Underlined Definitions: May be asked for on exam
Underlined Propositions or Theorems: Proofs may be asked for on exam
Underlined Homework Exercises: Problems may be asked for on exam
Double Underlined Homework Exercises: Similar problems will be asked for on exam
Double Underlined Named Theorems/Results: Statements may be asked for on exam

Chapter 7.1

Homework 7.1 Page 150 1, 2, 4, 5, 7, 8

Definition. Let $G$ be a region and let $(\Omega, d)$ be a complete metric space. Then, $\mathcal{C}(G, \Omega) = \ldots$

Proposition. Let $G$ be a region. Then there exists a sequence of subsets $\{K_n\}$ of $G$ such that
\begin{enumerate}[(i)]  
  \item $K_n \subseteq G$
  \item $K_n \subseteq \text{int}(K_{n+1})$
  \item $\bigcup_{n=1}^{\infty} K_n = G$
  \item $K \subseteq G$ implies $K \subseteq K_n$ for some $n \in \mathbb{N}$
\end{enumerate}

Lemma If $(S, d)$ is a metric space, then $(S, \mu)$ is a metric space, where $\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)}$. A set $O$ is open in $(S, d)$ if and only if $O$ is open in $(S, \mu)$.

Definition. For $K \subseteq G$ and $f, g \in \mathcal{C}(G, \Omega)$, let $\rho_K(f, g) = \sup_{z \in K} d(f(z), g(z))$,
\[
\sigma_K(f, g) = \frac{\rho_K(f, g)}{1 + \rho_K(f, g)} , \quad B_{\rho_K}(f, \delta) = \{g : \rho_K(f, g) < \delta\}.
\]

Definition. For $\{K_n\}$ a compact exhaustion of a region $G$ and for $f, g \in \mathcal{C}(G, \Omega)$ let $\rho(f, g) = \ldots$

Proposition. $(\mathcal{C}(G, \Omega), \rho)$ is a metric space.

Lemma 1.7  
(i) Given $\varepsilon > 0$ there exists $\delta > 0$ and $K \subseteq G$ such that for $f, g \in \mathcal{C}(G, \Omega)$
\[
\rho_K(f, g) < \delta \implies \rho(f, g) < \varepsilon
\]
(ii) Given $\delta > 0$ and there exists $\varepsilon > 0$ such that
\[
\rho(f, g) < \varepsilon \implies \rho_K(f, g) < \delta
\]
Lemma 1.10  (i) A set $O \subset \mathcal{C} (G, \Omega)$ is open if and only if for each $f \in O$ there exists $\delta > 0$ and $K \subset \subset G$ such that $O \supset B_{\rho_k} (f, \delta)$

(ii) A sequence $\{f_n\} \subset \mathcal{C} (G, \Omega)$ converges to $f$ (in the $\rho$ metric) if and only if for each $K \subset \subset G$ $\{f_n\}$ converges to $f$ in the $\rho_k$ metric.

**Proposition.** $(\mathcal{C} (G, \Omega), \rho)$ is a complete metric space.

**Definition.** A set $F \subset \mathcal{C} (G, \Omega)$ is normal . . .

**Proposition.** A set $F \subset \mathcal{C} (G, \Omega)$ is normal if and only if $\overline{F}$ is compact.

**Proposition.** A set $F \subset \mathcal{C} (G, \Omega)$ is normal if and only if for each $\delta > 0$ and $K \subset \subset G$ there exist functions $f_1, f_2, \ldots, f_n \in F$ such that $F \subset \bigcup_{k=1}^{n} B_{\rho_k} (f_k, \delta)$.

**Definition.** A set $F \subset \mathcal{C} (G, \Omega)$ is equicontinuous at a point $z_0 \in G$ if . . .

**Definition.** A set $F \subset \mathcal{C} (G, \Omega)$ is equicontinuous on a set $E \subset G$ if . . .

**Proposition.** Suppose a set $F \subset \mathcal{C} (G, \Omega)$ is equicontinuous at each point of $G$. Then, $F$ is equicontinuous on each $K \subset \subset G$.

**Arzela-Ascoli Theorem**

**Chapter 7.2**

**Homework**  7.2 Page 154  4, 6, 8, 10, 13

**Definition.** Let $G$ be a region. $A (G) = \cdots$

**Theorem.** Let $G$ be a region. Let $\{f_n\} \subset A (G)$ and let $f \in \mathcal{C} (G, \mathbb{C})$. If $f_n \rightarrow f$, then $f \in A (G)$ and $f_n^{(k)} \rightarrow f^{(k)}$ for each $k \geq 1$.

**Hurwitz's Theorem**

**Corollary.** Let $G$ be a region. Let $\{f_n\} \subset A (G)$ and $f \in A (G)$ be such that $f_n \rightarrow f$. If each $f_n$ is non-vanishing on $G$, then either $f$ is non-vanishing on $G$ or else $f \equiv 0$.

**Definition.** A set $F \subset A (G)$ is locally bounded if . . .
Lemma  A set $F \subset A(G)$ is locally bounded if and only if for each $K \subset G$ there exists a constant $M$ such that $|f(z)| \leq M$ for all $f \in F$ and for all $z \in K$.

Montel's Theorem

Chapter 7.4

Homework  7.4 Page 163 $4, 5, 6, 7$

Definition  A region $G_1$ is conformally equivalent to a region $G_2$ if . . .

Riemann Mapping Theorem

Chapter 7.5

Homework  7.5 Page 173 $4, 5, 6, 7, 9$

Definition  Let $\{z_n\} \subset \mathbb{C}$. Then, the infinite product $\prod_{n=1}^{\infty} z_n = \cdots$

Proposition  Let $\Re z_n > 0$ for all $n$. Then, the product $\prod_{n=1}^{\infty} z_n$ converges to a non-zero number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Proposition  Let $\Re z_n > 0$ for all $n$. Then, the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n - 1$ converges absolutely.

Definition  Let $\Re z_n > 0$ for all $n$. The product $\prod_{n=1}^{\infty} z_n$ converges absolutely if . . .

Corollary  Let $\Re z_n > 0$ for all $n$. Then, the product $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n - 1$ converges absolutely.
Theorem. Let $G$ be a region. Let $\{f_n\} \subset A(G)$ be such that no $f_n$ is identically 0. If 
$\sum [f_n(z) - 1]$ converges in $A(G)$, then $\prod f_n(z)$ converges in $A(G)$. Further, each zero of $\prod f_n(z)$ is a zero of one or more of the factors $f_n(z)$.

Definition. An elementary factor $E_p(z) = \cdots$

Lemma. If $|z| \leq 1$, then $|E_p(z) - 1| \leq |z|^{p+1}$

Theorem. Let $\{a_n\} \subset \mathbb{C}$ be such that $\lim_{n \to \infty} |a_n| = \infty$, $a_n \neq 0$ for all $n$. If $\{p_n\}$ is a sequence of integers such that

$$\sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n+1} < \infty$$

(*)

for all $r > 0$, then $\prod_{n=1}^{\infty} E_{p_n}(\frac{z}{a_n})$ converges to an entire function whose zero set is precisely $\{a_n\}$.

Furthermore, (*) is always satisfied if $p_n = n - 1$.

Weierstrass Factorization Theorem

Chapter 7.6

Theorem. $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \left(\frac{z}{2n}\right)^{2}\right)$. 

Chapter 7.7

Homework. 7.7 Page 185 1, 2, 3, 7, 8

Definition. The gamma function $\Gamma(z) = \cdots$

Gauss's Formula

Gauss’s Functional Equation. For $z \neq 0, -1, -2, \cdots$, \( \Gamma(z+1) = z\Gamma(z) \).

Bohr-Mollerup Theorem

Integral Representation. For $\Re z > 0$, $\Gamma(z) = \int_{0}^{\infty} e^{-t}t^{z-1} \, dt$.

Lemma. \( \left(1 + \frac{z}{n}\right)^n \) converges to $e^z$ in $A(G)$.
Chapter 7.8

Definition  The Riemann zeta function $\zeta(z) = \cdots$

Integral Representation 1. For $\Re z > 1$, $\zeta(z)\Gamma(z) = \int_0^\infty \frac{1}{e^t - 1} t^{z-1} \, dt$.

Extension 1. For $\Re z > 0$, $\zeta(z)\Gamma(z) = \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} \, dt + \frac{1}{z-1} + \int_1^\infty \frac{1}{e^t - 1} t^{z-1} \, dt$

Integral Representation 2. For $0 < \Re z < 1$, $\zeta(z)\Gamma(z) = \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} \, dt$.

Extension 2. For $-1 < \Re z < 1$,
$$\zeta(z)\Gamma(z) = \int_0^1 \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} \, dt - \frac{1}{2z} + \int_1^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1} \, dt$$

Integral Representation 3. For $-1 < \Re z < 0$, $\zeta(z)\Gamma(z) = \int_0^\infty \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1} \, dt$

Riemann's Functional Equation

Theorem $\zeta(z) \in \mathcal{A} \quad (\mathbb{C} \setminus \{1\})$ with a simple pole at $z = 1$ with residue 1. Outside of the strip $0 \leq \Re z \leq 1$, $\zeta(z)$ is non-vanishing except for simple zeros at $z = -2, -4, -6, \cdots$.

Riemann Hypothesis

Euler’s Theorem  For $\Re z > 0$, $\zeta(z) = \prod_{n=1}^\infty \left( \frac{1}{1 - p_n^{-z}} \right)$, where $\{p_n\}$ is an enumeration of the prime numbers.