# Review Exam III
## Complex Analysis

Underlined Definitions: May be asked for on exam
Underlined Propositions or Theorems: Proofs may be asked for on exam
Underlined Homework Exercises: Problems may be asked for on exam
Double Underlined Homework Exercises: Similar problems will be asked for on exam
Double Underlined Named Theorems/Results: Statements may be asked for on exam

**Chapter 7.2**

Homework 7.2 Page 154 4, 6, 8, 10, 13

Definition Let $G$ be a region. $\mathcal{A}(G) = \cdots$

Theorem Let $G$ be a region. Let $\{f_n\} \subset \mathcal{A}(G)$ and let $f \in \mathcal{C}(G, \mathbb{C})$. If $f_n \to f$, then $f \in \mathcal{A}(G)$ and $f_n^{(k)} \to f^{(k)}$ for each $k \geq 1$.

Hurwitz's Theorem

Corollary Let $G$ be a region. Let $\{f_n\} \subset \mathcal{A}(G)$ and $f \in \mathcal{A}(G)$ be such that $f_n \to f$. If each $f_n$ is non-vanishing on $G$, then either $f$ is non-vanishing on $G$ or else $f \equiv 0$.

Definition A set $\mathcal{F} \subset \mathcal{A}(G)$ is locally bounded if . . .

Lemma A set $\mathcal{F} \subset \mathcal{A}(G)$ is locally bounded if and only if for each $K \subset\subset G$ there exists a constant $M$ such that $|f(z)| \leq M$ for all $f \in \mathcal{F}$ and for all $z \in K$.

Montel's Theorem

**Chapter 7.4**

Homework 7.4 Page 163 4, 5, 6, 7

Definition A region $G_1$ is conformally equivalent to a region $G_2$ if . . .

Riemann Mapping Theorem

**Chapter 7.5**

Homework 7.5 Page 173 4, 5, 6, 7, 9

Definition Let $\{z_n\} \subset \mathbb{C}$. Then, the infinite product $\prod_{n=1}^{\infty} z_n = \cdots$

Proposition Let $\operatorname{Re} z_n > 0$ for all $n$. Then, the product $\prod_{n=1}^{\infty} z_n$ converges to a non-zero number if and only if the series $\sum_{n=1}^{\infty} \log z_n$ converges.

Proposition Let $\operatorname{Re} z_n > 0$ for all $n$. Then, the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n - 1$ converges absolutely.

<u>Definition</u> Let $\operatorname{Re} z_n > 0$ for all $n$. The product $\prod_{n=1}^{\infty} z_n$ converges absolutely if . . .

Corollary Let $\operatorname{Re} z_n > 0$ for all $n$. Then, the product $\prod_{n=1}^{\infty} z_n$ converges absolutely if and only if series $\sum_{n=1}^{\infty} z_n - 1$ converges absolutely.

Theorem Let $G$ be a region. Let $\{f_n\} \subset \mathcal{A}(G)$ be such that no $f_n$ is identically 0. If $\sum [f_n(z) - 1]$ converges in $\mathcal{A}(G)$, then $\prod f_n(z)$ converges in $\mathcal{A}(G)$. Further, each zero of $\prod f_n(z)$ is a zero of one or more of the factors $f_n(z)$.

<u>Definition</u> An elementary factor $E_p(z) = \cdots$

Lemma If $|z| \le 1$, then $|E_p(z) - 1| \le |z|^{p+1}$

Theorem Let $\{a_n\} \subset \mathbb{C}$ be such that $\lim_{n \to \infty} |a_n| = \infty$, $a_n \neq 0$ for all $n$. If $\{p_n\}$ is a sequence of integers such that

$$\sum_{n=1}^{\infty} \left( \frac{r}{|a_n|} \right)^{p_n + 1} < \infty \qquad (*)$$

for all $r > 0$, then $\prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ converges to an entire function whose zero set is precisely $\{a_n\}$. Furthermore, (*) is always satisfied if $p_n = n - 1$.

## <u>Weierstrass Factorization Theorem</u>

**Chapter 7.6**

Theorem. $\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$.

**Chapter 7.7**

Homework 7.7 Page 185 1, 2, 3, 7, 8

Definition The gamma function $\Gamma(z) = \cdots$

Gauss's Formula

Gauss's Functional Equation For $z \neq 0, -1, -2, \cdots$, $\Gamma(z+1) = z\Gamma(z)$.

Bohr-Mollerup Theorem

Integral Representation For $\operatorname{Re} z > 0$, $\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1}\, dt$.

Lemma $\left\{\left(1 + \frac{z}{n}\right)^n\right\}$ converges to $e^z$ in $\mathcal{A} \quad (G)$

**Chapter 7.8**

Definition The Riemann zeta function $\zeta(z) = \cdots$

Integral Representation 1. For $\operatorname{Re} z > 1$, $\zeta(z)\Gamma(z) = \int_0^{\infty} \frac{1}{e^t - 1} t^{z-1}\, dt$.

Extension 1. For $\operatorname{Re} z > 0$, $\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt + \frac{1}{z-1} + \int_1^{\infty} \frac{1}{e^t - 1} t^{z-1}\, dt$

Integral Representation 2. For $0 < \operatorname{Re} z < 1$, $\zeta(z)\Gamma(z) = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt$.

Extension 2. For $-1 < \operatorname{Re} z < 1$,

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\, dt - \frac{1}{2z} + \int_1^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t}\right) t^{z-1}\, dt$$

Integral Representation 3. For $-1 < \operatorname{Re} z < 0$, $\zeta(z)\Gamma(z) = \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) t^{z-1}\, dt$

Riemann's Functional Equation

Theorem $\zeta(z) \in \mathcal{A} \quad (\mathbb{C} \setminus \{1\})$ with a simple pole at $z = 1$ with residue 1. Outside of the strip $0 \leq \operatorname{Re} z \leq 1$,

$\zeta(z)$ is non-vanishing except for simple zeros at $z = -2, -4, -6, \cdots$.

## Riemann Hypothesis

Euler's Theorem For $\operatorname{Re} z > 0$, $\zeta(z) = \prod_{n=1}^{\infty}\left(\frac{1}{1-p_n^{-z}}\right)$, where $\{p_n\}$ is an enumeration of the prime numbers.