On a Coefficient Conjecture of Brannan

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Abstract

In 1972, D.A. Brannan conjectured that all of the odd coefficients, $a_{2n+1}$, of the power series $(1 + xz)\alpha/(1 - z)$ were dominated by those of the series $(1 + z)\alpha/(1 - z)$ for the parameter range $0 < \alpha < 1$, after having shown that this was not true for the even coefficients. He verified the case when $2n + 1 = 3$. The case when $2n + 1 = 5$ was verified in the mid-eighties by J.G. Milcetic. In this paper, we verify the case when $2n + 1 = 7$ using classical Sturm sequence arguments and some computer algebra.

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Introduction.

For $k \geq 2$ let $V_k$ denote the class of locally univalent analytic functions

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \quad (1)$$

which map $|z| < 1$ conformally onto a domain whose boundary rotation is at most $k\pi$. (See [Pa] for the definition and basic properties of the class $V_k$.)

The function

$$f_k(z) = \frac{1}{k} \left[ \left( \frac{1 + z}{1 - z} \right)^{\frac{k}{2}} - 1 \right] = \sum_{n=1}^{\infty} A_n z^n$$

belongs to $V_k$. The coefficient conjecture for the class $V_k$ was that for a function (1) in $V_k$ that

$$|a_n| \leq A_n, \quad (n \geq 1) \quad (2)$$


Using extreme point theory arguments, Brannan, Clunie and Kirwan [Br-ClKi] showed in 1973 that (2) can be reduced to showing that for

$$\Phi(\alpha, x; z) = \left( \frac{1 + xz}{1 - z} \right)^{\alpha} = \sum_{n=1}^{\infty} B_n(\alpha, x) z^n$$

that

$$|B_n(\alpha, x)| \leq B_n(\alpha, 1), \quad (n \geq 1) \quad (3)$$

for $\alpha \geq 1, \ |x| = 1$. Brannan, Clunie and Kirwan showed that (3) holds for $1 \leq n \leq 13$, which implies (2) for $2 \leq n \leq 14$.

In 1972 Aharonov and Friedland [AhFr] considered a related coefficient inequality. Let

$$\Psi(\alpha, x; z) = \frac{(1 + xz)^{\alpha}}{1 - z} = \sum_{n=1}^{\infty} A_n(\alpha, x) z^n.$$
In [AhFr] it was shown, by a long technical argument, that

\[ |A_n(\alpha, x)| \leq A_n(\alpha, 1), \quad (n \geq 1) \tag{4} \]

for \( \alpha \geq 1, |x| = 1 \), which implies (3) and, hence, by the work in [BrClKi], also implies (2). Later, in 1973 Brannan [Br2] gave a short, elegant proof that (4) holds for \( \alpha \geq 1, |x| = 1 \).

In [Br2] Brannan also considered the question about whether (4) holds for \( 0 < \alpha < 1, |x| = 1 \). He showed there the unexpected result that for each \( \alpha, 0 < \alpha < 1 \), there exists an \( n_\alpha \) such that

\[ \max_{|x|=1} \text{Re} A_{2n}(\alpha, x) > A_{2n}(\alpha, 1) \tag{5} \]

for \( n > n_\alpha \), that is, that (4) fails for even coefficients when \( 0 < \alpha < 1 \).

Brannan showed, using an inequality for quadratic trigonometric polynomials, that

\[ |A_3(\alpha, x)| \leq A_3(\alpha, 1) \]

for \( 0 < \alpha < 1 \) and he conjectured, based on numerical data, that

**Brannan’s Conjecture**

\[ |A_{2n+1}(\alpha, x)| \leq A_{2n+1}(\alpha, 1), \quad (n \geq 1) \tag{6} \]

for \( 0 < \alpha < 1, |x| = 1 \).

Brannan’s conjecture has been verified for \( n = 2 \), that is, for \( 2n + 1 = 5 \), by Milcetich [Mi], who employed a lengthy argument based on a result of Brown and Hewitt [BrHe] for positive trigonometric sums.

In this paper, we will establish Brannan’s conjecture for \( n = 3 \), that is, for \( 2n + 1 = 7 \). The method we will employ is based largely on (i) a judicious rearrangement of the coefficients \( A_n(\alpha, x) \) over carefully chosen subintervals of \((0,1)\), the domain of \( \alpha \), (ii) an application of Sturm sequences to verify the nonnegativity of those rearrangements and (iii) using a computer algebra
program (in this case Maple) to generate the coefficients \( A_n(\alpha, x) \) and the Sturm sequences.

Section 1.

Brannan’s coefficient inequality (6) is equivalent to

\[
A_{2n+1}^2(\alpha, 1) - |A_{2n+1}(\alpha, x)|^2 \geq 0
\]

(7)

for \( 0 < \alpha < 1, |x| = 1 \). We will let \( F_{2n+1}(\alpha, x) \) denote the left-hand side of (7) and we will show for \( 2n + 1 = 7 \) that \( F_{2n+1}(\alpha, x) \geq 0 \).

We note that

\[
\frac{(1 + x z)^\alpha}{(1 - z)} = \sum_{n=0}^\infty \frac{(-\alpha)_n (-1)^n x^n}{n!} z^n \sum_{n=0}^\infty z^n
\]

\[
= \sum_{n=0}^\infty \sum_{k=0}^n \frac{(-\alpha)_k (-1)^k x^k}{k!} z^n
\]

\[
= \sum_{n=0}^\infty A_n(\alpha, x) z^n,
\]

where \((a)_k\) denotes the Pockhammer symbol, which is defined as

\[
(a)_k = \begin{cases} 
1 & k = 0 \\
 a(a + 1) \cdots (a + k - 1) & k > 0 
\end{cases}
\]

Hence, we can write \( F_N(\alpha, x) \) as

\[
F_N(\alpha, x) = \sum_{k=0}^N \frac{(-\alpha)_k (-1)^k}{k!} \sum_{k=0}^N \frac{(-\alpha)_k (-1)^k}{k!} - \]

\[
\sum_{k=0}^N \frac{(-\alpha)_k (-1)^k x^k}{k!} \sum_{k=0}^N \frac{(-\alpha)_k (-1)^k x^k}{k!}
\]

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\[ \begin{align*}
&= \sum_{k=0}^{2N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k \delta_j \delta_{k-j}}{k!(k-j)!} - \\
&\sum_{k=0}^{2N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k \delta_j \delta_{k-j} x^{2j-k}}{k!(k-j)!}
\end{align*} \]

where

\[ \delta_j = \begin{cases} 
1 & 0 \leq j \leq N \\
0 & N + 1 \leq j \leq 2 \times N 
\end{cases} \]

Since \( F_N(\alpha, x) \) is real, we can write, setting \( x = e^{i\theta} \),

\[ F_N(\alpha, x) = \sum_{k=0}^{2N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k \delta_j \delta_{k-j}}{k!(k-j)!} - \\
\sum_{k=0}^{2N} \sum_{j=0}^{k} \frac{(-\alpha)_j(-\alpha)_{k-j}(-1)^k \delta_j \delta_{k-j} \cos((2 \times j - k)\theta)}{k!(k-j)!} \quad (8) \]

The following Maple Procedure can be used to generate the coefficients of \( F_N(\alpha, x) \), where \( x = e^{i\theta} \),

Procedure 1

\begin{verbatim}
F:=proc(N)
local i, j, a, csum, dsum, temp;
global c;
a[0]:=1;
for i from 1 to N do a[i]:=a[i-1]*(-alpha+i-1)*(-1)/i od;
for i from N+1 to 2*N do a[i]:=0 od;
csum:=0; dsum:=0;
for i from 0 to N do csum:= csum+a[i] od;
for i from 0 to N do dsum:= dsum+a[i] od;
end proc;
\end{verbatim}
for i from 0 to 2*N do
for j from 0 to i do
    dsum:= dsum + a[j]*a[i-j]* cos ((2*j-i)*theta);
od;
od;
temp:= collect(csum*csum-dsum,alpha);
for i from 0 to (2*N-1) do c[i]:= coeff(temp,alpha,i) od;
temp;
end;

Using Procedure 1, we obtain for \( N = 7 \) that
\[
F_7(\alpha, x) = \sum_{k=1}^{13} c_k(\theta)\alpha^k
\]
where each \( c_k(\theta) \) is a trigonometric polynomial of the form
\[
c_k(\theta) = \sum_{j=0}^{7} a_{kj} \cos(j\theta)
\]
with rational coefficients \( a_{kj} \). We will show that \( F_7(\alpha, x) \geq 0 \) for \( 0 < \alpha < 1 \) by subdividing the domain of \( \alpha \) into subintervals \( 0 < \alpha \leq t_0 \) and \( t_0 < \alpha < 1 \), where \( t_0 = 2/5 \). We will show that \( F_7(\alpha, x) \geq 0 \) on each subinterval.

First for the case \( 0 < \alpha \leq t_0 \) we will show the following:

\[
c_1(\theta) \geq 0, \quad \frac{7}{10} c_1(\theta) + c_2(\theta)t_0 \geq 0,
\]
\[
\frac{1}{10} c_1(\theta) + c_3(\theta)t_0^2 \geq 0, \quad \frac{1}{10} c_1(\theta) + c_3(\theta)t_0^2 + c_4(\theta)t_0^3 \geq 0,
\]
\[
\frac{1}{5} c_1(\theta) + c_5(\theta)t_0^4 \geq 0, \quad \frac{1}{5} c_1(\theta) + c_5(\theta)t_0^4 + c_6(\theta)t_0^5 \geq 0,
\]
\[
c_7(\theta) \geq 0, \quad c_7(\theta) + c_8(\theta)t_0 \geq 0,
\]
\[
c_9(\theta) \geq 0, \quad c_9(\theta) + c_{10}(\theta)t_0 \geq 0,
\]
\[
c_{11}(\theta) \geq 0, \quad c_{11}(\theta) + c_{12}(\theta)t_0 \geq 0,
\]
\[
c_{13}(\theta) \geq 0.
\]

It will follow then that for \( 0 < \alpha \leq t_0 \) we have
\[
F_7(\alpha, x) = \left[ \frac{7}{10} c_1(\theta) + c_2(\theta)\alpha \right] \alpha + \left[ \frac{1}{10} c_1(\theta) + c_3(\theta)\alpha^2 + c_4(\theta)\alpha^3 \right] \alpha \\
+ \left[ \frac{1}{5} c_1(\theta) + c_5(\theta)\alpha^4 + c_6(\theta)\alpha^5 \right] \alpha + [c_7(\theta) + c_8(\theta)\alpha] \alpha^7 \\
+ [c_9(\theta) + c_{10}(\theta)\alpha] \alpha^9 + [c_{11}(\theta) + c_{12}(\theta)\alpha] \alpha^{11} \\
+ c_{13}(\theta)\alpha^{13} \geq 0 \tag{10}
\]

The inequalities (9) imply (10) because they imply that each of the terms in brackets in (10) are non-negative. The non-negativity of the bracketed terms of the form \([c_i(\theta) + c_{i+1}(\theta)\alpha]\) follows from (9.1), (9.4), (9.5) and (9.6) because the terms are linear in \(\alpha\) and, hence they take their minimum at either \(\alpha = 0\) or else at \(\alpha = t_0\).

Since \([c_3(\theta) + c_4(\theta)\alpha]\) is linear in \(\alpha\), it takes its minimum at either \(\alpha = 0\) or else at \(\alpha = t_0\). Thus, we have

\[
\frac{1}{10} c_1(\theta) + (c_3(\theta) + c_4(\theta)\alpha)\alpha^2 \geq \frac{1}{10} c_1(\theta) + \min_{0 < s \leq t_0} \{c_3(\theta) + c_4(\theta)s\} \alpha^2. \tag{11}
\]

The right-hand side of (11) is linear in \(\alpha^2\), and hence takes its minimum at either \(\alpha = 0\) or else at \(\alpha = t_0\). Therefore, the right-hand side of (11) is non-negative by (9.2).

Similarly, since \([c_5(\theta) + c_6(\theta)\alpha]\) is linear in \(\alpha\), it takes its minimum at either \(\alpha = 0\) or else at \(\alpha = t_0\). Thus, we have

\[
\frac{1}{5} c_1(\theta) + (c_5(\theta) + c_6(\theta)\alpha)\alpha^4 \geq \frac{1}{5} c_1(\theta) + \min_{0 < s \leq t_0} \{c_5(\theta) + c_6(\theta)s\} \alpha^4. \tag{12}
\]

The right-hand side of (12) is linear in \(\alpha^4\), and hence takes its minimum at either \(\alpha = 0\) or else at \(\alpha = t_0\). Therefore, the right-hand side of (12) is non-negative by (9.3).

Thus, to complete the case \(0 < \alpha \leq t_0\) we will need to establish (9). We will transform each of the trigonometric coefficients \(c_i(\theta)\), which are polynomials in \(\cos(n\theta)\), to polynomials in \(\cos(\theta)\) and then by a change of variable to polynomials \(c_i(x), -1 \leq x \leq 1\). To verify the non-negativity of the linear combinations of trigonometric coefficients \(c_i(\theta)\) specified in (9), we will
establish the non-negativity of the same linear combinations of polynomials $e_i(x)$.

The following two Maple procedures can be used to: (i) transform the trigonometric coefficients $c_i(\theta)$ to the polynomials $e_i(x)$; and (ii) compute the number of roots of a polynomial $p$ on the interval $(-1, 1]$ via a Sturm sequence argument.

Procedure 2

\[
G:=\text{proc}(N) \ \
\text{local } i, t, \text{ temp;} \ \\ 
\text{global } c, e; \ 
\text{for } i \text{ from } 0 \text{ to } 2*N - 1 \text{ do} \ 
\quad t[i] := \text{expand}(c[i]); \ 
\quad e[i] := \text{subs}(\text{cos}(\theta) = x, t[i]); \ 
\quad \text{od;} \ 
\quad \text{temp;} \ 
\end{\text{proc}}
\]

Here $N$ is chosen the same as in Procedure 1.

Procedure 3

\[
H := \text{proc}(p) \ 
\text{local } s; \ 
\text{global } lc, nr; \ 
\quad lc := \text{roots}(p, x); \ 
\quad s := \text{sturmseq}(p, x); \ 
\quad nr := \text{sturm}(s, x, -1, 1) \ 
\end{\text{proc}}
\]

The library call \text{readlib(sturm)} must be loaded prior to applying the procedure.
If the polynomial $e(x)$, created from linear combinations of the $e_i(x)$ after applying Procedure 2, is assigned to the variable $p$, then Procedure 3 will compute both the number of roots of $e(x)$ on the interval $(-1, 1]$ and the location of the rational roots of $e(x)$. We will see that the conclusion of this application of Procedure 3 is that the polynomial $e(x)$ is non-negative on $[-1, 1]$ with $e(x) = 0$ only for $x = 1$. This check can be confirmed for each of the polynomials $e(x)$ which arise as linear combinations of the polynomials $e_i(x)$, where the linear combinations are specified as in (9), and, thus, complete the case $0 < \alpha \leq t_0$.

To illustrate the utility of using computer algebra software to establish the inequalities in (9) we will explicitly demonstrate the process for the inequality (9.1). Applying Procedure 1 to compute the trigonometric coefficients $c_i(\theta)$ of $F_7(\alpha, x)$, we obtain

$$c_1(\theta) = -\frac{2}{5} \cos(5\theta) + \cos(2\theta) - \frac{2}{3} \cos(3\theta) + \frac{1}{3} \cos(6\theta) + \frac{1}{2} \cos(4\theta) - 2 \cos(\theta) - \frac{2}{7} \cos(7\theta) + \frac{319}{210}$$

From Procedure 2 we obtain

$$e(x) = e_1(x) = \frac{128}{5} x^5 - \frac{32}{3} x^3 + 4 x^2 + \frac{24}{35} x + \frac{32}{3} x^6 - 12 x^4 - \frac{128}{7} x^7.$$ 

Then, Procedure 3 yields

$$\text{H}(p) = 1, \quad \text{lc} = [[1, 1]]$$

The value returned by the procedure call $\text{H}(p)$ is the number of roots of $p$ on $(-1, 1]$ and the value returned by $\text{lc}$ is the interval location of the rational roots of $p$. 

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For the second half of (9.1), we have

\[
\frac{7}{10} c_1(\theta) + c_2(\theta) t_0 = \frac{128}{525} \cos(5\theta) - \frac{37}{210} \cos(2\theta) + \frac{451}{1575} \cos(3\theta) - \frac{292}{1575} \cos(60) - \frac{197}{700} \cos(4\theta) + \frac{5}{7} \cos(\theta) + \frac{2}{25} \cos(7\theta) + \frac{523}{700}
\]

From Procedure 2 we obtain

\[
e(x) = \frac{7}{10} e_1(x) + e_2(x) t_0 = -\frac{2656}{525} x^5 + \frac{236}{315} x^3 - \frac{151}{105} x^2 + \frac{1303}{1575} 9344x^6 + \frac{698}{1575} x^4 + \frac{128}{25} x^5 - \frac{32}{35} x
\]

Then, Procedure 3 yields

\[
> H(p); \\
1; \\
> lc \\
[[1, 1]]
\]

We have that each linear combination \(e(x)\) has only one root on \((-1, 1]\) and that root is at \(x = 1\). Since we can explicitly observe that each \(e(0) > 0\), we can conclude that each \(e(x)\) is non-negative on \([-1, 1]\). Therefore, we have that (9.1) holds.

For the case \(t_0 < \alpha < 1\) we make the substitution \(\alpha = \beta + t_0\). Then, we have \(F_7(\alpha, x) = G_7(\beta, x) = \sum_{k=0}^{13} d_k(\theta) \beta^k\) where each \(d_k(\theta)\) is a trigonometric polynomial of the form \(d_k(\theta) = \sum_{j=0}^{7} b_{kj} \cos(j\theta)\) and \(0 < \beta < t_1 = 3/5\). It will suffice to show for this case that

\[
d_0(\theta) \geq 0, \quad d_0(\theta) + d_2(\theta) t_1^2 \geq 0 \quad (13.1)
\]

\[
d_1(\theta) \geq 0, \quad \frac{2}{3} d_1(\theta) + d_5(\theta) t_1^4 \geq 0 \quad (13.2)
\]

\[
\frac{1}{3} d_1(\theta) + d_6(\theta) t_1^5 \geq 0, \quad \frac{1}{3} d_1(\theta) + d_6(\theta) t_1^5 + d_8(\theta) t_1^7 \geq 0 \quad (13.3)
\]

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\[
d_3(\theta) \geq 0, \quad d_3(\theta) + d_4(\theta) t_1 \geq 0 \tag{13.4}
\]
\[
d_7(\theta) \geq 0 \tag{13.5}
\]
\[
d_9(\theta) \geq 0, \quad d_9(\theta) + d_{10}(\theta) t_1 \geq 0, \tag{13.6}
\]
\[
d_{11}(\theta) \geq 0, \quad d_{11}(\theta) + d_{12}(\theta) t_1 \geq 0, \tag{13.7}
\]
\[
d_{13}(\theta) \geq 0. \tag{13.8}
\]

For then, it will follow that for \(0 < \beta < t_1\) we have

\[
G_\gamma(\beta, x) = [d_0(\theta) + d_2(\theta) \beta^2] + [\frac{2}{3} d_1(\theta) + d_5(\theta) \beta^4] \beta
\]
\[
+ \left[\frac{1}{3} d_1(\theta) + d_6(\theta) \beta^5 + d_8(\theta) \beta^7\right] \beta + [d_3(t) + d_4(\theta) \beta] \beta^3
\]
\[
+ d_7(\theta) \beta^7 + [d_9(\theta) + d_{10}(\theta) \beta] \beta^9 + [d_{11}(\theta) + d_{12}(\theta) \beta] \beta^{11}
\]
\[
+ d_{13}(\theta) \beta^{13} \geq 0 \tag{14}
\]

The inequalities (13) imply (14) because they imply that each of the bracketed terms in (14) are non-negative for \(0 < \beta < t_1\). Procedure 2 can be adapted (by changing the global variable \(c\) to \(d\)) so that it can be applied to each of the trigonometric coefficients \(d_i(\theta)\) to generate new polynomials \(e_i(x)\). Then, Procedure 3 can be applied to each of the (transformed) linear combinations specified in (13) to verify (14) and thus, complete the case \(0 < \beta < t_1\).

**Remarks.**

1. We have verified the above constructions alternately using Mathematica for the computer algebra component of the construction.

2. The above process can be applied to \(F_3(\alpha, x)\) and \(F_5(\alpha, x)\) to give relatively straight-forward proofs of two cases of Brannan’s conjecture (6), specifically, the cases \(2n + 1 = 3\) and \(2n + 1 = 5\). In the latter case, the proof subdivides the interval \(0 < \alpha < 1\) into two cases \(0 < \alpha \leq 2/5\) and \(2/5 < \alpha < 1\). The argument here is substantially simpler than Milcetic’s proof.
3. This technique for verifying Brannan’s conjecture (6) for the case $2n + 1 = 7$ can be applied to an alternate, but closely related coefficient inequality. If in the series representation for $F_N(\alpha, x)$ in (8) the summation is extended to infinity, that is, if we write

$$F_N(\alpha, x) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-\alpha)_j (-\alpha)_{k-j} (-1)^k \delta_j \delta_{k-j}}{k!(k-j)!} -$$

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} \frac{(-\alpha)_j (-\alpha)_{k-j} (-1)^k \delta_j \delta_{k-j} \cos((2 \ast j - k)\theta)}{k!(k-j)!}$$

(15)

and where again

$$\delta_j = \begin{cases} 1 & 0 \leq j \leq N \\ 0 & N + 1 \leq j \end{cases},$$

then one can define the partial sums

$$F^m_N(\alpha, x) = \sum_{k=0}^{m} \sum_{j=0}^{k} \frac{(-\alpha)_j (-\alpha)_{k-j} (-1)^k \delta_j \delta_{k-j}}{k!(k-j)!} -$$

$$\sum_{k=0}^{m} \sum_{j=0}^{k} \frac{(-\alpha)_j (-\alpha)_{k-j} (-1)^k \delta_j \delta_{k-j} \cos((2 \ast j - k)\theta)}{k!(k-j)!}$$

Wheeler [Wh] considered the partial sums $F^m_N(\alpha, x)$. He showed that these partial sums have many properties which are analogous to the coefficient sums $F_N(\alpha, x)$. Specifically, he showed there that for each $\alpha$, $0 < \alpha < 1$, there exists an $m_\alpha$ such that

$$\max_{|x|=1} F^{2m}_N(\alpha, x) < 0$$

for $m > m_\alpha$. Furthermore, he devised the computer algebra technique described above, and applied it to show that for $m = 1, 3, 5$ and 7,

$$F^m_N(\alpha, x) \geq 0$$

for $0 < \alpha < 1$. 
References


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