Review Exam II

Complex Analysis

Underlined Definitions: May be asked for on exam
Underlined Propositions or Theorems: Proofs may be asked for on exam

Chapter 5.3

Definition. Let G be a region. A function f is meromorphic on G if . . .

Argument Principle Let G be a region and let f be meromorphic on G with poles $\{p_1, p_2, \cdots, p_m\}$ and zeros $\{z_1, z_2, \cdots, z_n\}$, counted according to multiplicity. Let γ be a closed rectifiable curve in G which does not pass through any of the points p_j and z_k and let $\gamma \approx 0$ in G. Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} n(\gamma; z_k) - \sum_{j=1}^{m} n(\gamma; p_j).$$

Corollary Let G be a region and let f be meromorphic on G with poles $\{p_1, p_2, \cdots, p_m\}$ and zeros $\{z_1, z_2, \cdots, z_n\}$, counted according to multiplicity. Let γ be a simple closed rectifiable curve in G which is positively oriented which does not pass through any of the points p_j and z_k and let $\gamma \approx 0$ in G. Let $\text{int}(\gamma)$ denote the bounded component of $\mathbb{C} \setminus \{\gamma\}$. Let $P_{f,\gamma}$ and $Z_{f,\gamma}$ denote the cardinality of $\{p_1, p_2, \cdots, p_m\} \cap \text{int}(\gamma)$ and $\{z_1, z_2, \cdots, z_n\} \cap \text{int}(\gamma)$, respectively . Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = Z_{f,\gamma} - P_{f,\gamma}.$$

Corollary Let G be a region and let f be analytic on G with zeros $\{z_1, z_2, \cdots, z_n\}$, counted according to multiplicity. Let γ be a simple closed rectifiable curve in G which is positively oriented which does not pass through any of the points z_k and let $\gamma \approx 0$ in G. Let $\text{int}(\gamma)$ denote the bounded component of $\mathbb{C} \setminus \{\gamma\}$. Let $Z_{f,\gamma}$ denote the cardinality of $\{z_1, z_2, \cdots, z_n\} \cap \text{int}(\gamma)$. Then, $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = Z_{f,\gamma}$.

Problems about counting the number of zeros of a given function in a given region using Argument Principle

Rouche's Theorem (Ver #1) Let G be a region and let f and g be mereomorphic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros or poles on $C(a,r)$. Let Z_f, Z_g, P_f, P_g denote the cardinality of the zeros of f and g and the cardinality of the poles of f and g inside $C(a,r)$, respectively. If $|f(z)+g(z)| < |f(z)| + |g(z)|$ on $C(a,r)$, then $Z_f - P_f = Z_g - P_g$.

Corollary Let G be a region and let f and g be analytic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros on $C(a,r)$. Let Z_f, Z_g denote the cardinality of the zeros of f and g inside $C(a,r)$, respectively. If $|f(z)+g(z)| < |f(z)| + |g(z)|$ on $C(a,r)$, then $Z_f = Z_g$.

Rouche's Theorem (Ver #2) Let G be a region and let f and g be mereomorphic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros or poles on $C(a,r)$. Let $Z_f, Z_{f+g}, P_f, P_{f+g}$ denote the cardinality of the zeros of f and g and the cardinality of the poles of f and $f+g$ inside $C(a,r)$, respectively. If $0 < |g(z)| < |f(z)| < \infty$ on $C(a,r)$, then $Z_f - P_f = Z_{f+g} - P_{f+g}$.

Corollary Let G be a region and let f and g be analytic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros on $C(a,r)$. Let Z_f, Z_g denote the cardinality of the zeros of f and g inside $C(a,r)$, respectively. If $0 < |g(z)| < |f(z)| < \infty$ on $C(a,r)$, then $Z_f = Z_{f+g}$.

Problems about counting the number of zeros of a given function in a given region using Rouche's Theorem

Chapter 6.1

Maximum Modulus Theorem (Ver #1) Let G be a region and let $f \in A(G)$. If there exists $a \in G$ such that $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

Maximum Modulus Theorem (Ver #2) Let G be a bounded region and let $f \in A(G) \cap C(\overline{G})$. Then, $\max_{z \in \overline{G}} |f(z)| = \max_{z \in \partial G} |f(z)|$.

Definition. Let G be a region and let $f : G \to \mathbb{R}$. Let $a \in \partial G$ or $a = \infty$. Then, $\limsup_{z \to a} f(z) = \ \ldots$

Definition. Let G be a region. Then, $\partial_\infty G = \ \ldots$

Maximum Modulus Theorem (Ver #3) Let G be a region and let $f \in A(G)$. Suppose there exists a constant M such that $\limsup_{z \to a} |f(z)| \leq M$ for all $a \in \partial_\infty G$. Then, $|f(z)| \leq M$ for all $z \in G$.

Supplement Maximum Modulus

Definition (for Supplement). Let $D = \{z : |z| < 1\}$ and for $0 < r \le 1$ let $D_r = \{z : |z| < r\}$

Theorem For $f \in A(D)$ let

a) $M(r, f) = \max_{|z|=r} |f(z)|$ b) $A(r, f) = \max_{|z|=r} \operatorname{Re} f(z)$

c) $I_p(r, f) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} |f(re^{i\theta})|^p \, d\theta$

Then,

(i) $M(r, f)$ is a strictly increasing function of r on (0,1) unless f is constant.

(i) $A(r, f)$ is a strictly increasing function of r on (0,1) unless f is constant.

(i) $I_p(r, f)$ is a strictly increasing function of r on (0,1) unless f is constant, for $p \in \mathbb{N}$

Parseval's Identity Let $f \in A(D)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. For $0 < r < 1$,

$$I_2(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Chapter 6.2

Let D denote the open unit disk (centered at 0) and let D_r denote open disk of radius r (centered at 0).

<u>Schwarz's Lemma</u> (Ver #1) Let $f \in A(D)$ such that (a) $|f(z)| \le 1$ for all $z \in D$ and (b) $f(0) = 0$. Then, (i) $|f(z)| \le |z|$ for all $z \in D$ and (ii) $|f'(0)| \le 1$. Moreover, equality occurs in (i) or (ii) if and only if $f(z) = \zeta z$ for some $\zeta, |\zeta| = 1$.

<u>Schwarz's Lemma</u> (Ver #2) Let $f \in A(D)$ such that (a) $f(D) \subset D$ and (b) $f(0) = 0$. Then, (i) $f(D_r) \subset D_r$ and (ii) $|f'(0)| \le 1$. Moreover, equality occurs in (i) or (ii) if and only if $f(z) = \zeta z$ for some $\zeta \in \partial D$.

Proposition. For $a \in D$ let $\varphi_a(z) = \dfrac{z - a}{1 - \bar{a}z}$. Then, $\varphi_a \in A(D)$, $\varphi_a : D \to D$ and, φ_a is one-to-one and onto.

Further, for $|z| = 1$, $|\varphi_a(z)| = 1$ and $\varphi_a'(0) = 1 - |a|^2$, $\varphi_a'(a) = (1 - |a|^2)^{-1}$.

Proposition Let $f \in A(D)$, $f : D \to D$. For $a \in D$ let $f(a) = \alpha$. Then, $| f'(a) | \le \dfrac{1-|\alpha|^2}{1-|a|^2}$. Equality occurs if and only if $f(z) = \varphi_{-\alpha}(\zeta \varphi_a(z)) \;\; for \;\; some \;\; \zeta \in \partial D$.

Theorem Let $f \in A(D)$, $f : D \to D$ such that f is one-to-one and onto. Then, there exists $a \in D$ and $\zeta \in \partial D$ such that $f = \zeta \varphi_a$.

Supplement Subordination

Definition (for Supplement). Let $D = \{z : |z| < 1\}$ and for $0 < r \le 1$ let $D_r = \{z : |z| < r\}$

Definition Let $f, g \in A(D)$. Then, f is subordinate to g ($f \prec g$) if . . .

Theorem. Let $f \prec g$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then,

i) $f(0) = g(0), \; | f'(0) | \le | g'(0) |$

ii) $f(D) \subset g(D)$

iii) $f(D_r) \subset g(D_r)$

iv) $M(r, f) \le M(r, g)$

v) $I_p(r, f) \le I_p(r, g)$

vi) $\sum_{k=0}^{m} |a_k|^2 \le \sum_{k=0}^{m} |b_k|^2$, $m = 0, 1, 2, 3, \cdots$

vii) $\max_{z \in D_r} (1-|z|^2) | f'(z) | \le \max_{z \in D_r} (1-|z|^2) | g'(z) |$

Proposition. Let $f, g \in A(D)$. Suppose (a) $f(0) = g(0)$, (b) $f(D) \subset g(D)$, (c) g is one-to-one. Then, $f \prec g$.

Example. Let $P = \{ f \in A(D) : \text{Re}\, f(z) > 0 \;\text{ for all }\; z \in D, \;\; f(0) = 1\}$. Let $p(z) = \dfrac{1+z}{1-z}$. Then, $f \in P$ implies $f \prec p$.

Chapter 10.1

Definition. Let G be a region and let $f : G \to \mathbb{R}$. f is harmonic on G if . . .

Definition. Let G be a region and let $f : G \to \mathbb{R}$. f satisfies the Mean Value Property (MVP) on G if . . .

Proposition. Let G be a region. Let f be harmonic on G. Then, f satisfies the MVP on G.

<u>Maximum Principle</u> <u>(Ver. #1)</u> Let G be a region and let $u : G \to R$ satisfy the MVP on G. If there exists $a \in G$ such that $u(a) \ge u(z)$ for all $z \in G$, then u is constant.

Maximum Principle (Ver. #2) Let G be a region and let $u, v : G \to \mathbb{R}$ be bounded functions satisfying the MVP on G. Suppose for each $a \in \partial_\infty G$ that $\limsup_{z \to a} u(z) \le \limsup_{z \to a} v(z)$, then either $u(z) < v(z)$ for all $z \in G$ or else $u \equiv v$.

<u>Minimum Principle</u> <u>(Ver. #1)</u> Let G be a region and let $u : G \to R$ satisfy the MVP on G. If there exists $a \in G$ such that $u(a) \le u(z)$ for all $z \in G$, then u is constant.

Chapter 10.2

<u>Definition</u>. The Poisson kernel $P_r(\theta) = \ . \ . \ .$

Proposition $P_r(\theta) = \mathrm{Re}\dfrac{1+re^{i\theta}}{1-re^{i\theta}} = \dfrac{1-r^2}{1-2r\cos(\theta)+r^2} = \mathrm{Re}\dfrac{e^{i\theta}+r}{e^{i\theta}-r}$

Proposition. The Poisson kernel satisfies

(i) $\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1, \ 0 < r < 1$

(ii) $P_r(\theta) > 0$, $P_r(\theta) = P_r(-\theta)$

(iii) $P_r(\theta) < P_r(\delta)$ for $0 < \delta < |\theta| < \pi$, i.e., $P_r(\theta)$ is strictly decreasing on $(0, \pi)$

(iv) $\lim_{r \to 1^-} P_r(\delta) = 0 \quad for \ \ each \ \ \delta, 0 < \delta \le \pi$

Theorem Let $f \in C(\partial D)$, $f : \partial D \to \mathbb{R}$. Then, there exists $u \in C(\overline{D}) \cap Har(D)$ such that

(i) $u(e^{i\theta}) = f(e^{i\theta})$

(ii) u is unique

Corollary Let $u \in C(\overline{D}) \cap Har(D)$. Then, $u(re^{i\theta}) = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r(\theta - t) u(e^{it})\, dt$, $0 < r < 1$

Corollary Let $h \in C(C(a,R))$, $h : C(a,R) \to \mathbb{R}$. Then, there exists $w \in C(\overline{B(a,R)}) \cap Har(B(a,R))$ such that $w(z) = h(z)$ on $C(a,R)$.

Corollary Let $u \in C(\overline{B(a,R)}) \cap Har(B(a,R))$. Then, $u(a+re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_{r/R}(\theta - t)u(a+re^{it})\,dt$, $0<r<R$

Harnack's Inequality Let $u \in C(\overline{B(a,R)}) \cap Har(B(a,R))$, $u(z) \geq 0$. Then,
$\frac{R-r}{R+r}u(a) \leq u(a+re^{i\theta}) \leq \frac{R+r}{R-r}u(a)$, $0 \leq r < R$

Chapter 7.1

Definition. Let G be a region and let (Ω, d) be a complete metric space. Then, $C(G,\Omega) =$ . . .

Proposition. Let G be a region. Then there exists a sequence of subsets $\{K_n\}$ of G such that

(i) $K_n \subset\subset G$
(ii) $K_n \subset \text{int}(K_{n+1})$
(iii) $\bigcup_{n=1}^{\infty} K_n = G$
(iv) $K \subset\subset G$ implies $K \subset K_n$ for some $n \in \mathbb{N}$

Lemma If (S,d) is a metric space, then (S,σ) is a metric space, where $\sigma(s,t) = \frac{d(s,t)}{1+d(s,t)}$. A set O is open in (S,d) if and only if O is open in (S,σ) .

Definition. For $K \subset\subset G$ and $f,g \in C(G,\Omega)$, let $\rho_K(f,g) = \cdots$, $\sigma_K(f,g) = \cdots$, $B_{\rho_K}(f,\delta) = \cdots$.

Definition. For $\{K_n\}$ a compact exhaustion of a region G and for $f,g \in C(G,\Omega)$ let $\rho(f,g) =$ . . .

Proposition. $(C(G,\Omega),\rho)$ is a metric space.

Lemma 1.7 (i) Given $\varepsilon > 0$ there exists $\delta > 0$ and $K \subset\subset G$ such that for $f,g \in C(G,\Omega)$

$$\rho_K(f,g) < \delta \Rightarrow \rho(f,g) < \varepsilon$$

(ii) Given $\delta > 0$ and $K \subset\subset G$ there exists $\varepsilon > 0$ such that

$$\rho(f,g) < \varepsilon \Rightarrow \rho_K(f,g) < \delta$$

Lemma 1.10 (i) A set $O \subset C(G,\Omega)$ is open if and only if for each $f \in O$ there exists $\delta > 0$ and $K \subset\subset G$ such that $O \supset B_{\rho_K}(f,\delta)$

(ii) A sequence $\{f_n\} \subset C(G,\Omega)$ converges to f (in the ρ metric) if and only if for each $K \subset\subset G$ $\{f_n\}$ converges to f in the ρ_K metric.

Proposition $(C(G,\Omega),\rho)$ is a complete metric space.

Definition. A set $F \subset C(G,\Omega)$ is normal . . .

Proposition. A set $F \subset C(G,\Omega)$ is normal if and only if $\overline{F}$ is compact.

Proposition. A set $F \subset C(G,\Omega)$ is normal if and only if for each $\delta > 0$ and $K \subset\subset G$ there exist functions $f_1, f_2, \cdots, f_n \in F$ such that $F \subset \bigcup_{k=1}^{n} B_{\rho_K}(f_k, \delta)$.

Definition. A set $F \subset C(G,\Omega)$ is equicontinuous at a point $z_0 \in G$ if . . .

Definition. A set $F \subset C(G,\Omega)$ is equicontinuous on a set $E \subset G$ if . . .

Proposition. Suppose a set $F \subset C(G,\Omega)$ is equicontinuous at each point of G. Then, F is equicontinuous on each $K \subset\subset G$.

Arzela-Ascoli Theorem A set $F \subset C(G,\Omega)$ is normal if and only if

(a) for each $z \in G$ the orbit of z under F, *i.e.*, $\{f(z) : f \in F\}$, has compact closure and

(b) $F \subset C(G,\Omega)$ is equicontinuous at each point of G.

Chapter 7.2

Theorem Let G be a region and let $A(D)$ denote the set of analytic functions on G. Then,

1'. Let $f \in C(G,\Omega)$. If f is a limit point of $A(D)$, then $f \in A(D)$.

1. Let $\{f_n\} \subset A(D)$ and let $f \in C(G,\Omega)$. If $f_n \to f$ (in $C(G,\Omega)$), then $f \in A(D)$.
2. Let $\{f_n\} \subset A(D)$ and let $f \in C(G,\Omega)$. If $f_n \to f$ (in $C(G,\Omega)$), then $f_n^{(k)} \to f^{(k)}$ (in $C(G,\Omega)$) for each $k > 0$.

Corollary. $A(D)$ is closed in $C(G,\Omega)$

Corollary. $A(D)$ is a complete metric space

Corollary. Let $\{f_n\} \subset A(D)$ and let $f \in C(G,\Omega)$. If $\sum_{n=1}^{\infty} f_n(z) \to f(z)$ (in $C(G,\Omega)$), then $f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z)$.