Tail Mean Estimation is More Efficient than Tail Median: Evidence From the Exponential Power Distribution

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Abstract
We investigate the relative efficiency of the empirical “tail median” vs. ”tail mean” as estimators of location under random sampling from the exponential power distribution (EPD). By considering appropriate probabilities so that the quantile of the untruncated EPD (population tail median) and mean of the left-truncated EPD (population tail mean) coincide, limiting results are established concerning the ratio of asymptotic variances of the corresponding estimators. The most remarkable finding is that in the limit of the right tail, the asymptotic variance of the tail median is approximately 36% larger than that of the tail mean, irrespective of the EPD shape parameter. This discovery has important repercussions for quantitative risk management practitioners, where the tail median and tail mean correspond to value-at-risk and expected shortfall, respectively. From a purely academic standpoint, the findings also offer a generalized solution to the age-old statistical quandary: which of the sample median vs. sample mean is the most efficient estimator of centrality?

Keywords: value-at-risk; expected shortfall; asymptotic relative efficiency; rate of convergence; generalized quantile function.

1 Introduction
The quantification of risk is an increasingly important exercise carried out by risk management professionals in a variety of disciplines. These may range from assessing the likelihood of structural failure in machined parts, and catastrophic floods in hydrology and storm management, to predicting economic loss in insurance, portfolio management, and credit lending companies. In the area of (financial) quantitative risk management (QRM), increasing demand by regulatory bodies such as the Basel Committee on Banking Supervision† to implement methodically sound credit risk assessment practices, has fostered much recent research into measures of risk. See McNeil et al. (2005) for a classic treatment of QRM, and Embrechts and Hofert (2014) for a current survey of the field.

As the term risk is synonymous with extreme event, it is naturally desirable to determine or estimate high percentiles of the (often potential) distribution of losses. To be useful in quick decision making, a single number is required, and two common measures in use by credit lenders include Value-at-Risk (VaR), and Conditional Value-at-Risk (CVaR), along with the 95th and 1

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1[http://www.bis.org/bcbs/]: A forum for regular cooperation on worldwide banking supervisory matters based in Basel (Switzerland).
99th percentiles. The term CVaR was coined by Rockafellar and Uryasev (2000), but synonyms (and closely related variants) for it also in common usage include: “expected shortfall” (Acerbi and Tasche, 2002), “tail-conditional expectation” (Artzner et al., 1999), and “average Value-at-Risk” (Chun et al., 2012), or simply “tail mean”.

VaR and CVaR are widely used to measure and manage risk in the financial industry (Jorion, 2003; Duffie and Singleton, 2003). With the random variable $X$ representing the distribution of losses, given a probability level $0 < \beta < 1$, VaR$_\beta$ is simply the $\beta$ quantile of $X$, and hence is the value beyond which higher losses only occur with probability $1 - \beta$. Using similar notation, CVaR$_\beta$ is the average of this $1 - \beta$ fraction of worst losses. To define these precisely, let $X$ be a continuous real-valued random variable defined on some probability space $(\Omega, \mathcal{A}, P)$, with cumulative distribution function (CDF) $F(\cdot)$ and probability density function (PDF) $f(\cdot)$. The quantities $\mu$ and $\sigma^2$ will denote, respectively, the mean and variance of $X$, and both are assumed to be finite. The VaR and CVaR of $X$ at probability level $\beta$ are then defined as follows.

**Definition 1 (VaR).**

$$\text{VaR}_\beta(X) \equiv \xi_\beta(X) = F^{-1}(\beta).$$  \hspace{1cm} (1)

**Definition 2 (CVaR).**

$$\text{CVaR}_\beta(X) \equiv \mu_\beta(X) = \mathbb{E}(X|X \geq \xi_\beta) = \frac{1}{1 - \beta} \int_{\xi_\beta}^{\infty} xf(x)dx = \frac{1}{1 - \beta} \int_{\beta}^{1} F^{-1}(u)du.$$  \hspace{1cm} (2)

When no ambiguity arises we write simply $\xi_\beta$ and $\mu_\beta$. Typical values for $\beta$ are in the range $0.90 \leq \beta \leq 0.99$. Note that although the quantile function $F^{-1}(\cdot)$ is well-defined here due to the assumed strict monotonocity on $F(\cdot)$, in general $\xi_\beta$ has to be defined through the generalized quantile function $F^{-}(\cdot)$ (Embretchen and Hofert, 2013). In addition, the definition of $\mu_\beta$ implicitly assumes the existence of the first absolute moment, $\mathbb{E}|X|$.

There has been much debate and research over the past decade over which of these two measures should be the industry standard\(^2\). Some of the issues at stake concern trade-offs between conceptual simplicity (VaR), versus better axiomatic adherence and mathematical properties (CVaR), since these measures are often used in intricate optimization schemes; see e.g., Follmer and Schied (2011), and Pflug and Romisch (2007). While there is a general consensus that CVaR is more easily optimized (primarily due to convexity), Yamai and Yoshiba (2002, 2005) report that “...expected shortfall needs a larger size of sample than VaR for the same level of accuracy.”. This basic fact is echoed in the review paper of Embretchen and Hofert (2014) when comparing confidence limits for VaR vs. CVaR based on extreme value theory (EVT). This an expected consequence of their definition that ensures $\xi_\beta \leq \mu_\beta$, and as such we would point out that it is not an entirely fair comparison since for a given $\beta$ the two measures are in effect measuring different parts of the tail of $X$.

A fairer comparison of the estimation “accuracy” of VaR versus CVaR would result if the two measures were forced to coincide, a study that to the best of our knowledge has not yet been undertaken. This situation is depicted in Figure 1, which shows the relative positions of the quantiles $\xi_\beta$ and $\xi_\alpha$ for a PDF with $\beta < \alpha$ so that $\mu_\beta = \xi_\alpha$. This implicitly assumes the existence of a function $g(\cdot)$, such that

$$\alpha = g(\beta) \equiv g_\beta.$$  \hspace{1cm} (3)

Loosely speaking, we then have the “tail mean” ($\mu_\beta$) coinciding with the “tail median” ($\xi_\alpha$), where the usage of these terms is meant to convey that $\mu_\beta$ is a “mean-like” estimator and $\xi_\alpha$ a “median-like” estimator. Note that we are effectively adjusting the probability levels so that the mean of the left-truncated EPD at $\xi_\beta$ coincides with the untruncated $\alpha$-quantile.

\(^2\)See for example [http://gloria-mundi.com/], a website serving as a resource for Value-at-Risk and more generally financial risk management.
Figure 1: Illustration of relative positions of $VaR_\beta = \xi_\beta$ and $VaR_\alpha = \xi_\alpha$ in the right tail of a PDF, so that $CVaR_\beta = \mu_\beta$ coincides with $VaR_\alpha$.

The primary aim of this paper is to shed light on this matter, by considering the asymptotic relative efficiencies of the empirical (or nonparametric) estimators. When a random sample of size $n$ with order statistics $X(1) \leq \cdots \leq X(n)$ is available, consistent estimators of VaR and CVaR are respectively,

$$\hat{\xi}_\beta = X(k_\beta), \quad \text{and} \quad \hat{\mu}_\beta = \frac{1}{n - k_\beta + 1} \sum_{r=k_\beta}^{n} X(r), \quad (4)$$

where $k_\beta = \lfloor n\beta \rfloor$ can denote either of the two integers closest to $n\beta$ (or any interpolant thereof). In the process we will also generalize the age-old statistical quandary of which of the sample median or sample mean is the “best” estimator of centrality, since these estimators are obtained in the limit as $\beta \to 0$ in (4).

To effect this comparison, we will consider sampling from the exponential power distribution (EPD); a flexible model for distributions with exponentially declining tails and finite moments of all orders. The EPD is also variously called Subbotin, Generalized Error Distribution (Mineo and Ruggieri, 2005), and Generalized Normal Distribution (Nadarajah, 2005), with slight differences in the parametrizations. The version we adopt here is similar to the EPD of Gomez et al. (1998), but following the simpler parametrization of Sherman (1997). This means the standard member of the family has mean and median zero, with a PDF\(^3\) given by

$$f(x; p) = \frac{p}{2\Gamma(1/p)} \exp\{-|x|^p\}, \quad p \in (0, \infty). \quad (5)$$

\(^3\)The PDF, CDF, quantiles, and random values from the standard EPD in (5) can be obtained from the R package **normalp** by setting the scale parameter $\text{sigmap} = p^{-1/p}$.
The EPD has finite moments of all orders. Parameter $p$ controls the shape, so that we obtain for $p = 2$ a normal with variance $1/2$, and for $p = 1$ a classical Laplace with variance $2$. For $p < 2$ and $p > 2$ we obtain respectively, leptokurtic (heavier than Gaussian) and platikurtic (lighter than Gaussian) tail characteristics. In the limit as $p \to \infty$ the EPD becomes $\mathcal{U}[-1,1]$, a uniform distribution on $[-1,1]$. As $p \to 0$ the limit is degenerate, as the PDF converges to zero everywhere on the real line.

The non-Gaussian members of the elliptical family of distributions (which includes the EPD) have recently been investigated by Landsman and Valdez (2003) as providing, from a QRM perspective, a more realistic model than the Gaussian. Note also that the Laplace rendering of the EPD ($p = 1$) does coincide with Gumbel-like behavior far out in the right tail (large $x$); the Gumbel being an important case in an EVT-based analysis. In any case, the basic question we are posing concerning the relative efficiency of VaR vs. CVaR, or tail median vs. tail mean, is interesting also from a purely academic point of view.

The rest of the paper is organized as follows. Notation and necessary preliminary results are established in Section 2. This is followed by the main theorems and discussion of their meaning and implications in Section 3. The proof of the theorems appear in Sections 4 and 5. The paper concludes with a short Discussion.

2 Preliminary Results

Before stating our main findings, we derive some preliminary results that will be useful in subsequent sections. Let $\gamma(a,x)$ and $\Gamma(a,x)$ denote the incomplete Gamma functions defined by
\[
\gamma(a,x) = \int_0^x e^{-t} t^{a-1} dt, \quad \text{and} \quad \Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} dt, \tag{6}
\]
and note that $\gamma(a,x) + \Gamma(a,x) = \Gamma(a)$. For notational convenience, define for $f(x)$ the EPD PDF in (5) and $n = 0, 1, 2, \ldots$,
\[
F_n(x) = \int_{-\infty}^x t^n f(t) dt, \quad G_n(x) = \int_x^\infty t^n f(t) dt, \quad A_n = \int_{-\infty}^\infty t^n f(t) dt. \tag{7}
\]
Using this notation, the EPD CDF satisfies $F \equiv F_0$, and consequently $A_0 = 1$. We note that for all $x$ we have
\[
F_n(x) + G_n(x) = A_n \tag{8}
\]
Now, for $n$ odd or even, we can, by making the change of variable $t^p = u$, use (6) to rewrite $G_n$ as
\[
G_n(x) = \frac{1}{2\Gamma(1/p)} \Gamma\left(\frac{n+1}{p}, x^p\right), \quad \text{for } x > 0, \tag{9}
\]
and note that $G_n(0) = [2\Gamma(1/p)]^{-1} \Gamma((n+1)/p)$. For the case that $n$ is odd, the integrand in (7) is odd, and hence, we have
\[
G_n(x) = \int_{-x}^{-x} t^n f(t) dt + G_n(-x) = \frac{1}{2\Gamma(1/p)} \Gamma\left(\frac{n+1}{p}, |x|^p\right), \quad \text{for } x < 0. \tag{10}
\]
For the case that $n$ is even, the integrand in (7) is even, and hence $G_n(0) = \frac{1}{2}A_n$. Now, using (8) and (9) yields

$$F_n(x) = \begin{cases} G_n(-x), & \text{for all } x; \\ A_n - G_n(x) = \frac{\Gamma\left(\frac{n+1}{p}\right)}{\Gamma(1/p)} - \frac{1}{2\Gamma(1/p)} \Gamma\left(\frac{n+1}{p}, x^p\right), & \text{for } x > 0; \\ G_n(-x) = \frac{1}{2\Gamma(1/p)} \Gamma\left(\frac{n+1}{p}, |x|^p\right), & \text{for } x < 0, \\ \end{cases}$$

$$G_n(x) = A_n - F_n(x) = \frac{\Gamma\left(\frac{n+1}{p}\right)}{\Gamma(1/p)} - \frac{1}{2\Gamma(1/p)} \Gamma\left(\frac{n+1}{p}, |x|^p\right), \quad \text{for } x < 0.$$ Consequently, and noting that $F_0(\xi_\beta) = \beta$, we can rewrite $\mu_\beta$, making a change of variable $u = F_0(t)$, as

$$\mu_\beta = \frac{1}{1 - F_0(\xi_\beta)} G_1(\xi_\beta) = \frac{G_1(\xi_\beta)}{G_0(\xi_\beta)}, \quad (11)$$

whence the functional relationship between $\alpha$ and $\beta$ in (3) can be written in closed form as

$$\alpha = g_\beta = F_0(\mu_\beta) = F_0\left(\frac{G_1(\xi_\beta)}{G_0(\xi_\beta)}\right). \quad (12)$$

For use in subsequent sections, we define the variance of the truncated version of $X$, whose mean is $\mu_\beta$, as

$$\sigma_\beta^2 = \frac{1}{1 - \beta} \int_{\xi_\beta}^{\infty} (x - \mu_\beta)^2 f(x)dx = \frac{1}{1 - \beta} \int_{\beta}^{1} [F^{-1}(u) - \mu_\beta]^2du, \quad (13)$$

which, using (11) and (12), can be rewritten as

$$\sigma_\beta^2 = \frac{1}{G_0(\xi_\beta)} \int_{\xi_\beta}^{\infty} \left(x - \frac{G_1(\xi_\beta)}{G_0(\xi_\beta)}\right)^2 f(x)dx = \left(\frac{G_2(\xi_\beta)}{G_0(\xi_\beta)}\right) - \left(\frac{G_1(\xi_\beta)}{G_0(\xi_\beta)}\right)^2. \quad (14)$$

Defining for $n = 1, 2$,

$$h_n = \frac{G_n(\xi_\beta)}{G_0(\xi_\beta)}, \quad (15)$$

leads to notationally simpler expressions for several of the above equations.

### 3 Asymptotic Efficiency of VaR Relative to CVaR

Appealing to well-known results for order statistics (David and Nagaraja, 2003) we obtain asymptotic normality for $\hat{\xi}_\beta$ in the quantile case of interest here ($k_\beta/n \to \beta$ as $n \to \infty$). Corresponding results for $\hat{\mu}_\beta$ are more complex, as this estimator involves the extreme order statistic $X_{(n)}$, but were derived by Trindade et al. (2007) in the continuous case of interest here; see also Giurcanu and Trindade (2007) for joint asymptotic normality of $(\hat{\xi}_\beta, \hat{\mu}_\beta)$. Appealing to these results we obtain the asymptotic relative efficiency (ARE) of the empirical estimator of VaR$_\alpha$ with respect to that of CVaR$_\beta$, as

$$H(\hat{\xi}_\alpha, \hat{\mu}_\beta) = \frac{\text{asymptotic variance of } \hat{\xi}_\alpha}{\text{asymptotic variance of } \hat{\mu}_\beta} = \frac{\alpha(1 - \alpha)}{f^2(\xi_\alpha)} \cdot \frac{1 - \beta}{\sigma_\beta^2 + \beta(\mu_\beta - \xi_\beta)^2}, \quad (16)$$

where $\sigma_\beta^2$ is as defined in (13). Choosing $\alpha = g_\beta$ as in (3) to force VaR$_\alpha$ and CVaR$_\beta$ to coincide, and defining the resulting expression in (16) to be $H(\beta, p)$, leads to the following closed form for this ARE under random sampling from the EPD.
Lemma 1. If $\alpha = \beta$ is as in (12), then the ARE of $\hat{\xi}_\alpha$ with respect to $\hat{\mu}_\beta$ as given in (16) is

$$H(\beta, p) = \frac{F_0(h_1)G_0(h_1)}{\left(\frac{p}{2(1/p)}\right)^2} \cdot \frac{G_0(\xi_\beta)}{h_2 - (h_1)^2 + F_0(\xi_\beta)}.$$  

(17)

Proof. This follows immediately by combining (11), (12), and (14), using the notational simplification of (15).

Several interesting questions now arise, but primarily, when is $H(\beta, p) > 1$ so that CVaR is a more efficient estimator than VaR? Figure 2 shows a plot of $H(\beta, p)$ as a function of $0 < \beta < 1$, for different values of $p$. Although neither estimator is uniformly better, we do see that for $p$ greater than about 1.4 (solid line), CVaR is uniformly more efficient. This uniformity is difficult to establish rigorously; but we can make concrete statements by investigating the limiting values of $H(\beta, p)$ separately for each variable while holding the other fixed. Consider first the cases $\beta \to 0$ and $\beta \to 1$, for fixed $p$. This yields the interesting results of the following theorem.

Theorem 2 (Limiting behavior of the ARE in $\beta$). For $H(\beta, p)$ as defined in Lemma 1,

$$\lim_{\beta \to 0} H(\beta, p) \equiv H(0, p) = \frac{3\Gamma^3(1 + 1/p)}{\Gamma(1 + 3/p)}, \quad \text{and} \quad \lim_{\beta \to 1} H(\beta, p) \equiv H(1, p) = \frac{e}{2}.$$  

(18)

Proof. See section 4.

The result for $H(0, p)$ agrees with Sherman (1997), where his eff($\bar{x}, \tilde{x}$) corresponding to the ARE of the sample mean relative to the sample median, is our $1/H(0, p)$. We also note (as did
Sherman, 1997) that $H(0, p)$ is an increasing function of $p$ which maps the interval $(0, \infty)$ onto the interval $(0, 3)$. Consequently, there exists a unique $p^* \approx 1.4074$ such that for $p < p^*$, $H(0, p) < 1$ and for $p > p^*$, $H(0, p) > 1$. Thus, for heavy (light) tailed distributions, the median (mean) is a more efficient estimator, where the boundary between “heavy” and “light” is exactly $p = p^*$.

Furthermore, Figure 2 suggests this result holds more generally for the tail median and tail mean, regardless of $\beta$.

On the other hand, the result $H(1, p) = e/2 \approx 1.36$ is remarkable! It says that in the limit of the right tail, the asymptotic variance of the tail median (VaR) is approximately 36% larger than that of the tail mean (CVaR); a result that holds uniformly for all $p$. Equating “efficiency” with “reduction in variance”, this translates equivalently into the tail mean being approximately 26% more efficient than the tail median. The implications of this finding are clear: CVaR is a more "reduction in variance", this translates equivalently into the tail mean being approximately 26% that of the tail mean (CVaR); a result that holds uniformly for all $p$.

Although not as interesting from a practical standpoint, the next theorem considers also the cases $p \to 0$ and $p \to \infty$, for fixed $\beta$. In the former case we obtain zero; a consequence of the degeneracy of the PDF in the limit. But the useful part of this result is an expression for the rate of convergence to zero, obtained by establishing asymptotic equivalence with a given sequence of $x = 1/p$. In this regard, we write $a(x) \approx b(x)$ if $a(x)/b(x) \to 1$ as $x \to \infty$, as the definition of asymptotic equivalence for the sequences $a(x)$ and $b(x)$.

**Theorem 3** (Limiting behavior of the ARE in $p$). For $H(\beta, p)$ as defined in Lemma 1,

$$\lim_{p \to 0} H(\beta, p) \equiv H(\beta, 0) = 0, \quad \text{and} \quad \lim_{p \to \infty} H(\beta, p) \equiv H(\beta, \infty) = \frac{1 + \beta}{1/3 + \beta}. \tag{19}$$

Moreover, defining $x = 1/p$, we obtain the following rate of convergence to zero of $H(\beta, x)$ as $x \to \infty$, which is given by:

$$H(\beta, x) \approx \left(1 - \frac{(\frac{4}{3} x - \frac{3}{2} \log 2)x e^{-\frac{3}{2} x + \frac{3}{2} \log 2}}{2\Gamma(x)}\right) \left(\frac{(\frac{4}{3} x - \frac{3}{2} \log 2)x e^{-\frac{3}{2} x + \frac{3}{2} \log 2}}{2\Gamma(x)}\right) x^2 \Gamma(x)$$

$$\frac{\Gamma(3x)}{\Gamma(x)} - \frac{1}{2} \left(\frac{\Gamma(2x)}{\Gamma(x)}\right)^2 \exp \left\{-2 \left(\frac{4}{3} x - \frac{3}{2} \log 2\right)\right\}.$$

**Proof.** See section 5. \qed

Interestingly, and in parallel with the second result of Theorem 2, the limiting expression for $H(\beta, 0)$ and associated rate of convergence are also independent of the other parameter ($\beta$). Finally, note that the expression for $H(\beta, \infty)$ is a decreasing function of $\beta$ mapping the interval $(0, 1)$ onto the interval $(3/2, 3)$. Thus, for the limiting $\mathcal{U}[-1, 1]$ distribution obtained when $p \to \infty$, the tail mean would be the (uniformly) preferred measure for all $\beta$.

## 4 Proof of Theorem 2

To assess the limiting behavior of $H(\beta, p)$, we take the approach of considering the limits of its individual components, separately for the cases $\beta \to 0$ (Case 1) and $\beta \to 1$ (Case 2). A key idea is to make the change of variable $\beta = F_0(\xi_\beta)$ early on in each case. This sheds light on the connections to the Gamma and incomplete Gamma, thus naturally allowing one to invoke the properties and asymptotics of these functions.
Case 1: \( \beta \to 0 \) (for fixed \( p \))

If we let \( \beta \to 0 \), then \( \mu_\beta \to 0 \) which is the average value of \( F^{-1}_0 \) over \((0, 1)\). Hence, as \( \beta \to 0 \), we have \( g_\beta \to 1/2 \) which is the value of \( F_0 \) at \( x = 0 \). Clearly as \( \beta \to 0 \), \( f(\mu_\beta) \to f(0) = \frac{p}{2(1/p)} \).

Define \( \xi_\beta \) by \( F_0(\xi_\beta) = \beta \). Then, \( \beta \to 0 \) implies \( \xi_\beta = F^{-1}_0(\beta) \to -\infty \). Considering the term \( \beta(\mu_\beta - \xi_\beta)^2 \), we have, since \( \mu_\beta \to 0 \) as \( \beta \to 0 \),

\[
\lim_{\beta \to 0} \beta(\mu_\beta - \xi_\beta)^2 = \lim_{\beta \to 0} -2\mu_\beta\beta F^{-1}_0(\beta) + \beta F^{-1}_0(\beta)^2.
\] (20)

Making a change of variable \( \beta = F_0(\xi_\beta) \) and applying l’Hopital’s rule, we note, for \( k = 1, 2 \),

\[
\lim_{\beta \to 0} \beta[F^{-1}(\beta)]^k = \lim_{\xi_\beta \to -\infty} F_0(\xi_\beta)\xi_\beta^k = \lim_{\xi_\beta \to -\infty} \frac{F_0(\xi_\beta)}{\xi_\beta^{-k}} = \lim_{\xi_\beta \to -\infty} \frac{p}{2(1/p)} e^{-|\xi_\beta|^p} = 0.
\]

Hence, the limit in (20) is 0. Finally, we note that as \( \beta \to 0 \), \( \sigma_\beta^2 \to \sigma_0^2 = \int_0^1 [F^{-1}(u)]^2 du \). Making a change of variable, \( u = F_0(t) \), we can write

\[
\sigma_0^2 = \int_{-\infty}^{\infty} \frac{p}{2(1/p)} t^2 e^{-|t|^p} dt = \frac{p}{\Gamma(1/p)} \int_{0}^{\infty} t^2 e^{-t} dt = \frac{\Gamma(3/p)}{\Gamma(1/p)}.
\]

Hence, we have as \( \beta \to 0 \),

\[
\lim_{\beta \to 0} H(\beta, p) = \frac{1/4}{\left( \frac{p}{2(1/p)} \right)^2} \cdot \frac{1}{\Gamma(3/p)} = \frac{1}{p^2} \frac{\Gamma(1/p)^3}{\Gamma(3/p)} = \frac{3\Gamma(1+1/p)^3}{\Gamma(1+3/p)}.
\] (21)

Case 2: \( \beta \to 1 \) (for fixed \( p \))

Define \( \xi_\beta \) by \( F_0(\xi_\beta) = \beta \). Then, \( \beta \to 1 \) implies \( \xi_\beta \to \infty \). In particular, we have, since \( \xi_\beta > 0 \), from (9) that

\[
G_0(\xi_\beta) = \frac{\Gamma(n+1/p)}{2\Gamma(1/p)} \xi_\beta^p h_n = \frac{\Gamma(n+1/p)}{2\Gamma(1/p)} \xi_\beta^p h_n = \frac{\Gamma(n+1/p)}{\Gamma(1/p)} \xi_\beta^p h_n, \quad \text{for } n = 1, 2.
\]

Defining,

\[
s(a, z) = 1 + a - \frac{1}{z} + \frac{(a-1)(a-2)}{z^2} + o\left(\frac{1}{z^2}\right), \quad z \to \infty,
\] (22)

we can represent \( \Gamma(a, z) \), see Erdelyi et al. (1953, p. 135, (6)), as

\[
\Gamma(a, z) = \frac{z^a e^{-z}}{z} s(a, z), \quad z \to \infty.
\] (23)

Consequently, we have

\[
G_n(\xi_\beta) = \frac{1}{2\Gamma(1/p)} \xi_\beta^p \left(1 + \frac{(n+1)/p - 1}{\xi_\beta^p} \left(1 + \frac{(n+1)/p - 1}{\xi_\beta^p} \right) + o\left(\frac{1}{\xi_\beta^2}\right)\right),
\]

\[
= \frac{1}{2\Gamma(1/p)} \xi_\beta^p \left(1 + \frac{(n+1)/p - 1}{\xi_\beta^p} \left(1 + \frac{(n+1)/p - 1}{\xi_\beta^p} \right) + o\left(\frac{1}{\xi_\beta^2}\right)\right),
\]

\[
h_n = \frac{G_n(\xi_\beta)}{G_0(\xi_\beta)} = \xi_\beta^p \left(1 + \frac{(n+1)/p - 1}{\xi_\beta^p} \left(1 + \frac{(n+1)/p - 1}{\xi_\beta^p} \right) + o\left(\frac{1}{\xi_\beta^2}\right)\right).
\]
In particular,
\[ G_0(\xi_\beta) = \frac{1}{2\Gamma(1/p)} \frac{\xi_\beta e^{-\xi_\beta^p}}{\xi_\beta^p} \left( 1 + \frac{1/p - 1}{\xi_\beta^p} + \frac{(1/p - 1)(1/p - 2)}{(\xi_\beta^p)^2} \right) + o\left( \frac{1}{\xi_\beta^{2p}} \right), \]
\[ h_1 = \xi_\beta \left( 1 + \frac{1/p}{\xi_\beta^p} + \frac{2/p^2 - 2/p}{(\xi_\beta^p)^2} + o\left( \frac{1}{\xi_\beta^{2p}} \right) \right), \]
which implies \( G_0(\xi_\beta) \to 0 \), \( F_0(\xi_\beta) \to 1 \), and \( h_1 \to \infty \), as \( \xi_\beta \to \infty \). Furthermore, we have
\[ G_0(h_1) = \frac{1}{2\Gamma(1/p)} \frac{h_1 e^{-(h_1)^p}}{(h_1)^p} \left( 1 + \frac{1/p - 1}{\xi_\beta^p} + o\left( \frac{1}{\xi_\beta^{2p}} \right) \right). \]
Consequently, we have \( G_0(h_1) \to 0 \) and \( F_0(h_1) \to 1 \) as \( \xi_\beta \to \infty \). Considering the individual terms of (17) we then have, as \( \xi_\beta \to \infty \),
\[ \frac{G_0(h_1)G_0(\xi_\beta)}{(\frac{1}{2\Gamma(1/p)})^2 \exp\{-2(h_1)^p\}} \approx \frac{h_1}{e^{(h_1)^p}} \frac{\xi_\beta}{(h_1)^p} e^{(h_1)^p} = (h_1)^{1-p} \xi_\beta^{1-p} e^{(h_1)^p} = \xi_\beta^{2-2p} e^{(h_1)^p} - \xi_\beta^p. \]
Now, note that for the exponent of the exponential in (25) we have
\[ (h_1)^p - \xi_\beta^p = \xi_\beta^{p} \left( 1 + \frac{1/p}{\xi_\beta^p} + o\left( \frac{1}{\xi_\beta^{2p}} \right) \right)^p - \xi_\beta^p = \xi_\beta^p \left( 1 + p \frac{1/p}{\xi_\beta^p} + o\left( \frac{1}{\xi_\beta^{2p}} \right) \right) - \xi_\beta^p = 1 + o(1), \]
whence, \( e^{(h_1)^p - \xi_\beta^p} \to e \) as \( \xi_\beta \to \infty \). Considering remaining terms of (17) we have, as \( \xi_\beta \to \infty \),
\[ (h_1 - \xi_\beta)^2 = \left[ \xi_\beta \left( 1 + \frac{1/p}{\xi_\beta^p} + \frac{2/p^2 - 2/p}{(\xi_\beta^p)^2} + o\left( \frac{1}{(\xi_\beta^p)^p} \right) \right) - \xi_\beta \right]^2 = \xi_\beta^p \left( \frac{1}{\xi_\beta^p} + \frac{2/p^2 - 2/p}{(\xi_\beta^p)^2} + o\left( \frac{1}{(\xi_\beta^p)^p} \right) \right)^2 \]
\[ = \frac{\xi_\beta^{2-2p}}{p^2} \left( 1 + \frac{4/p - 4}{\xi_\beta^p} + o\left( \frac{1}{(\xi_\beta^p)^p} \right) \right), \]
and finally,
\[ h_2 - (h_1)^2 \approx \xi_\beta^p \left[ s(3/p, \xi_\beta^p) s(1/p, \xi_\beta^p) - s(2/p, \xi_\beta^p) s(2/p, \xi_\beta^p) \right]. \]
Since,
\[ s(3/p, z) s(1/p, z) - s(2/p, z) s(2/p, z) = \frac{1/p^2}{z^2} + o\left( \frac{1}{z^2} \right), \]
\[ s(1/p, z) s(1/p, z) = 1 + \frac{2/p - 2}{z} + \frac{3/p^2 - 8/p + 5}{z^2} + o\left( \frac{1}{z^2} \right), \]
and
\[ \frac{s(3/p, z) s(1/p, z) - s(2/p, z) s(2/p, z)}{s(1/p, z) s(1/p, z)} = \frac{1/p^2}{z^2} + \frac{6/p^3 - 6/p^2}{z^3} + o\left( \frac{1}{z^2} \right), \]
we have, as \( \xi_\beta \to \infty \),
\[ h_2 - (h_1)^2 \approx \frac{\xi_\beta^{2-2p}}{p^2}. \]
Hence, as $\xi \rightarrow \infty$,

$$H(\beta, p) \approx \frac{[1 - 0]^{2-2p} e^{(h_1)p - \xi_0}}{p^2 \left[ \frac{\xi_0^{2-2p}}{p^2} + (1 - 0) \frac{\xi_0^{2-2p}}{p^2} \right]},$$

and we have $H(\beta, p) \rightarrow \frac{e^0}{2}$ as $\xi \rightarrow \infty$ ($\Leftrightarrow \beta \rightarrow 1$).

## 5 Proof of Theorem 3

We consider separately the cases $p \rightarrow 0$ (Case 3) and $p \rightarrow \infty$ (Case 4). In Case 3 we use a power series expansion for the incomplete Gamma function, together with Stirling’s formula for the Gamma function. The desired result then follows by careful consideration of a result of Tricomi (1950), which handles the subtle asymptotic behavior of the incomplete Gamma function when both arguments tend to infinity. Case 4 is elegantly tackled by considering the limiting distribution that results when $p \rightarrow \infty$. The computation of $H(\beta, \infty)$ is then straightforward for the limiting $U[-1, 1]$. Usage of the (well-defined) generalized quantile function to invert the CDF permits the interchange of limits and integrals, via the Lebesgue Dominated Convergence theorem, thus justifying the $U[-1, 1]$ computation. The change of variable $\beta = F_0(\xi)$ idea is used only in Case 3. Note that since the limiting EPD PDF when $p \rightarrow 0$ is zero everywhere, it would be meaningless to inquire about the rate of convergence of $H(\beta, p)$ if the method of Case 4 (interchange of limits and integrals) were employed.

### Case 3: $p \rightarrow 0$ (for fixed $\beta$)

Simplify the notation by setting $B \equiv \xi$. The fact that the EPD PDF and CDF converge to zero as $p \rightarrow 0$ (for all $\beta$), immediately gives $H(\beta, p) \rightarrow 0$. Although the representation of the terms in $H(\beta, p)$ depends on whether $B > 0$ or $B < 0$, the rate at which $H(\beta, p) \equiv H(B, x)$ approaches zero as $x = p^{-1} \rightarrow \infty$ is independent of the sign of $B$.

First note that from section 2 we have the following basic results:

\[
G_n(B) = \begin{cases} 
\frac{1}{2T(x)} \Gamma((n + 1)x, B^{1/x}), & \text{for } B > 0 \text{ and } n = 0, 1, 2; \\
1 - \frac{1}{2T(x)} \Gamma(x, |B|^{1/x}), & \text{for } B < 0 \text{ and } n = 0; \\
\frac{1}{2T(x)} \Gamma(2x, |B|^{1/x}), & \text{for } B < 0 \text{ and } n = 1; \\
\frac{1}{2T(x)} \Gamma(3x) - \frac{1}{2T(x)} \Gamma(3x, |B|^{1/x}), & \text{for } B < 0 \text{ and } n = 2. 
\end{cases}
\]

\[
h_n = \begin{cases} 
\frac{1}{2T(x)} \Gamma((n+1)x, B^{1/x}), & \text{for } B > 0 \text{ and } n = 1, 2; \\
\frac{1}{2T(x)} \Gamma(x, B^{1/x}), & \text{for } B < 0 \text{ and } n = 1; \\
\frac{1}{2T(x)} \Gamma(3x) - \frac{1}{2T(x)} \Gamma(3x, |B|^{1/x}), & \text{for } B < 0 \text{ and } n = 2. 
\end{cases}
\]

Now, letting

$$t(a, z) = 1 + \frac{z}{a + 1} + \frac{z^2}{(a + 1)(a + 2)} + o\left(\frac{1}{a^2}\right),$$

from Temme (1996, p. 186) we have $\gamma(a, z) = z^a e^{-z} t(a, z) / a$, whence, $\Gamma(a, z) = \Gamma(a) - z^a e^{-z} t(a, z) / a$. 

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Thus, for $s > 0$

$$\Gamma((n + 1)x, s^{1/x}) = \Gamma((n + 1)x) - \frac{e^{-s^{1/x}(n+1)x}}{(n + 1)x} \left( 1 + \frac{s^{1/x}}{(n + 1)x + 1} + o\left(\frac{1}{x}\right)\right)$$

$$= \Gamma((n + 1)x) - \frac{s^{n+1}e^{-s^{1/x}}}{(n + 1)x} \left( 1 + \frac{s^{1/x}}{(n + 1)x + 1} + o\left(\frac{1}{x}\right)\right)$$

$$= \Gamma((n + 1)x) - \frac{s^{n+1}e^{-B^{1/x}}}{(n + 1)x} + o\left(\frac{1}{x}\right).$$

(27)

From this it follows that for $B > 0$, we have, using the asymptotic behavior given in (27) for the incomplete Gamma function,

$$h_n = \frac{\Gamma(n + 1)x, B^{1/x}}{\Gamma(x, B^{1/x})} = \frac{\Gamma((n + 1)x) - \frac{B^{n+1}e^{-B^{1/x}}}{(n + 1)x} + o\left(\frac{1}{x}\right)}{\Gamma(x) - \frac{Be^{-B^{1/x}}}{x} + o\left(\frac{1}{x}\right)} = \frac{\Gamma((n + 1)x) + o\left(\frac{1}{x}\right)}{\Gamma(x)}$$

(28)

and, for $B < 0$,

$$h_1 = \frac{1}{2\Gamma(x)} \Gamma(2x, |B|^{1/x}) = \frac{1}{2\Gamma(x)} \left[ \Gamma(2x) - \frac{|B|^2 e^{-|B|^{1/x}}}{2x} + o\left(\frac{1}{x}\right) \right] = \frac{\Gamma(2x) - \frac{|B|^2 e^{-|B|^{1/x}}}{2x} + o\left(\frac{1}{x}\right)}{\Gamma(x) + \frac{|B|^2 e^{-|B|^{1/x}}}{2x} + o\left(\frac{1}{x}\right)}$$

$$= \frac{\Gamma(2x) + o\left(\frac{1}{x}\right)}{\Gamma(x)} + o\left(\frac{1}{x}\right).$$

(29)

$$h_2 = \frac{\Gamma(3x) - \frac{1}{2}\Gamma(3x, |B|^{1/x})}{1 - \frac{1}{2\Gamma(x)} \Gamma(x, |B|^{1/x})} = \frac{\Gamma(3x) - \frac{1}{2\Gamma(x)} \left[ \Gamma(3x) - \frac{|B|^3 e^{-|B|^{1/x}}}{3x} + o\left(\frac{1}{x}\right) \right]}{1 - \frac{1}{2\Gamma(x)} \left[ \Gamma(x) - \frac{|B|^2 e^{-|B|^{1/x}}}{x} + o\left(\frac{1}{x}\right) \right]}$$

$$= \frac{\Gamma(3x) + \frac{|B|^3 e^{-|B|^{1/x}}}{3x} + o\left(\frac{1}{x}\right)}{\Gamma(x) + \frac{|B|^2 e^{-|B|^{1/x}}}{x} + o\left(\frac{1}{x}\right)} = \frac{\Gamma(3x) + o\left(\frac{1}{x}\right)}{\Gamma(x)} + o\left(\frac{1}{x}\right).$$

(30)

Thus, from (28), (29) and (30), we note that $h_1$ and $h_2$ have the same asymptotic behavior irrespective of whether $B > 0$ or $B < 0$. Also,

$$1 - F_0(B) = G_0(B) = \begin{cases} \frac{1}{2\Gamma(x)} \Gamma(x, B^{1/x}), & B > 0; \\ 1 - \frac{1}{2\Gamma(x)} \Gamma(x, |B|^{1/x}), & B < 0. \end{cases}$$

so that in either case, we have, using the asymptotic behavior given in (27) for the incomplete Gamma function, $\lim_{x \to \infty} 1 - F_0(B) = \lim_{x \to \infty} G_0(B) = 1/2$. Finally, we have for the notational terms introduced in section 2,

$$\mu_\beta = h_1, \quad f(\mu_\beta) = \frac{e^{-(h_1)^{1/x}}}{2x\Gamma(x)}, \quad \mu_\beta - B = h_1 - B,$$

$$g_\beta \equiv g(\beta, p) = g(B, x) = 1 - G_0(h_1) = F_0(h_1), \quad \sigma^2_\beta \equiv \sigma^2(\beta, p) = \sigma^2(B, x) = h_2 - (h_1)^2,$$

which enables us to rewrite (17) as

$$H(\beta, p) = H(B, x) = \frac{g(B, x)(1 - g(B, x))G_0(B)}{(f(h_1))^2 \left[ \sigma^2(B, x) + F_0(B)(h_1 - B)^2 \right]}.$$

(31)
Using Prudnikov et al. (1986, p. 760) to represent the quotient $\Gamma(2x)/\Gamma(x)$ for large $x$, and using Stirling’s formula to also approximate $\Gamma(x)$ for large $x$, we have

$$
(h_1)^{1/x} \approx \left( \frac{\Gamma(2x)}{\Gamma(x)} \right)^{1/x} \approx \left( \frac{4^x \Gamma(x + 1/2)}{2\sqrt{\pi}} \right)^{1/x} \approx \left( \frac{4^x (x + 1/2)^x \sqrt{2\pi}}{2\sqrt{\pi} e^{x+1/2}} \right)^{1/x} \approx \frac{4(x + 1/2)}{2^{1/2} e^{1+1/2x}}
$$

This gives rise to the following corresponding approximations for the above notational terms:

$$
g(B, x) \approx 1 - \frac{\Gamma(x, \frac{4}{e} x - \frac{2}{e} \log 2)}{2\Gamma(x)}, \quad f(\mu_\beta) \approx \frac{e^{-\frac{1}{2} x + \frac{2}{e} \log 2}}{2x \Gamma(x)}, \quad 1 - F_0(B) = G_0(B) \approx \frac{1}{2},
$$

$$
\mu_\beta - B = h_1 - B \approx \frac{\Gamma(2x)}{\Gamma(x)} - B \approx \frac{\Gamma(3x)}{\Gamma(x)} - \left( \frac{\Gamma(2x)}{\Gamma(x)} \right)^2.
$$

These lead to the asymptotic estimate, as $x \to \infty$,

$$
H(B, x) \approx \frac{1 - \frac{\Gamma(x, \frac{4}{e} x - \frac{2}{e} \log 2)}{2\Gamma(x)}}{\left( \frac{\Gamma(3x)}{\Gamma(x)} - \frac{\Gamma(2x)}{\Gamma(x)} - B \right)^2}
$$

$$
\approx \frac{1 - \frac{\Gamma(x, \frac{4}{e} x - \frac{2}{e} \log 2)}{2\Gamma(x)}}{\left( \frac{\Gamma(3x)}{\Gamma(x)} - \frac{\Gamma(2x)}{\Gamma(x)} - B \right)^2}
$$

$$
\approx \frac{1 - \frac{\Gamma(x, \frac{4}{e} x - \frac{2}{e} \log 2)}{2\Gamma(x)}}{\left( \frac{\Gamma(3x)}{\Gamma(x)} - \frac{\Gamma(2x)}{\Gamma(x)} - B \right)^2}
$$

$$
\approx \frac{1 - \frac{\Gamma(x, \frac{4}{e} x - \frac{2}{e} \log 2)}{2\Gamma(x)}}{\left( \frac{\Gamma(3x)}{\Gamma(x)} - \frac{\Gamma(2x)}{\Gamma(x)} - B \right)^2}
$$

$$
\approx \frac{1 - \frac{\Gamma(x, \frac{4}{e} x - \frac{2}{e} \log 2)}{2\Gamma(x)}}{\left( \frac{\Gamma(3x)}{\Gamma(x)} - \frac{\Gamma(2x)}{\Gamma(x)} - B \right)^2}
$$

$$
\approx \frac{1 - \frac{\Gamma(x, \frac{4}{e} x - \frac{2}{e} \log 2)}{2\Gamma(x)}}{\left( \frac{\Gamma(3x)}{\Gamma(x)} - \frac{\Gamma(2x)}{\Gamma(x)} - B \right)^2}
$$

Following Tricomi (1950), we can now estimate $\Gamma(x, 4e^{-1}x - 2e^{-1}\log 2)$, since both arguments tend to infinity with comparable magnitude. Writing

$$
\Gamma \left( x, \frac{4}{e} x - \frac{2}{e} \log 2 \right) = \Gamma(1 + a, z),
$$

Tricomi (1950) noted that if $w = a^{1/2}/(z - a)$ tended to 0, then, defining

$$
u(a, z) = 1 - \frac{1}{(x - a)^2} + \frac{2a}{(x - a)^3} + O\left( \frac{|a|^2}{(z - a)^4} \right),
$$

results in the estimate

$$
\Gamma(1 + a, z) = \frac{z^{1+a} e^{-z}}{z - a} u(a, z).
$$

Hence, applying (33) with $a = x - 1$ and $z = 4e^{-1}x - 2e^{-1}\log 2$ to (34), we obtain

$$
\Gamma \left( x, \frac{4}{e} x - \frac{2}{e} \log 2 \right) \approx \frac{\left( \frac{4}{e} x - \frac{2}{e} \log 2 \right) e^{-\frac{1}{2} x + \frac{2}{e} \log 2}}{\frac{4}{e} x - \frac{2}{e} \log 2 - x + 1},
$$

and substitution of (35) into (32) gives the desired result.
Case 4: $p \to \infty$ (for fixed $\beta$)

Denote by $f(x; p)$ and $F(x; p)$ the EPD PDF and CDF, respectively, for given shape parameter $p$. Denote by $1_A(x)$ the indicator function for set $A$, which takes on the value 1 if $x \in A$, and 0 otherwise. Formally define $f(x; \infty) = \lim_{p \to \infty} f(x; p)$ and $F(x; \infty) = \lim_{p \to \infty} F(x; p)$. Note that $f(x; \infty) = \frac{1}{2} 1_{[-1,1]}(x)$ is a uniform distribution on $[-1, 1]$. Thus, $F(x; \infty) = \frac{1 + x}{2} 1_{[-1,1]}(x) + 1_{(1,\infty)}(x)$, and we have that $F(x; p)$ converges uniformly to $F(x; \infty)$ on $\mathbb{R}$ as $p \to \infty$. Throughout, we will let $F^-(x; \infty)$ denote the generalized inverse of $F(x; \infty)$, also known as the generalized quantile function (Embrechts and Hofert, 2013). Thus, we have $F^-(x; \infty) = 2x - 1$ for $0 < x \leq 1$.

Straightforward calculations yield

$\mu_\beta(\infty) \equiv \mu(\beta, \infty) = \frac{1}{1-\beta} \int_\beta^1 F^-(u; \infty) du = \beta$, \quad $g_\beta(\infty) \equiv g(\beta, \infty) = F(\mu(\beta, \infty), \infty) = \frac{1 + \beta}{2}$, \quad $\sigma_\beta^2(\infty) \equiv \sigma^2(\beta, \infty) = \frac{1}{1-\beta} \int_\beta^1 [F^-(u; \infty) - \mu(\beta, \infty)]^2 du = \frac{(1-\beta)^2}{3}$,

and hence,

$H(\beta, \infty) \equiv \frac{g_\beta(\infty)(1 - g_\beta(\infty))}{[f(\mu_\beta(\infty); \infty)]^2}. \quad \frac{1 - \beta}{\sigma_\beta^2(\infty) + \beta[\mu_\beta(\infty) - F^-(\beta; \infty)]^2} = \frac{1 + \beta}{1/3 + \beta}$.

On the other hand, recall

$\mu_\beta = \mu(\beta, p) = \frac{1}{1-\beta} \int_\beta^1 F^-(u; p) du, \quad g_\beta = g(\beta, p) = F(\mu(\beta, p); p)$, \quad $\sigma_\beta^2 = \sigma^2(\beta, p) = \frac{1}{1-\beta} \int_\beta^1 [F^-(u; p) - \mu(\beta, p)]^2 du$.

For $0 < \delta < 1$, define

$e_\alpha(\delta) = \lim_{p \to \infty} \int_\delta^\infty t^\alpha f(t, p) dt$, \quad and note that $\lim_{p \to \infty} \frac{p}{2^\alpha(1/p)} = 1/2$. Since, for $n = 0, 1, 2$ and for $p \geq 2$ and $t \geq 1$, we have $t^n e^{-\theta t} \leq e^{-\theta t/2} \leq e^{-t}$, we see, applying the Lebesgue Dominated Convergence theorem, that

$e_\alpha(\delta) = \lim_{p \to \infty} \int_\delta^\infty t^n f(t, p) dt = 1/2 \int_\delta^\infty t^n 1_{[-1,1]}(t) dt = \frac{1 - \delta^{n+1}}{2(n+1)}$. \quad (36)$

Define $B = B(p)$ by $F(B; p) = \beta$. Then, making a change of variable $u = F(t; p)$ we obtain

$\mu_\beta = \mu(\beta, p) = \frac{1}{1-\beta} \int_{B(p)}^\infty t f(t; p) dt, \quad g_\beta = g(\beta, p) = F(\mu(\beta, p); p)$, \quad $\sigma_\beta^2 = \sigma^2(\beta, p) = \frac{1}{1-\beta} \int_{B(p)}^\infty \nu(t) \mu(\beta, p) f(t; p) dt$.

Since $e_\alpha(\delta)$ given by (36) is continuous in $\delta$, and since for $0 < \beta \leq 1$ the uniform convergence of $F(x; p)$ to $F(x; \infty)$ implies

$\lim_{p \to \infty} B(p) = \lim_{p \to \infty} F^-(\beta; p) = F^-(\beta; \infty) = 2\beta - 1$,
then we have

\[
\lim_{p \to \infty} \mu(\beta, p) = \lim_{p \to \infty} \frac{1}{1 - \beta} \int_{\beta}^{1} F^-(u; p) du = \lim_{p \to \infty} \frac{1}{1 - \beta} \int_{B(p)}^{\infty} tf(t; p) dt
\]

\[
\lim_{p \to \infty} \mu(\beta, p) = \lim_{p \to \infty} \frac{1}{1 - \beta} e_1(B(p)) = \frac{1}{1 - \beta} e_1(2\beta - 1) = \beta,
\]

\[
\lim_{p \to \infty} g(\beta, p) = \lim_{p \to \infty} F(\mu(\beta; p); p) = F(\beta; \infty) = \frac{1 + \beta}{2},
\]

\[
\lim_{p \to \infty} \sigma^2(\beta, p) = \lim_{p \to \infty} \frac{1}{1 - \beta} \int_{\beta}^{1} [F^-(u; p) - \mu(\beta, p)]^2 du = \lim_{p \to \infty} \frac{1}{1 - \beta} \int_{B(p)}^{\infty} (t - \mu(\beta, p))^2 f(t; p) dt
\]

\[
\lim_{p \to \infty} \sigma^2(\beta, p) = \lim_{p \to \infty} \frac{1}{1 - \beta} \left[ e_2(B(p)) - 2\mu(\beta, p)e_1(B(p)) + \mu^2(\beta, p)e_0(B(p)) \right]
\]

\[
= \frac{1}{1 - \beta} \left[ e_2(2\beta - 1) - 2\beta e_1(2\beta - 1) + \beta^2 e_0(2\beta - 1) \right] = \frac{(1 - \beta)^2}{3}.
\]

Thus, we obtain

\[
\lim_{p \to \infty} H(\beta, p) = H(\beta, \infty) = \frac{1 + \beta}{1/3 + \beta}.
\]

(37)

6 Discussion

We established limiting results concerning the ratio of asymptotic variances of the classical empirical estimators of location, tail median vs. tail mean, in the context of the flexible EPD family of distributions. The most remarkable result concerned the fact that in the limit of the right tail, the asymptotic variance of the tail median is approximately 36% larger than that of the tail mean, irrespective of the EPD shape parameter. Equating “efficiency” with “reduction in variance”, this translates equivalently into the tail mean being approximately 26% more efficient than the tail median. This is the central message of this paper, with possible important repercussions for QRM practitioners with regard to choice of risk measure: VaR or CVaR. We speculate that this result may hold more generally for a large class of distributions with finite variance. The findings also offer a generalization of the solution to the age-old statistical quandary: which of the sample median vs. sample mean is the most efficient estimator of centrality?

References


