On a Coefficient Conjecture of Brannan

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Abstract

In 1972, D.A. Brannan conjectured that all of the odd coefficients, a_{2n+1}, of the power series $(1+xz)^\alpha/(1-z)$ were dominated by those of the series $(1+z)^\alpha/(1-z)$ for the parameter range $0 < \alpha < 1$, after having shown that this was not true for the even coefficients. He verified the case when $2n+1 = 3$. The case when $2n+1 = 5$ was verified in the mid-eighties by J.G. Milcetich. In this paper, we verify the case when $2n+1 = 7$ using classical Sturm sequence arguments and some computer algebra.

1980 AMS Subject Classification: 30C50

Keywords: coefficient conjecture, computer algebra

Introduction.

For $k \geq 2$ let V_k denote the class of locally univalent analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \tag{1}$$

which map $|z| < 1$ conformally onto a domain whose boundary rotation is at most $k\pi$. (See [Pa] for the definition and basic properties of the class V_k.)

The function

$$f_k(z) = \frac{1}{k}\left[\left(\frac{1+z}{1-z}\right)^{\frac{k}{2}} - 1\right] = \sum_{n=1}^{\infty} A_n z^n$$

belongs to V_k. The coefficient conjecture for the class V_k was that for a function (1) in V_k that

$$|a_n| \leq A_n, \quad (n \geq 1). \tag{2}$$

This conjecture was verified for $n = 2$ by Pick (see [Le]), for $n = 3$ by Lehto [Le] in 1952 and for $n = 4$ by Schiffer and Tammi [ScTa] in 1967, Lonka and Tammi [LoTa] in 1968 and Brannan [Br1] in 1969.

Using extreme point theory arguments, Brannan, Clunie and Kirwan [BrClKi] showed in 1973 that (2) can be reduced to showing that for

$$\Phi(\alpha, x; z) = \left(\frac{1+xz}{1-z}\right)^{\alpha} = \sum_{n=1}^{\infty} B_n(\alpha, x) z^n$$

that

$$|B_n(\alpha, x)| \leq B_n(\alpha, 1), \quad (n \geq 1) \tag{3}$$

for $\alpha \geq 1$, $|x| = 1$. Brannan, Clunie and Kirwan showed that (3) holds for $1 \leq n \leq 13$, which implies (2) for $2 \leq n \leq 14$.

In 1972 Aharonov and Friedland [AhFr] considered a related coefficient inequality. Let

$$\Psi(\alpha, x; z) = \frac{(1+xz)^{\alpha}}{1-z} = \sum_{n=1}^{\infty} A_n(\alpha, x) z^n.$$

In [AhFr] it was shown, by a long technical argument, that

$$|A_n(\alpha, x)| \leq A_n(\alpha, 1), \quad (n \geq 1) \tag{4}$$

for $\alpha \geq 1, |x| = 1$, which implies (3) and, hence, by the work in [BrClKi], also implies (2). Later, in 1973 Brannan [Br2] gave a short, elegant proof that (4) holds for $\alpha \geq 1$, $|x| = 1$.

In [Br2] Brannan also considered the question about whether (4) holds for $0 < \alpha < 1$, $|x| = 1$. He showed there the unexpected result that for each α, $0 < \alpha < 1$, there exists an n_α such that

$$\max_{|x|=1} Re\ A_{2n}(\alpha, x) > A_{2n}(\alpha, 1) \tag{5}$$

for $n > n_\alpha$, that is, that (4) fails for even coefficients when $0 < \alpha < 1$.

Brannan showed, using an inequality for quadratic trigonometric polynomials, that

$$|A_3(\alpha, x)| \leq A_3(\alpha, 1)$$

for $0 < \alpha < 1$ and he conjectured, based on numerical data, that

Brannan's Conjecture

$$|A_{2n+1}(\alpha, x)| \leq A_{2n+1}(\alpha, 1), \quad (n \geq 1) \tag{6}$$

for $0 < \alpha < 1$, $|x| = 1$.

Brannan's conjecture has been verified for $n = 2$, that is, for $2n + 1 = 5$, by Milcetich [Mi], who employed a lengthy argument based on a result of Brown and Hewitt [BrHe] for positive trigonometric sums.

In this paper, we will establish Brannan's conjecture for $n = 3$, that is, for $2n + 1 = 7$. The method we will employ is based largely on (i) a judicious rearrangement of the coefficients $A_n(\alpha, x)$ over carefully chosen subintervals of (0,1), the domain of α, (ii) an application of Sturm sequences to verify the nonnegativity of those rearrangements and (iii) using a computer algebra program (in this case Maple) to generate the coefficients $A_n(\alpha, x)$ and the Sturm sequences.

Section 1.

Brannan's coefficient inequality (6) is equivalent to

$$A_{2n+1}^2(\alpha, 1) - |A_{2n+1}(\alpha, x)|^2 \geq 0 \tag{7}$$

for $0 < \alpha < 1$, $|x| = 1$. We will let $F_{2n+1}(\alpha, x)$ denote the left-hand side of (7) and we will show for $2n+1 = 7$ that $F_{2n+1}(\alpha, x) \geq 0$.

We note that

$$\begin{aligned} \frac{(1+xz)^\alpha}{(1-z)} &= \sum_{n=0}^{\infty} \frac{(-\alpha)_n (-1)^n x^n}{n!} z^n \sum_{n=0}^{\infty} z^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-\alpha)_k (-1)^k x^k}{k!} z^n \\ &= \sum_{n=0}^{\infty} A_n(\alpha, x) z^n, \end{aligned}$$

where $(a)_k$ denotes the Pockhammer symbol, which is defined as

$$(a)_k = \begin{cases} 1 & k = 0 \\ a(a+1)\cdots(a+k-1) & k > 0 \end{cases}.$$

Hence, we can write $F_N(\alpha, x)$ as

$$\begin{aligned} F_N(\alpha, x) &= \sum_{k=0}^{N} \frac{(-\alpha)_k (-1)^k}{k!} \sum_{k=0}^{N} \frac{(-\alpha)_k (-1)^k}{k!} - \\ & \sum_{k=0}^{N} \frac{(-\alpha)_k (-1)^k x^k}{k!} \sum_{k=0}^{N} \frac{(-\alpha)_k (-1)^k \bar{x}^k}{k!} \end{aligned}$$

$$
\begin{aligned}
= & \sum_{k=0}^{2*N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}}{k!(k-j)!} - \\
& \sum_{k=0}^{2*N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}x^{2*j-k}}{k!(k-j)!}
\end{aligned}
$$

where

$$
\delta_j = \begin{cases} 1 & 0 \le j \le N \\ 0 & N+1 \le j \le 2*N \end{cases}
$$

Since $F_N(\alpha, x)$ is real, we can write, setting $x = e^{i\theta}$,

$$
\begin{aligned}
F_N(\alpha, x) \;=\; & \sum_{k=0}^{2*N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}}{k!(k-j)!} - \\
& \sum_{k=0}^{2*N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}\cos((2*j-k)\theta)}{k!(k-j)!} \qquad (8)
\end{aligned}
$$

The following Maple Procedure can be used to generate the coefficients of $F_N(\alpha, x)$, where $x = e^{i\theta}$,

Procedure 1

```
F:=proc(N)
local i, j, a, csum, dsum, temp;
global c;
    a[0]:=1;
    for i from 1 to N do a[i]:=a[i-1]*(-alpha+i-1)*(-1)/i od;
    for i from N+1 to 2*N do a[i]:=0 od;
    csum:=0; dsum:=0;
    for i from 0 to N do csum:= csum+a[i] od;
```

```
    for i from 0 to 2*N do
    for j from 0 to i do
        dsum:= dsum + a[j]*a[i-j]* cos ((2*j-i)*theta);
    od;
    od;
    temp:= collect(csum*csum-dsum,alpha);
    for i from 0 to (2*N-1) do c[i]:= coeff(temp,alpha,i) od;
    temp;
end;
```

Using Procedure 1, we obtain for $N = 7$ that $F_7(\alpha, x) = \sum_{k=1}^{13} c_k(\theta)\alpha^k$ where each $c_k(\theta)$ is a trigonometric polynomial of the form $c_k(\theta) = \sum_{j=0}^{7} a_{kj} \cos(j\theta)$ with rational coefficients a_{kj}. We will show that $F_7(\alpha, x) \geq 0$ for $0 < \alpha < 1$ by subdividing the domain of α into subintervals $0 < \alpha \leq t_0$ and $t_0 < \alpha < 1$, where $t_0 = 2/5$. We will show that $F_7(\alpha, x) \geq 0$ on each subinterval.

First for the case $0 < \alpha \leq t_0$ we will show the following:

$$c_1(\theta) \geq 0, \quad \frac{7}{10}c_1(\theta) + c_2(\theta)t_0 \geq 0, \tag{9.1}$$

$$\frac{1}{10}c_1(\theta) + c_3(\theta)t_0^2 \geq 0, \quad \frac{1}{10}c_1(\theta) + c_3(\theta)t_0^2 + c_4(\theta)t_0^3 \geq 0, \tag{9.2}$$

$$\frac{1}{5}c_1(\theta) + c_5(\theta)t_0^4 \geq 0, \quad \frac{1}{5}c_1(\theta) + c_5(\theta)t_0^4 + c_6(\theta)t_0^5 \geq 0, \tag{9.3}$$

$$c_7(\theta) \geq 0, \quad c_7(\theta) + c_8(\theta)t_0 \geq 0, \tag{9.4}$$

$$c_9(\theta) \geq 0, \quad c_9(\theta) + c_{10}(\theta)t_0 \geq 0, \tag{9.5}$$

$$c_{11}(\theta) \geq 0, \quad c_{11}(\theta) + c_{12}(\theta)t_0 \geq 0, \tag{9.6}$$

$$c_{13}(\theta) \geq 0. \tag{9.7}$$

It will follow then that for $0 < \alpha \leq t_0$ we have

$$
\begin{aligned}
F_7(\alpha, x) &= \left[\tfrac{7}{10}c_1(\theta) + c_2(\theta)\alpha\right]\alpha + \left[\tfrac{1}{10}c_1(\theta) + c_3(\theta)\alpha^2 + c_4(\theta)\alpha^3\right]\alpha \\
&\quad + \left[\tfrac{1}{5}c_1(\theta) + c_5(\theta)\alpha^4 + c_6(\theta)\alpha^5\right]\alpha + \left[c_7(\theta) + c_8(\theta)\alpha\right]\alpha^7 \\
&\quad + \left[c_9(\theta) + c_{10}(\theta)\alpha\right]\alpha^9 + \left[c_{11}(\theta) + c_{12}(\theta)\alpha\right]\alpha^{11} \\
&\quad + c_{13}(\theta)\alpha^{13} \geq 0
\end{aligned}
\tag{10}
$$

The inequalities (9) imply (10) because they imply that each of the terms in brackets in (10) are non-negative. The non-negativity of the bracketed terms of the form $[c_i(\theta) + c_{i+1}(\theta)\alpha]$ follows from (9.1), (9.4), (9.5) and (9.6) because the terms are linear in α and, hence they take their minimum at either $\alpha = 0$ or else at $\alpha = t_0$.

Since $[c_3(\theta) + c_4(\theta)\alpha]$ is linear in α, it takes its minimum at either $\alpha = 0$ or else at $\alpha = t_0$. Thus, we have

$$
\frac{1}{10}c_1(\theta) + (c_3(\theta) + c_4(\theta)\alpha)\alpha^2 \geq \frac{1}{10}c_1(\theta) + \min_{0<s\leq t_0}\{c_3(\theta) + c_4(\theta)s\}\alpha^2. \tag{11}
$$

The right-hand side of (11) is linear in α^2, and hence takes its minimum at either $\alpha = 0$ or else at $\alpha = t_0$. Therefore, the right-hand side of (11) is non-negative by (9.2).

Similarly, since $[c_5(\theta) + c_6(\theta)\alpha]$ is linear in α, it takes its minimum at either $\alpha = 0$ or else at $\alpha = t_0$. Thus, we have

$$
\frac{1}{5}c_1(\theta) + (c_5(\theta) + c_6(\theta)\alpha)\alpha^4 \geq \frac{1}{5}c_1(\theta) + \min_{0<s\leq t_0}\{c_5(\theta) + c_6(\theta)s\}\alpha^4. \tag{12}
$$

The right-hand side of (12) is linear in α^4, and hence takes its minimum at either $\alpha = 0$ or else at $\alpha = t_0$. Therefore, the right-hand side of (12) is non-negative by (9.3).

Thus, to complete the case $0 < \alpha \leq t_0$ we will need to establish (9). We will transform each of the trigonometric coefficients $c_i(\theta)$, which are polynomials in $\cos(n\theta)$, to polynomials in $\cos\theta$ and then by a change of variable to polynomials $e_i(x)$, $-1 \leq x \leq 1$. To verify the non-negativity of the linear combinations of trigonometric coefficients $c_i(\theta)$ specified in (9), we will establish the non-negativity of the same linear combinations of polynomials $e_i(x)$.

The following two Maple procedures can be used to: (i) transform the trigonometric coefficients $c_i(\theta)$ to the polynomials $e_i(x)$; and (ii) compute the number of roots of a polynomial p on the interval $(-1, 1]$ via a Sturm sequence argument.

Procedure 2

```
G:=proc(N)
local i, t, temp;
global c, e;
    for i from 0 to 2*N - 1 do
        t[i] := expand(c[i]);
        e[i] := subs(cos(theta) = x, t[i]);
    od;
    temp;
end;
```

Here N is chosen the same as in Procedure 1.

Procedure 3

```
H := proc(p)
local s;
global lc, nr;
    lc := roots(p, x);
    s := sturmseq(p, x);
    nr := sturm(s, x, -1, 1)
end;
```

The library call *readlib(sturm)* must be loaded prior to applying the procedure.

If the polynomial $e(x)$, created from linear combinations of the $e_i(x)$ after applying Procedure 2, is assigned to the variable p, then Procedure 3 will compute both the number of roots of $e(x)$ on the interval $(-1, 1]$ and the location of the rational roots of $e(x)$. We will see that the conclusion of this application of Procedure 3 is that the polynomial $e(x)$ is non-negative on $[-1, 1]$ with $e(x) = 0$ only for $x = 1$. This check can be confirmed for each of the polynomials $e(x)$ which arise as linear combinations of the polynomials $e_i(x)$, where the linear combinations are specified as in (9), and, thus, complete the case $0 < \alpha \leq t_0$.

To illustrate the utility of using computer algebra software to establish the inequalities in (9) we will explicitly demonstrate the process for the inequality (9.1). Applying Procedure 1 to compute the trigonometric coefficients $c_i(\theta)$ of $F_7(\alpha, x)$, we obtain

$$\begin{aligned} c_1(\theta) &= -\frac{2}{5}\cos(5\theta) + \cos(2\theta) - \frac{2}{3}\cos(3\theta) + \frac{1}{3}\cos(6\theta) \\ &+ \frac{1}{2}\cos(4\theta) - 2\cos(\theta) - \frac{2}{7}\cos(7\theta) + \frac{319}{210} \end{aligned}$$

From Procedure 2 we obtain

$$\begin{aligned} e(x) &= e_1(x) = \frac{128}{5}x^5 - \frac{32}{3}x^3 + 4\,x^2 + \frac{24}{35} \\ &+ \frac{32}{3}x^6 - 12\,x^4 - \frac{128}{7}x^7. \end{aligned}$$

Then, Procedure 3 yields

```
> H(p);
                1
> lc
             [[1, 1]]
```

The value returned by the procedure call H(p) is the number of roots of p on $(-1, 1]$ and the value returned by lc is the interval location of the rational roots of p.

For the second half of (9.1), we have

$$\begin{aligned}\tfrac{7}{10}c_1(\theta)+c_2(\theta)t_0 &= \frac{128}{525}\cos(5\theta)-\frac{37}{210}\cos(2\theta)+\frac{451}{1575}\cos(3\theta)\\ &-\frac{292}{1575}\cos(60)-\frac{197}{700}\cos(4\theta)-\frac{5}{7}\cos(\theta)+\frac{2}{25}\cos(7\theta)+\frac{523}{700}\end{aligned}$$

From Procedure 2 we obtain

$$\begin{aligned}e(x)=\tfrac{7}{10}e_1(x)+e_2(x)t_0 &= -\frac{2656}{525}x^5+\frac{236}{315}x^3-\frac{151}{105}x^2\\ &+\frac{1303}{1575}-\frac{9344}{1575}x^6+\frac{698}{105}x^4+\frac{128}{25}x^7-\frac{32}{35}x\end{aligned}$$

Then, Procedure 3 yields

```
> H(p);
                    1;
> lc
                  [[1, 1]]
```

We have that each linear combination $e(x)$ has only one root on $(-1,1]$ and that root is at $x=1$. Since we can explicitly observe that each $e(0)>0$, we can conclude that each $e(x)$ is non-negative on $[-1,1]$. Therefore, we have that (9.1) holds.

For the case $t_0<\alpha<1$ we make the substitution $\alpha=\beta+t_0$. Then, we have $F_7(\alpha,x)=G_7(\beta,x)=\sum_{k=0}^{13}d_k(\theta)\beta^k$ where each $d_k(\theta)$ is a trigonometric polynomial of the form $d_k(\theta)=\sum_{j=0}^{7}b_{kj}\cos(j\theta)$ and $0<\beta<t_1=3/5$. It will suffice to show for this case that

$$d_0(\theta)\geq 0,\ \ d_0(\theta)+d_2(\theta)t_1^2\geq 0 \tag{13.1}$$

$$d_1(\theta)\geq 0,\ \ \frac{2}{3}d_1(\theta)+d_5(\theta)t_1^4\geq 0 \tag{13.2}$$

$$\frac{1}{3}d_1(\theta)+d_6(\theta)t_1^5\geq 0,\ \ \frac{1}{3}d_1(\theta)+d_6(\theta)t_1^5+d_8(\theta)t_1^7\geq 0 \tag{13.3}$$

$$d_3(\theta) \geq 0, d_3(\theta) + d_4(\theta)t_1 \geq 0 \tag{13.4}$$

$$d_7(\theta) \geq 0 \tag{13.5}$$

$$d_9(\theta) \geq 0, \quad d_9(\theta) + d_{10}(\theta)t_1 \geq 0, \tag{13.6}$$

$$d_{11}(\theta) \geq 0, \quad d_{11}(\theta) + d_{12}(\theta)t_1 \geq 0, \tag{13.7}$$

$$d_{13}(\theta) \geq 0. \tag{13.8}$$

For then, it will follow that for $0 < \beta < t_1$ we have

$$\begin{aligned} G_7(\beta, x) \quad &= [d_0(\theta) + d_2(\theta)\beta^2] + \left[\tfrac{2}{3}d_1(\theta) + d_5(\theta)\beta^4\right]\beta \\ &+ \left[\tfrac{1}{3}d_1(\theta) + d_6(\theta)\beta^5 + d_8(\theta)\beta^7\right]\beta + [d_3(t) + d_4(\theta)\beta]\beta^3 \\ &+ d_7(\theta)\beta^7 + [d_9(\theta) + d_{10}(\theta)\beta]\beta^9 + [d_{11}(\theta) + d_{12}(\theta)\beta]\beta^{11} \\ &+ d_{13}(\theta)\beta^{13} \geq 0 \end{aligned} \tag{14}$$

The inequalities (13) imply (14) because they imply that each of the bracketed terms in (14) are non-negative for $0 < \beta < t_1$. Procedure 2 can be adapted (by changing the global variable `c` to `d`) so that it can be applied to each of the trigonometric coefficients $d_i(\theta)$ to generate new polynomials $e_i(x)$. Then, Procedure 3 can be applied to each of the (transformed) linear combinations specified in (13) to verify (14) and thus, complete the case $0 < \beta < t_1$.

Remarks.

1. We have verified the above constructions alternately using Mathematica for the computer algebra component of the construction.

2. The above process can be applied to $F_3(\alpha, x)$ and $F_5(\alpha, x)$ to give relatively straight-forward proofs of two cases of Brannan's conjecture (6), specifically, the cases $2n + 1 = 3$ and $2n + 1 = 5$. In the latter case, the proof subdivides the interval $0 < \alpha < 1$ into two cases $0 < \alpha \leq 2/5$ and $2/5 < \alpha < 1$. The argument here is substantially simpler than Milcetich's proof.

3. This technique for verifying Brannan's conjecture (6) for the case $2n+1=7$ can be applied to an alternate, but closely related coefficient inequality. If in the series representation for $F_N(\alpha, x)$ in (8) the summation is extended to infinity, that is, if we write

$$F_N(\alpha, x) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}}{k!(k-j)!} - \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}\cos((2*j-k)\theta)}{k!(k-j)!} \quad (15)$$

and where again

$$\delta_j = \begin{cases} 1 & 0 \le j \le N \\ 0 & N+1 \le j \end{cases},$$

then one can define the partial sums

$$F_N^m(\alpha, x) = \sum_{k=0}^{m}\sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}}{k!(k-j)!} - \sum_{k=0}^{m}\sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k\delta_j\delta_{k-j}\cos((2*j-k)\theta)}{k!(k-j)!}$$

Wheeler [Wh] considered the partial sums $F_N^m(\alpha, x)$. He showed that these partial sums have many properties which are analogous to the coefficient sums $F_N(\alpha, x)$. Specifically, he showed there that for each α, $0 < \alpha < 1$, there exists an m_α such that

$$\max_{|x|=1} F_N^{2*m}(\alpha, x) < 0$$

for $m > m_\alpha$. Furthermore, he devised the computer algebra technique described above, and applied it to show that for $m = 1$, 3, 5 and 7,

$$F_N^m(\alpha, x) \ge 0$$

for $0 < \alpha < 1$.

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