Review Exam I
Complex Analysis

Underlined Definitions: May be asked for on exam
Underlined Propositions or Theorems: Proofs may be asked for on exam

Chapter 4.6

<u>Definition</u>. $\gamma_0 \sim \gamma_1$ for two closed rectifiable paths γ_0, γ_1 lying in a region G if ...

Proposition. "$\sim$" is an equivalence relation

<u>Examples</u> of pairs of curves γ_0, γ_1 (and the underlying region G) which are homotopic and of pairs of curves γ_0, γ_1 (and the underlying region G) which are not homotopic
<u>Problems</u> about constructing the homotopy function for a pair of curves γ_0, γ_1 (and the underlying region G) which are homotopic

<u>Definition</u>. A set S is convex if ...

<u>Definition</u>. A set S is starlike (w.r.t. a) if ...

<u>Examples</u> of sets which are convex / starlike and of sets which are not convex / starlike

Cauchy's Theorem (#2). Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let γ be a closed rectifiable curve in G. If $\gamma \sim 0$, then $\int_\gamma f = 0$.

Cauchy's Theorem (#3). Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let Let γ_0, γ_1 be closed rectifiable curves in G such that $\gamma_0 \sim \gamma_1$. Then, $\int_{\gamma_0} f = \int_{\gamma_1} f$.

<u>Definition</u>. $\gamma_0 \sim_{FEP} \gamma_1$ for two rectifiable paths γ_0, γ_1 lying in a region G if ...

<u>Examples</u> of pairs of curves γ_0, γ_1 (and the underlying region G) which are FEP-homotopic and of pairs of curves γ_0, γ_1 (and the underlying region G) which are not FEP-homotopic

Independence of Path Theorem. Let G be a region in $\mathbb{C}$ and let $f \in A(G)$. Let Let γ_0, γ_1 be rectifiable curves in G such that $\gamma_0 \sim_{FEP} \gamma_1$. Then, $\int_{\gamma_0} f = \int_{\gamma_1} f$.

<u>Definition</u>. A region G is simply connected if ...

<u>Examples</u> of regions which are simply connected and of regions which are not simply connected

Cauchy's Theorem (#4). Let G be a region in $\mathbb{C}$ which is simply connected and let $f \in A$ (G). Let γ be a closed rectifiable curve in G. Then, $\displaystyle\int_{\gamma} f = 0$.

Corollary. Let G be a region in $\mathbb{C}$ which is simply connected and let $f \in A$ (G). Then, f has a primitive on G.

Definition. A function f has a branch-of-log f on a region G if $\ldots$

Corollary. Let G be a region in $\mathbb{C}$ which is simply connected and let $f \in A$ (G). If $f(z) \neq 0$ on G, then f has a branch-of-log f on G.

Chapter 4.7

Theorem 7.2. Let G be a region in $\mathbb{C}$ and let $f \in A$ (G). Suppose $Z_f = \{a_1, a_2, \cdots, a_m\}$. Let γ be a closed rectifiable curve in G such that $\gamma \approx 0$ and such that $Z_f \cap \{\gamma\} = \varnothing$. Then $\displaystyle\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \sum_{k=1}^{m} n(\gamma; a_k)$

Corollary 7.3. Let G be a region in $\mathbb{C}$ and let $f \in A$ (G). Suppose $Z_{f-\alpha} = \{a_1, a_2, \cdots, a_m\}$. Let γ be a closed rectifiable curve in G such that $\gamma \approx 0$ and such that $Z_{f-\alpha} \cap \{\gamma\} = \varnothing$. Then $\displaystyle\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)-\alpha}dz = \sum_{k=1}^{m} n(\gamma; a_k)$

Proposition. Let G be a region in $\mathbb{C}$ and let $f \in A$ (G). Let γ be a closed rectifiable curve in G such that $\gamma \approx 0$. Let $\sigma = f \circ \gamma$. Let α, β belong to the same component of $\mathbb{C} \setminus \{\sigma\}$. Then,

$$
\begin{array}{ccc}
n(\sigma;\alpha) & = & n(\sigma;\beta) \\
\| & & \| \\
\displaystyle\sum_{Z_{f-\alpha}} n(\gamma; z_k(\alpha)) & & \displaystyle\sum_{Z_{f-\beta}} n(\gamma; z_k(\beta))
\end{array}
$$

Definition. A function f has a simple zero at $z = a$ if $\ldots$

Theorem 7.4. Let G be a region in $\mathbb{C}$, let $B(a,r) \subset G$ and let $f \in A$ (G). Let $f(a) = \alpha$. If $f(z) - \alpha$ has a zero of order m at $z = a$, then there exists $\varepsilon > 0$ and $\delta > 0$ such that for each $\zeta \in B(a,\delta)'$ the equation $f(z) = \zeta$ has exactly m simple roots in $B(a,\varepsilon)$.

Corollary. Open Mapping Theorem

Chapter 4.8

Cauchy-Goursat Theorem. Let G be a region. If $f \in D$ (G), then $f \in A$ (G).

Chapter 5.1

<u>Definition.</u> A function f has an isolated singularity at $z = a$ if ...

<u>Examples</u> of functions which have isolated singularities

<u>Definition.</u> Removable Singularity, Pole, Essential Singularity

<u>Examples</u> of functions which have removable singularities, poles, essential singularities

<u>Problems</u> about classifying singularities as removable singularities, poles, essential singularities

Proposition. Let G be a region in $\mathbb{C}$ and let f have an isolated singularity at $z = a$, $a \in G$. The following are equivalent.

     a. f has a removable singularity at $z = a$

     b. $\lim\limits_{z \to a} f(z)$ exists

     c. f is bounded (in modulus) on a punctured neighborhood of a

     d. $\lim\limits_{z \to a} (z - a) f(z) = 0$

<u>Definition.</u> A function f has a pole of order m at $z = a$ if ...

Proposition. Let G be a region in $\mathbb{C}$ and let f have an isolated singularity at $z = a$. f has a pole of order m at $z = a$ if and only if there exists a function g analytic and non-vanishing on a neighborhood of $z = a$ such that
$$f(z) = \frac{g(z)}{(z-a)^m} \, .$$
<u>Definition.</u> The singular part of a function f which has a pole of order m at $z = a$ is ...

<u>Problems</u> about identifying the singular parts of functions with poles

<u>Definition.</u> A double infinite series $\sum\limits_{n=-\infty}^{\infty} z_n$ is absolutely convergent if ...

<u>Definition.</u> A double infinite series $\sum\limits_{n=-\infty}^{\infty} u_n(s)$ is uniformly convergent on a region S if ...

<u>Examples</u> of double infinite series which are absolutely convergent / uniformly convergent.

<u>Definition.</u> $ann(a; r_1, r_2) =$

Theorem (Laurent Series Expansion). Let $f \in A \ (ann(a; R_1, R_2))$. Let $G = \mathbb{C} \setminus \overline{B(a, R_1)}$. Then there exist functions $f_2 \in A \ (B(a, R_2))$ and $f_1 \in A \ (G)$ such that $f = f_2 + f_1$ on $ann(a; R_1, R_2)$. Furthermore,

$$f_2(z) = \sum_{n=0}^{\infty} a_n (z - a)^n \text{ on } B(a, R_2) \tag{2}$$

and

$$f_1(z) = \sum_{n=1}^{\infty} a_{-n}(z - a)^{-n} \text{ on } G \tag{1}$$

where

$$a_n = \frac{1}{2\pi i} \int_{C(a,\rho)} \frac{f(w)}{(w - a)^{n+1}} dw, \ R_1 < \rho < R_2 \tag{*}$$

and the coefficient a_n in (*) is independent of the choice of ρ. The series for f_2 in (2) converges absolutely on $B(a, R_2)$ and uniformly on subsets $\overline{B(a, r_2)}$ of $B(a, R_2)$, $0 < r_2 < R_2$. The series for f_1 in (1) converges absolutely on G and uniformly on subsets $\overline{ann(a; r_1, \infty)}$ of G, $R_1 < r_1 < \infty$.

Furthermore, the series representation $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$ on $ann(a; R_1, R_2)$ is unique.

Corollary. Let G be a region in $\mathbb{C}$ and let f have an isolated singularity at $z = a$. Let
$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - a)^n$ be the Laurent Series expansion of f on a punctured neighborhood of a. Then,

    a. $z = a$ is removable singularity of f in and only if $a_n = 0$ for all $n \leq -1$

    b. $z = a$ is a pole of f of order m if and only if $a_{-m} \neq 0$ and $a_{-n} = 0$ for all $n \leq -m - 1$

    c. $z = a$ is an essential singularity if and only if $a_n \neq 0$ for infinitely many negative n

Problems constructing Laurent Series expansions.

Problems constructing Laurent Series expansions for rational functions.

<u>Casorati-Weierstrass Theorem.</u> Let G be a region in $\mathbb{C}$ and let f have an isolated singularity at $z = a$. If $z = a$ is an essential singularity of f, then for every $\delta > 0$, $\overline{f(B(a,\delta)')} = \mathbb{C}$.

Chapter 5.2

<u>Definition</u>. Let f have an isolated singularity at $z = a$. Then the residue of f at $z = a$ is $\ldots$

Residue Theorem Let G be a region and let $f \in A\ (G)$ except for isolated singularities $a_1, a_2, \cdots, a_m$. If γ is a closed rectifiable curve in G which does not pass through any of the points a_k and if $\gamma \approx 0$ in G, then

$$\frac{1}{2\pi}\int_\gamma f = \sum_{k=1}^m n(\gamma; a_k)\operatorname{Res}(f; a_k).$$

Computation of Residues:

Lemma 1.   Suppose f has a pole of order m at $z = a$. Let $g(z) = (z-a)^m f(z)$. Then,

$$\operatorname{Res}(f; a) = \lim_{z \to a} \frac{1}{(m-1)!} g^{(m-1)}(z).$$

Lemma 2.   Suppose f has a simple pole at $z = a$. Then, $\operatorname{Res}(f, a) = \lim_{z \to a} (z-a)f(z)$

Lemma 3.   Suppose $f = g/h$ where g, h are analytic on a neighborhood of $z = a$. If $g(a) \neq 0$ and

h has a simple zero at $z = a$, then, $\operatorname{Res}(f, a) = \dfrac{g(a)}{h'(a)}$.

Problems about computing residues.

Theorems for Calculation of Integrals using the Residue Theorem

See PDF file: Integration Topics

Problems about computing integrals using the Residue Theorem