Notes for April 28

Function

A rule, usually given by formula or equation, for assigning or calculating values. Typically, represented in the form $y = f(x)$. Examples,

$$y = 6x - 3 \qquad y = x^3 - 6x^2 + 18 \qquad y = 9t^2 + 2$$

$$y = \sqrt{1 - x^2} \qquad y = \frac{6x^3 - x}{x^2 - x - 6} \qquad s = -16t^2 + 60t + 20$$

In the form, $y = f(x)$, x is the so called independent variable and y is the dependent variable.

y Dependent variable Output	←	Rule Formula Calculation	←	x Independent Variable Input

For a function, $y = f(x)$, the set of all permissible x values is called the domain of the function. E.g.,

$y = 6x - 3$ The domain is all real numbers x or $\mathbb{R}$ or $(-\infty, \infty)$

$y = x^3 - 6x^2 + 18$ The domain is all real numbers x or $\mathbb{R}$ or $(-\infty, \infty)$

$y = \sqrt{1 - x^2}$ The domain is $\{ x \mid -1 < x < 1\}$ or $(-1,1)$

$y = \dfrac{6x^3 - x}{x^2 - x - 6}$ The domain is the set of real numbers not equal to –2,3 or $(-\infty, -2) \cup (-2, 3) \cup (3, \infty)$

Graph

The graph of a function $y = f(x)$ is the set $\{ (x,y) \mid y = f(x) \}$. The graph is a subset of the Cartesian plane $\mathbb{R}^2$. Typically, one thinks of the graph as a curve in $\mathbb{R}^2$ which lies "above" the domain, considered as a subset of the x-axis. We look at graphs because we can visually interpret the graph to tell us about properties of the function.

To create a graph for a function $y = f(x)$,

First, one creates a table of x,y values. E.g.

$y = 6x - 3$

x	y
-3	-21
-2	-15
-1	-9
0	-3
1	3
2	9
3	15

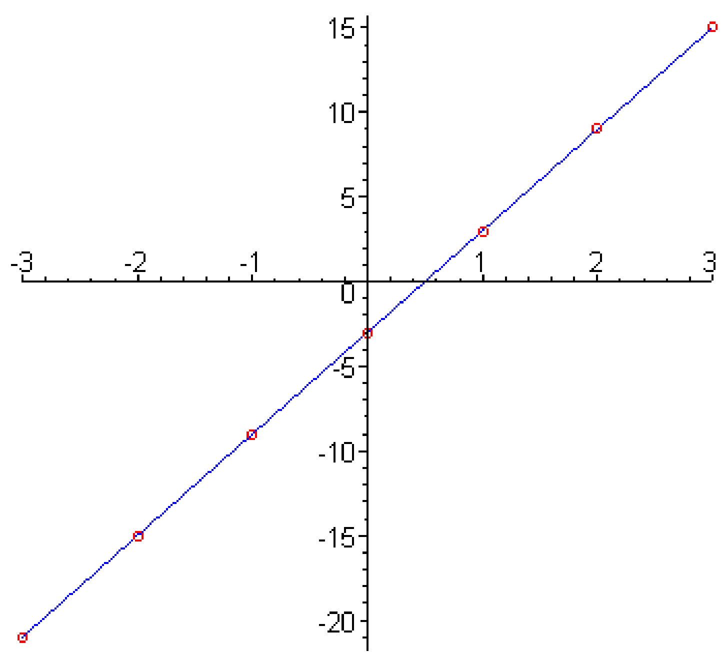
$y = x^3 - 6x^2 + 18$

x	y
-3	-63
-2	-14
-1	11
0	18
1	13
2	2
3	-9
4	-14
5	-7
6	18
7	67

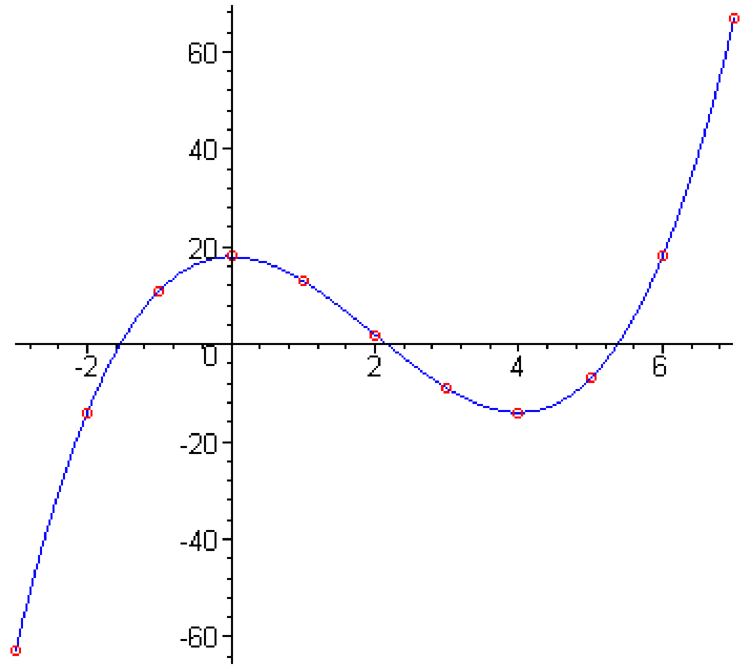
Second, one plots the table points (x,y) in the Cartesian plane $\mathbb{R}^2$.

Third, one connects the plotted (x,y) in $\mathbb{R}^2$, as ordered by the first coordinate.

For, $y = 6x - 3$



For $y = x^3 - 6x^2 + 18$



Plotting in Maple

The command for generating plots in Maple is

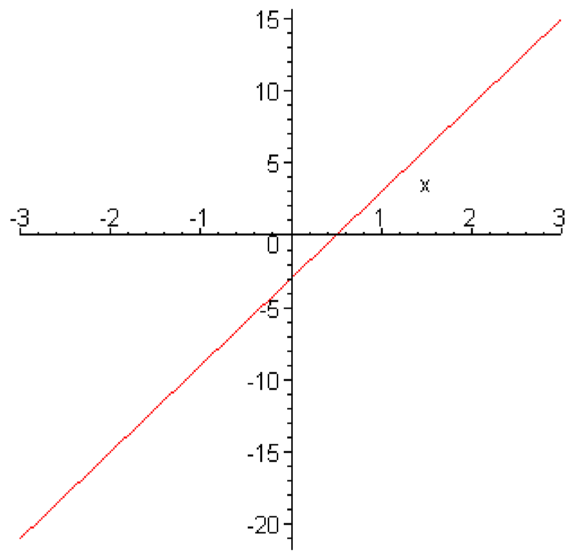
plot(**expr**, **x=a .. b**, [**y=c .. d**, **opt1**,...])

where expr is a symbolic expression in the variable x. E.g.,

```
> f1 := 6*x - 3;
```

$$f1 := 6\,x - 3$$

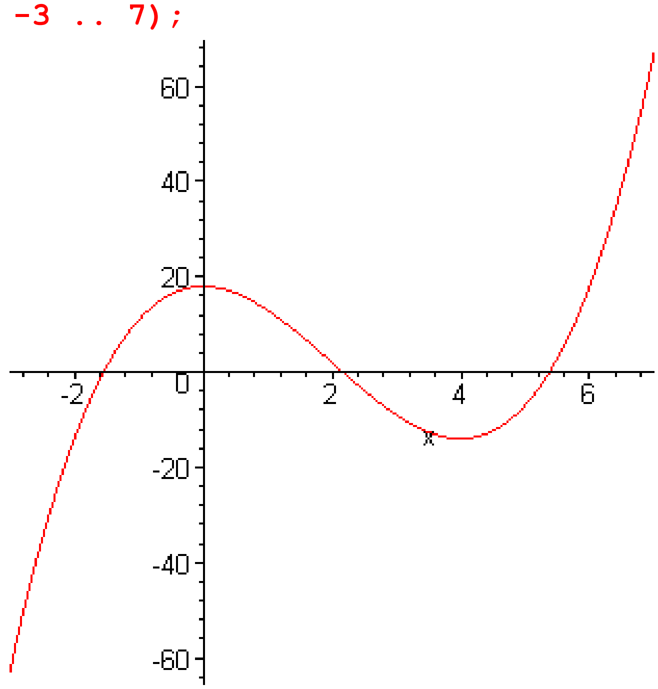
```
> plot(f1,x = -3 .. 3);
```




```
> f2 := x^3-6*x^2+18;
```

$$f2 := x^3 - 6\,x^2 + 18$$

```
> plot(f2,x = -3 .. 7);
```



Options include:

Restricting the vertical range in the plot:
y = c . . d

Specifying the thickness of the curve:
thickness = n,   where n = 1, 2, 3, 4 or 5

Specifying the color of the curve:
color = value,   where value = red, green, blue, etc.

Plotting two curves on the same graph:

plot ([f1, f2], x = a .. b, options);

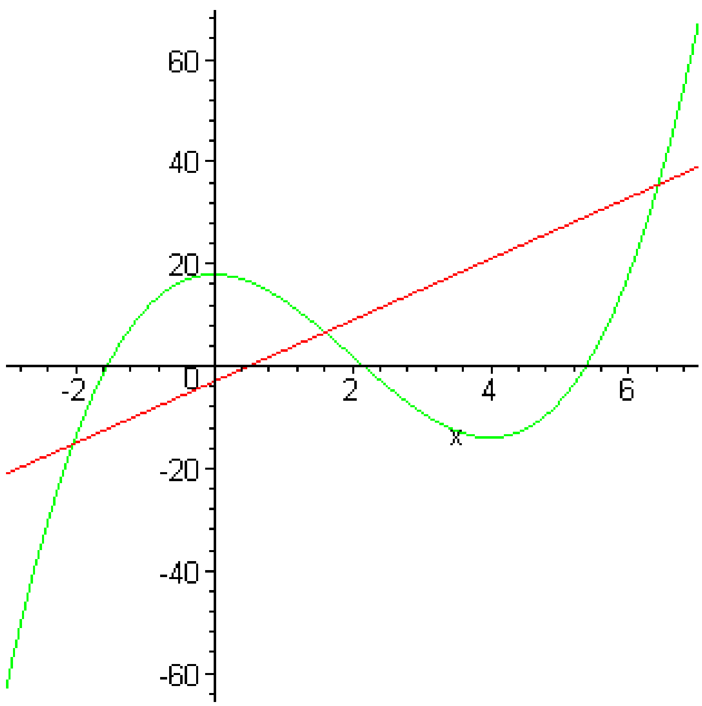
Note: using brackets forces Maple to plot f1 as the "first" function and f2 as the "second" function. By default the first function is plotted in red and the second function is plotted in green. E.g.

```
> f1 := 6*x - 3;
  f2 := x^3-6*x^2+18;
```

$$f1 := 6\,x - 3$$

$$f2 := x^3 - 6\,x^2 + 18$$

```
> plot([f1,f2],x = -3 .. 7);
```



Polynomials - Intercepts and Sign:

In graphing polynomials, the following quantities are standardly identified on the graph, to provide quantitative information about the behavior of the polynomial.

x-intercepts	Points on the x-axis where the graph crosses the axis Obtained by solving f(x) = 0 Maple: > `solve(f=0,x);` or Maple: > `fsolve(f=0,x);`
Positivity	Intervals in the domain (subsets of the real line) where the graph of the polynomial lies above the x-axis Bounded by the x-intercepts and may stretch to $\pm\infty$
Negativity	Intervals in the domain (subsets of the real line) where the graph of the polynomial lies below the x-axis Bounded by the x-intercepts and may stretch to $\pm\infty$
y-intercept	Point on the y-axis where the graph crosses the axis Obtained by setting x=0 in the function Maple: > `subs(x=0,f);`
End Limits	Behavior of the values of the function (y values) for large (positive or negative) values if the independent variable x.

E.g., for $f2 = x^3 - 6x^2 = 18$

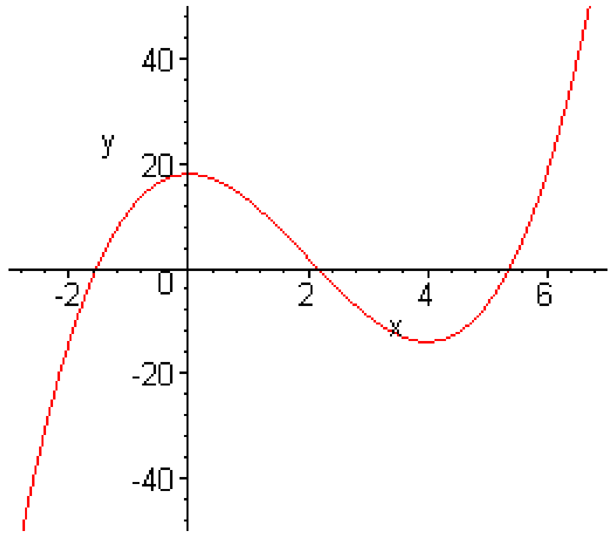
```
> f2 := x^3-6*x^2+18;
```

$$f2 := x^3 - 6\,x^2 + 18$$

```
> fsolve(f2=0,x);
```

$$-1.544606815,\ 2.167055173,\ 5.377551642$$

```
> plot(f2,x=-3..7,y=-50..50);
```




```
> subs(x=0,f2);
```

$$18$$

\>

a. x-intercepts: -1.544606815, 2.167055173, 5.377551642

b. Intervals on which f2 is positive: (-1.544606815 , 2.167055173) and (5.377551642 , ∞)

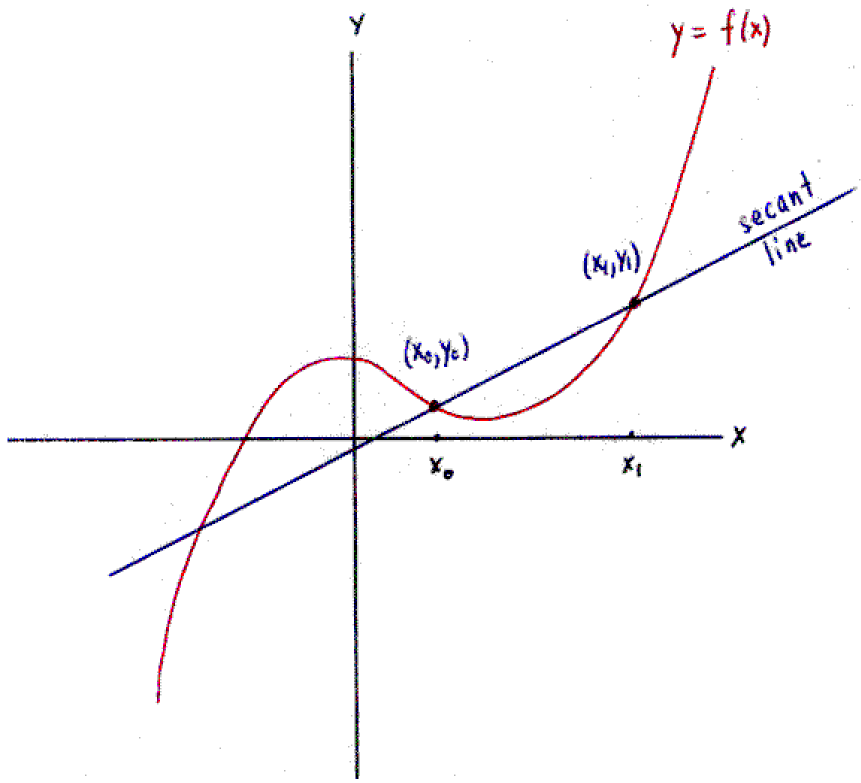
c. Intervals on which f2 is negative: (- ∞ , -1.544606815) and (2.167055173 , 5.377551642)

d. y-intercept: 18

e. For large positive x, f2 tends to + ∞
For large negative x, f2 tends to - ∞

Calculus - Interpretation of the Derivative

Consider the function $y = f(x)$. Fix a base point x_0 in the domain and select a nearby point x_1. Consider the secant line to the graph of $y = f(x)$ determined by the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. See the figure below.



There are two useful, specific ways to interpret the information represented in the above graph.

1. Slope of the secant line. The slope of the secant line to the graph of $y = f(x)$ determined by the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is given by

$$slope = \frac{y_1 - y_0}{x_1 - x_0}$$

The slope measures the steepness of the secant line: the larger the slope, the steeper the line. The sign of the slope identifies whether the secant line is rising or falling. If the slope is positive, then the secant line is rising (as you move from left to right). If the slope is negative, then the secant line is falling (as you move from left to right). If the slope is 0, then the secant line is flat or horizontal.

2. Average rate of change of $y = f(x)$. The difference $y_1 - y_0 = \Delta y$ is the change in the values of the function $y = f(x)$ over the interval (x_0, x_1). The difference $x_1 - x_0 = \Delta x$ is the change in the values of x over the interval (x_0, x_1). Their ratio

$$\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$$

is called the <u>difference quotient</u> and measures the average rate change of the function $y = f(x)$ over the interval (x_0, x_1).

Of course, the calculated value in 1. above, the slope of the secant line to the graph of $y = f(x)$ determined by the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$, and the calculated value in 2. above, the difference quotient measuring the average rate change of the function $y = f(x)$ over the interval (x_0, x_1)., are the same.

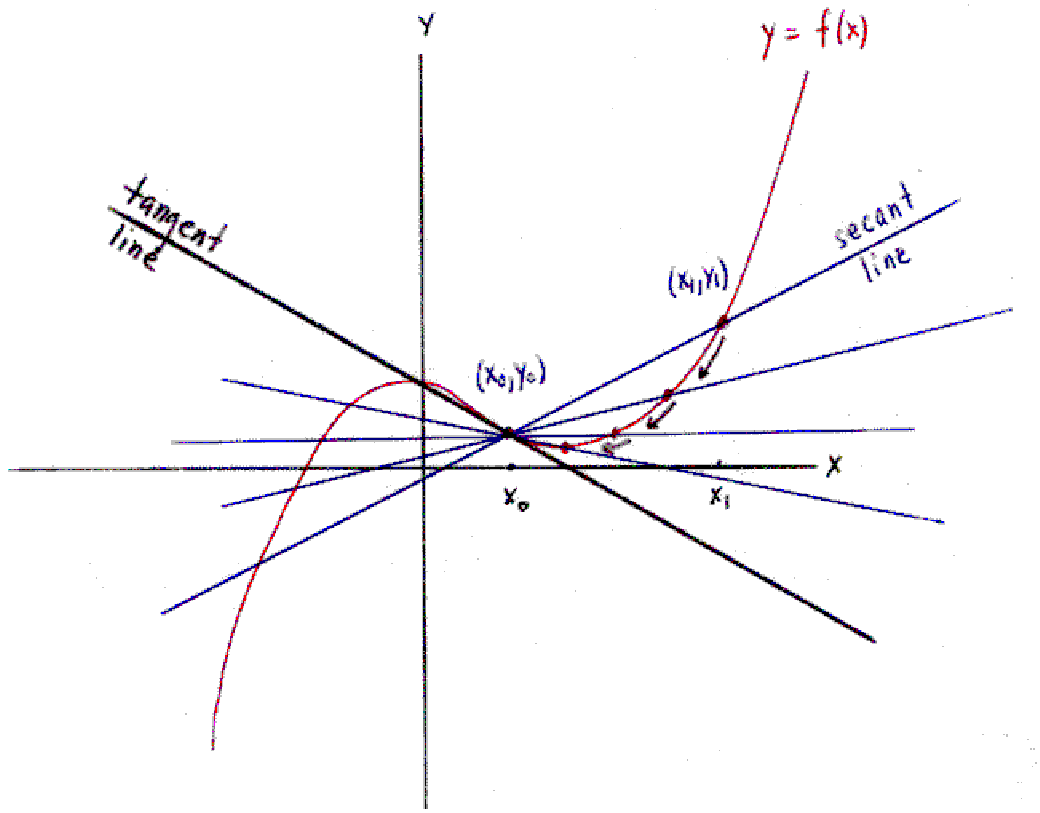
<u>Limiting Process</u>

Suppose that we now let the nearby point x_1 approach the base point x_0. Then, two things happen. See figure below.

First, the secant line shifts with x_1 and approaches the <u>tangent line</u> to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$; consequently, the slope of the secant line shifts with x_1 and approaches the <u>slope of the tangent line</u> to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$.

Second, the interval (x_0, x_1) over which difference quotient was being computed, which measured the average rate change of the function

$y = f(x)$, shrinks to the point x_0; consequently, the average rate change of the function $y = f(x)$ over the interval (x_0, x_1) approaches the instantaneous rate of change of the function $y = f(x)$ at the point x_0.



It would be important, useful, advantageous, mathematically significant if there were a way to calculate the limiting value arising in the process above (which is the same in either case). This limiting process, which is described above, is the basis for what is done in Calculus. We call this limiting value the derivative of the function $y = f(x)$ at the point x_0. There are two interpretations of what the derivative of the function $y = f(x)$ at the point x_0 means.

A. The derivative of the function $y = f(x)$ at the point x_0 can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$.

B. The derivative of the function $y = f(x)$ at the point x_0 can be interpreted as approaches the instantaneous rate of change of the function $y = f(x)$ at the point x_0.

In a regular Calculus course, students learn a suite of rules for computing the derivative of a function $y = f(x)$. Then, having mastered those rules they go on to interpreting them in applications. Maple has built into it a function for computing derivatives. If **f** is a symbolic expression in **x**, then the syntax for computing the derivative of **f** with respect to **x** is:

- **diff(f,x);**

Consider the following example:

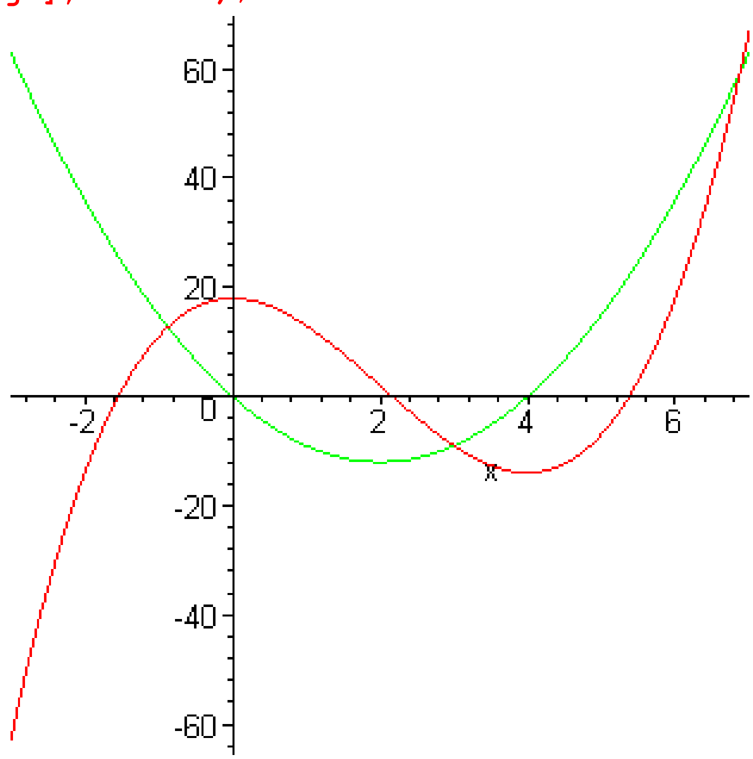
```
> f1 := x^3 - 6*x^2 + 18;
```

$$f1 := x^3 - 6\,x^2 + 18$$

```
> g1 := diff(f1,x);
```

$$g1 := 3\,x^2 - 12\,x$$

```
> plot([f1,g1],x=-3..7);
```



In the above graph, $f1$ is plotted in red and its derivative $g1$ is plotted in green. At each point x the value of $g1$, tells about the slope of the tangent line to the graph of $f1$. Where $g1$ is positive, then the slope to the tangent line to the graph of $f1$ is positive or alternately the graph of $f1$ is locally increasing. Where $g1$ is negative, then the slope to the tangent line to the graph of $f1$ is negative or alternately the graph of $f1$ is locally decreasing. Where $g1$ is 0 (where $g1$ crosses the x-axis), the slope of the tangent line to the graph of $f1$ is 0 (the tangent line to the graph of $f1$ is horizontal or flat).

In general, we can use the derivative of a function $y = f(x)$ to tell us information about the function $y = f(x)$. Specifically, we can deduce the following three points:

1. Let (a,b) be an interval in the domain of $y = f(x)$ on which the derivative of $y = f(x)$ is positive, then the function $y = f(x)$ is increasing on that interval (a,b) [rising as you move from left to right].

2. Let (a,b) be an interval in the domain of $y = f(x)$ on which the derivative of $y = f(x)$ is negative, then the function $y = f(x)$ is decreasing on that interval (a,b) [falling as you move from left to right].

3. Let x^* be a point in the domain of $y = f(x)$ at which the derivative of $y = f(x)$ is 0. We will call such a point a <u>critical point</u> of $y = f(x)$. Then, at x^* there is a horizontal tangent line to the graph of $y = f(x)$.

<u>Critical Points and Local Extreme Values and Monotonicity</u>

We say that a point $(x^*, f(x^*))$ is a <u>local maximum</u> point on the graph of $y = f(x)$ if the value $f(x^*) \geq f(x)$ for all x near x^*, i.e., the value $f(x^*)$ is the largest (or highest) value of $y = f(x)$ for nearby x. We say that a point $(x^*, f(x^*))$ is a <u>local minimum</u> point on the graph of $y = f(x)$ if the value $f(x^*) \leq f$

(x) for all x near x^*, i.e., the value $f(x^*)$ is the smallest (or lowest) value of $y = f(x)$ for nearby x.

Suppose that a function $y = f(x)$ has a derivative on an interval (a,b). If $y = f(x)$ has a local extreme value (local maximum or local minimum) on the interval (a,b), then because of the three points deduced above, the local extreme value **must** occur at a critical point, i.e., at a local extreme point the derivative of the function must be 0.

In graphing polynomials, we can, using the information from their derivatives, add the following quantities to our standard list of identified characteristics of the graph, to provide quantitative information about the behavior of the polynomial.

Critical Points	Points on the x-axis where the graph of the derivative crosses the axis. If g is the derivative of f, then the critical points are obtained by solving g(x) = 0 Maple: > `solve(g=0,x);` or Maple: > `fsolve(g=0,x);`
Local Maximums	Points (x^*, y^*) on the graph of $y = f(x)$ which are locally the highest points. If x^* is a critical point (obtained above), then the value y^* is obtained by substituting x^* into the formula for the function $y = f(x)$ Maple: > `subs(x=`x^*`,f);`

Local Minimums	Points (x^* , y^*) on the graph of $y = f(x)$ which are locally the lowest points. If x^* is a critical point (obtained above), then the value y^* is obtained by substituting x^* into the formula for the function $y = f(x)$ Maple: > `subs(x=`x^*`,f);`
Increasing	Intervals in the domain (subsets of the real line) where the graph of the polynomial is increasing which can be identified by finding intervals in the domain where the graph of the derivative of the polynomial is positive. Bounded by the critical points of the function and may stretch to $\pm\infty$
Decreasing	Intervals in the domain (subsets of the real line) where the graph of the polynomial is decreasing which can be identified by finding intervals in the domain where the graph of the derivative of the polynomial is negative. Bounded by the critical points of the function and may stretch to $\pm\infty$

E.g., for $f2 = x^3 - 6x^2 = 18$

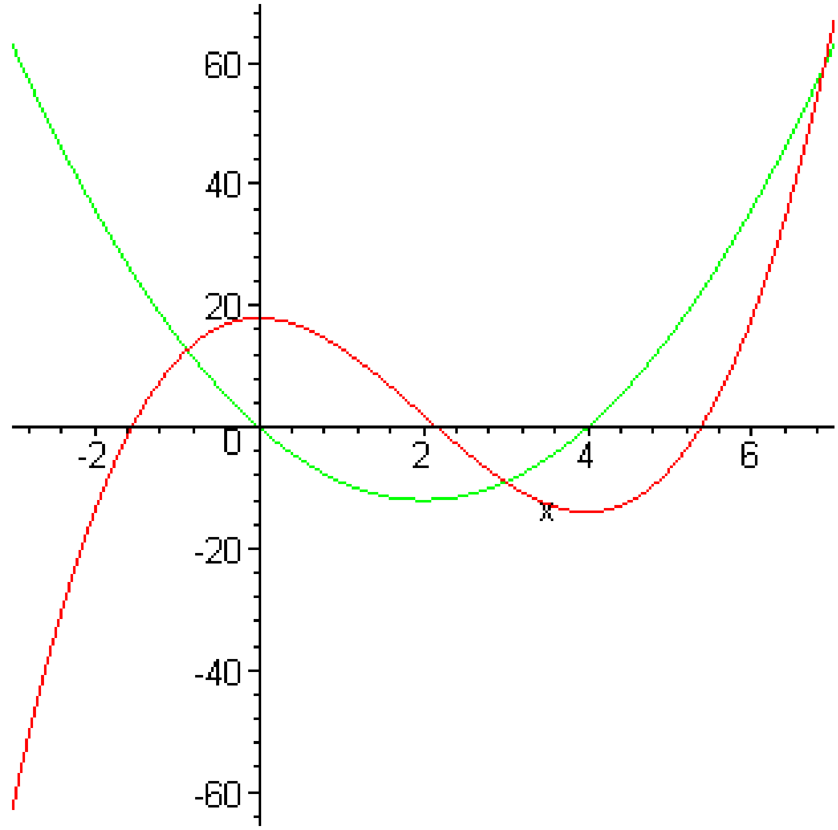
```
> f2 := x^3 - 6*x^2 + 18;
```

$$f2 := x^3 - 6\,x^2 + 18$$

```
> g2 := diff(f1,x);
```

$$g2 := 3\,x^2 - 12\,x$$

```
> plot([f2,g2],x=-3..7);
```




```
> fsolve(g2,x);
```

0., 4.

```
> subs(x=0,f2);
```

18

```
> subs(x=4,f2);
```

-14

A. critical points : 0., 4.
B. local maximum: (0,18)
C. local minimum: (4,-14)
D. increasing: (- ∞ ,0) and (4, ∞)
E. decreasing: (0,4)