ whenever $0 < t < 1$ and $b_1, b_2 \in \mathcal{HB}$.

Let $\mathcal{S}$ be the usual class of univalent functions f in $\mathcal{A}$ normalized by $f(z) = z + a_2z^2 + \cdots$. A. Pfluger [10] and L. Brickman and D. R. Wilken [3] have shown that if f is a support point of $\mathcal{S}$, then f maps the open unit disk to the complement of an analytic arc Γ, which tends to ∞ with increasing modulus. Furthermore, Γ satisfies the $\pi/4$-property, i.e., if Γ is oriented so that Γ is (positively) traversed from the finite tip to ∞, then the angle between the oriented tangent vector to Γ and the radius vector to Γ at any point is less than or equal to $\pi/4$, with strict inequality at each point of Γ except possibly at the finite tip.

In an early paper [1] L. Brickman proved that if f in $\mathcal{S}$ is an extreme point of $\mathcal{HS}$, then f maps the open unit disk to the complement of an arc which tends to ∞ with increasing modulus. Later W. E. Kirwan and R. W. Pell [9] improved Brickman's result. A special case of their result states that if f in $\mathcal{S}$ is an extreme point of $\mathcal{HS}$ and if the omitted arc of f is smooth, then the omitted arc of f satisfies the $\pi/4$-property, albeit, not necessarily with strict inequality.

Since $\mathcal{S}$ and $\mathcal{HS}$ are compact a lemma in Dunford and Schwartz [5, p. 440] implies that if f is an extreme point of $\mathcal{HS}$, then $f \in \mathcal{S}$. The following lemma shows that in certain cases we can identify support points of $\mathcal{S}$ as extreme points of $\mathcal{HS}$.

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LEMMA. *Let J be a continous linear functional on $\mathcal{A}$ such that* $\operatorname{Re} J$ *is nonconstant on* $\mathbb{S}$. *If there exist at most two support points of* $\mathbb{S}$ *which maximize* $\operatorname{Re} J$ *over* $\mathbb{S}$, *then each such support point of* $\mathbb{S}$ *is an extreme point of* $\mathcal{H}\mathbb{S}$.

It is well known that the Koebe functions $k_x(z) = z/(1 - xz)^2$, $|x| = 1$, uniquely maximize $\operatorname{Re} J_x$ over $\mathbb{S}$, where $J_x g = \bar{x}g''(0)$, $|x| = 1$. Thus, the Koebe functions k_x, $|x| = 1$, are both support points of $\mathbb{S}$ and extreme points of $\mathcal{H}\mathbb{S}$. Until recently, no other support points of $\mathbb{S}$ or extreme points of $\mathcal{H}\mathbb{S}$ were explicitly known. However, J. Brown [4] has determined the support points of $\mathbb{S}$ which maximize $\operatorname{Re} J$ over $\mathbb{S}$, where $Jg = g(z_0)$, $0 < |z_0| < 1$, and that each such support point of $\mathbb{S}$ is an extreme point of $\mathcal{H}\mathbb{S}$.

The class $\mathcal{C}$. Let $\mathcal{C}$ be the subclass of $\mathbb{S}$ of close-to-convex functions. In [2] L. Brickman, T. H. MacGregor, and D. R. Wilken showed that the extreme points of $\mathcal{H}\mathcal{C}$ are the functions

$$f_{x,y}(z) = \frac{z - \frac{1}{2}(x + y)z^2}{(1 - yz)^2}, \qquad |x| = |y| = 1, x \neq y. \tag{1}$$

Later E. Grassman, W. Hengartner, and G. Schober [7] proved that each support point of $\mathcal{C}$ is a function of the form (1). In [8] D. R. Wilken and R. Hornblower showed that each extreme point of $\mathcal{H}\mathcal{C}$ is a support point of $\mathcal{C}$.

A natural question arises as to whether the functions (1) are support points of $\mathbb{S}$ or extreme points of $\mathcal{H}\mathbb{S}$. Each function $f_{x,y}$ in (1) maps the open unit disk to the complement of a half-line. Let $\Gamma_{x,y}$, the omitted half-line of $f_{x,y}$, be oriented so that $\Gamma_{x,y}$ is traversed from $P_{x,y}$, the finite tip of $\Gamma_{x,y}$, to ∞. A computation shows that $|\arg(-x/y)|$ is the angle between the tangent vector to $\Gamma_{x,y}$ and the radius vector to $\Gamma_{x,y}$ at $P_{x,y}$. It is easily seen that the angle between the tangent vector to $\Gamma_{x,y}$ and the radius vector to $\Gamma_{x,y}$ decreases (monotonically) to 0 as $\Gamma_{x,y}$ is traversed (monotonically) from $P_{x,y}$ to ∞. Thus, if $\pi/4 < |\arg(-x/y)| < \pi$, then $f_{x,y}$ can be neither a support point of $\mathbb{S}$ nor an extreme point of $\mathcal{H}\mathbb{S}$ (because $\Gamma_{x,y}$ fails to satisfy the $\pi/4$-property). If $|\arg(-x/y)| = 0$, i.e., if $-x = y$, then $f_{x,y}$ is the Koebe function k_y and is both a support point of $\mathbb{S}$ and an extreme point of $\mathcal{H}\mathbb{S}$. In the remaining case–$0 < |\arg(-x/y)| \leq \pi/4$–$\Gamma_{x,y}$ does not violate the $\pi/4$-property. We will show for $0 < |\arg(-x/y)| \leq \pi/4$ that $f_{x,y}$ is both a support point of $\mathbb{S}$ and an extreme point of $\mathcal{H}\mathbb{S}$.

To prove the main result of this paper, we recall the bound on $|\arg f'(z_0)|$ for f in $\mathbb{S}$ given by G. M. Goluzin [6, p. 115]. Namely, Goluzin showed that if $f \in \mathbb{S}$, then

$$|\arg f'(z_0)| \leq 4 \arcsin|z_0|, \qquad |z_0| \leq \frac{1}{\sqrt{2}}. \tag{2}$$

We now prove

THEOREM. *Let $f_{x,y}$ be given by* (1). *If* $0 < |\arg(-x/y)| \leq \pi/4$, *then $f_{x,y}$ is both a support point of* $\mathbb{S}$ *and an extreme point of* $\mathcal{H}\mathbb{S}$.

PROOF. If we differentiate $f_{x,y}$ and then evaluate at z_0, we have

$$f'_{x,y}(z_0) = \frac{1 - xz_0}{(1 - yz_0)^3}.$$

An easy argument shows for $0 < |z_0| < 1$ that

$$|\arg f'_{x,y}(z_0)| \leqslant 4 \arcsin|z_0| \tag{3}$$

and that equality occurs in (3) if and only if

$$\arg xz_0 = -\arccos|z_0|, \quad \arg yz_0 = \arccos|z_0| \tag{4}$$

or

$$\arg xz_0 = \arccos|z_0|, \quad \arg yz_0 = -\arccos|z_0|. \tag{5}$$

If (4) holds, then $\arg f'_{x,y}(z_0) = 4 \arcsin|z_0|$ and if (5) holds, then $\arg f'_{x,y}(z_0) = -4 \arcsin|z_0|$. We note that for each pair $\{x, y\}$, $|x| = |y| = 1$, $x^2 \neq y^2$, there exists a unique z_0, $0 < |z_0| < 1$, such that exactly one of (4) or (5) holds.

Let $0 < |\arg(-x/y)| \leqslant \pi/4$ and suppose z_0 satisfies (4). Then (4) implies $0 < |z_0| \leqslant \sin \pi/8$. Goluzin's bound (2) on $|\arg f'(z_0)|$ implies that the region of variability of $f'(z_0)$ for f in $\mathcal{S}$ lies in a closed sector in the closed right half-plane. Together (2)–(4) imply that $f'_{x,y}(z_0)$ lies on the upper edge of the region of variability of $f'(z_0)$ for f in $\mathcal{S}$. By rotating the region of variability of $f'(z_0)$ for f in $\mathcal{S}$ we can realize a continuous linear functional $J_{x,y}$ whose real part is maximized over $\mathcal{S}$ by $f_{x,y}$; namely

$$J_{x,y}g = -e^{i(\pi/2 - 4 \arcsin|z_0|)}g'(z_0).$$

Similarly, if $0 < |\arg(-x/y)| \leqslant \pi/4$ and z_0 satisfies (5), then $f_{x,y}$ maximizes $\operatorname{Re} J_{x,y}$ over $\mathcal{S}$ where

$$J_{x,y}g = -e^{-i(\pi/2 - 4 \arcsin|z_0|)}g'(z_0).$$

We will show now that if $0 < |\arg(-x/y)| < \pi/4$, then $\operatorname{Re} J_{x,y}$ is uniquely maximized over $\mathcal{S}$ by $f_{x,y}$, and if $|\arg(-x/y)| = \pi/4$, then $\operatorname{Re} J_{x,y}$ is maximized over $\mathcal{S}$ (only) by $f_{x,y}$ and $f_{y,x}$. The lemma will then imply if $0 < |\arg(-x/y)| \leqslant \pi/4$, then $f_{x,y}$ is an extreme point of $\mathcal{HS}$.

As in the first part, we can see that if $0 < |z_0| \leqslant \sin \pi/8$ and f^* in $\mathcal{S}$ maximizes (minimizes) $\arg f'(z_0)$ over $\mathcal{S}$, then f^* is a support point of $\mathcal{S}$ and, hence, in particular, a slit mapping. Goluzin's argument [**6**, p. 115], which shows that (2) is sharp, also shows that for $0 < |z_0| \leqslant 1/\sqrt{2}$ there exists a unique slit mapping which maximizes (minimizes) $\arg f'(z_0)$ over $\mathcal{S}$.

Let $0 < |\arg(-x/y)| < \pi/4$ and let z_0 satisfy (4). Since determining the functions which maximize $\operatorname{Re} J_{x,y}$ over $\mathcal{S}$ is equivalent to determining the functions which maximize $\arg f'(z_0)$ over $\mathcal{S}$, we conclude from the above that $\operatorname{Re} J_{x,y}$ is uniquely maximized over $\mathcal{S}$ by $f_{x,y}$. Similarly, if $0 < |\arg(-x/y)| < \pi/4$ and z_0 satisfies (5), then $\operatorname{Re} J_{x,y}$ is uniquely maximized over $\mathcal{S}$ by $f_{x,y}$.

Let $|\arg(-x/y)| = \pi/4$ and let z_0 satisfy (4) or (5). It is easily seen, from (2)–(5) that one of $f_{x,y}$ and $f_{y,x}$ maximizes $\arg f'(z_0)$ over $\mathcal{S}$ and the other minimizes $\arg f'(z_0)$ over $\mathcal{S}$. Since, in this case, we have $|z_0| = \sin \pi/8$, it follows that

$J_{x,y} = J_{y,x}$. Thus, determining the functions which maximize $\operatorname{Re} J_{x,y}$ over $\mathcal{S}$ is equivalent to determining the functions which maximize or minimize $\arg f'(z_0)$ over $\mathcal{S}$. Consequently, $\operatorname{Re} J_{x,y}$ is maximized over $\mathcal{S}$ (only) by $f_{x,y}$ and $f_{y,x}$.

REMARK. For the functions $f_{x,y}$ with $|\arg(-x/y)| = \pi/4$, the known bound of $\pi/4$ for the acute angle between the omitted arc of a support point of $\mathcal{S}$ and the radius vector is achieved (at the finite tip).

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