Extra Credit Problems

Let Prop_LUB be the proposition that in $\mathbb{R}$ that every set S that is bounded above has a least upper bound. Let Prop_Cauchy be the proposition that in $\mathbb{R}$ that every Cauchy sequence is convergent.

Extra Credit Problem #1. Show that if Prop_Cauchy is taken as an axiom, then Prop_LUB follows as a theorem.

Extra Credit Problem #2. Show that if Prop_LUB is taken as an axiom, then Prop_Cauchy follows as a theorem.

Extra Credit Problem #3. In Cantor's Theorem (Thm II.3.7), there are three (Cantor) conditions on a family of sets $\{F_n\}$, n=1, 2, 3, ... :

1. Each set F_n is non-empty and closed.
2. The sets $\{F_n\}$ are nested, i.e., for each n we have $F_n \supset F_{n+1}$.
3. $diam\, F_n \to 0$

The conclusion of Cantor's Theorem is that in a complete metric space that if all of the Cantor conditions hold, then

$\bigcap_{n=1}^{\infty} F_n$ is non-empty and consists of a single point.

Give examples (in $\mathbb{R}$ or $\mathbb{C}$), for each Cantor condition, which show that if the Cantor condition does not hold (but the other two Cantor conditions do hold), then either $\bigcap_{n=1}^{\infty} F_n$ will be empty or consist of more than one point. If possible give examples, for each Cantor condition, which show that both cases are possible.