$\mathbb{R}^n$, $\{v_1, v_2, v_3, \cdots, v_k\} \subset \mathbb{R}^n$

Case i) $k < n$ Case ii) $k = n$ Case iii) $k > n$

Example

$$\mathbb{R}^4,\ v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -2 \end{bmatrix}, v_5 = \begin{bmatrix} 0 \\ -1 \\ 3 \\ -1 \end{bmatrix}, v_6 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

Do there exist $c_1, c_2, c_3, c_4, c_5, c_6$ not all zero so that

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 + c_6v_6 = 0$$

Equivalent to question does the matrix equation

$$\begin{bmatrix} -1 & 2 & -1 & 3 & 0 & -1 \\ 2 & -3 & 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 1 & 3 & 1 \\ 2 & -1 & 0 & -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = 0$$

have non-trivial solutions. However, the coefficient matrix can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{-10}{9} \\ 0 & 1 & 0 & 0 & 1 & \frac{-2}{3} \\ 0 & 0 & 1 & 0 & 2 & \frac{13}{9} \\ 0 & 0 & 0 & 1 & 0 & \frac{2}{9} \end{bmatrix}$$

Example

$$\mathbb{R}^4,\ v_1 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ -3 \\ 1 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, v_4 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -2 \end{bmatrix}$$

Do there exist c_1, c_2, c_3, c_4 not all zero so that

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$$

Equivalent to question does the matrix equation

$$\begin{bmatrix} -1 & 2 & -1 & 3 \\ 2 & -3 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$$

have non-trivial solutions. However, the coefficient matrix can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Alternatively, the determinant of the coefficient matrix is 18

Example

$$\mathbb{R}^4, \quad v_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \\ -1 \end{bmatrix}, v_3 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

Do there exist c_1, c_2, c_3 not all zero so that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

Equivalent to question does the matrix equation

$$\begin{bmatrix} 3 & 0 & -1 \\ -1 & -1 & 1 \\ 1 & 3 & 1 \\ -2 & -1 & -2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0$$

have non-trivial solutions. However, the coefficient matrix can be row reduced to

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

P_n , $\{v_1, v_2, v_3, \cdots, v_k\} \subset P_n$

Case i) $k < n$ Case ii) $k = n$ Case iii) $k > n$

Example

$$P_4 \text{ , } \begin{array}{l} v_1 = 2 - 3x + 4x^3, v_2 = -1 + 2x - x^2 + 2x^3, v_3 = x - 3x^2 + x^3, \\ v_4 = -1 + 2x + 4x^2, v_5 = 4 - x + 2x^2 + 5x^3, v_6 = -3 + 2x + x^2 - 7x^3 \end{array}$$

Do there exist $c_1, c_2, c_3, c_4, c_5, c_6$ not all zero so that

$$c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 + c_5v_5 + c_6v_6 = 0$$

is equivalent to the formulation do there exists $c_1, c_2, c_3, c_4, c_5, c_6$ not all zero so that

const $2c_1 - c_2 - c_4 + 4c_5 - 3c_6 = 0$

x $-3c_1 + 2c_2 + c_3 + 2c_4 - c_5 + 2c_6 = 0$

x^2 $-c_2 - 3c_3 + 4c_4 + 2c_5 + c_6 = 0$

x^3 $4c_1 + 2c_2 + c_3 + 5c_5 - 7c_6 = 0$

which is equivalent to the question does matrix equation

$$\begin{bmatrix} 2 & -1 & 0 & -1 & 4 & -3 \\ -3 & 2 & 1 & 2 & -1 & 2 \\ 0 & -1 & -3 & 4 & 2 & 1 \\ 4 & 2 & 1 & 0 & 5 & -7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix} = 0$$

have non-trivial solutions. However, the coefficient matrix is row equivalent to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{68}{37} & \frac{-61}{37} \\ 0 & 1 & 0 & 0 & \frac{-139}{37} & \frac{36}{37} \\ 0 & 0 & 1 & 0 & \frac{191}{37} & \frac{-87}{37} \\ 0 & 0 & 0 & 1 & \frac{127}{37} & \frac{-47}{37} \end{bmatrix}$$

$\mathbb{R}^{m\times n}$, $\{v_1, v_2, v_3, \cdots, v_k\} \subset \mathbb{R}^{m\times n}$

Case i) $k < m\times n$ Case ii) $k = m\times n$ Case iii) $k > m\times n$

Example

$$\mathbb{R}^{2\times 3}, \quad \begin{array}{l} v_1 = \begin{bmatrix} -2 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & -1 \end{bmatrix}, \\ v_3 = \begin{bmatrix} 0 & -1 & 1 \\ -2 & 1 & 0 \end{bmatrix}, \; v_4 = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \end{array}$$

Do there exist c_1, c_2, c_3, c_4 not all zero so that

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

is equivalent to the formulation do there exist c_1, c_2, c_3, c_4 not all zero so that

$a_{1,1} \quad -2c_1 + c_2 - c_4 = 0$

$a_{1,2} \quad c_2 - c_3 + c_4 = 0$

$a_{1,3} \quad 2c_1 - 3c_2 + c_3 - c_4 = 0$

$a_{2,1} \quad c_1 - 2c_3 - c_4 = 0$

$a_{2,3} \quad -c_1 + 2c_2 + c_3 = 0$

$a_{2,3} \quad c_1 - c_2 + 2c_4 = 0$

which is equivalent to the question does the matrix equation

$$\begin{bmatrix} -2 & 1 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 2 & -3 & 1 & -1 \\ 1 & 0 & -2 & -1 \\ -1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = 0$$

The coefficient matrix is row reducible to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$