Review Exam II

Complex Analysis

Underlined Definitions: May be asked for on exam
Underlined Propositions or Theorems: Proofs may be asked for on exam

Chapter 5.2

Definition. Let f have an isolated singularity at $z = a$. Then the residue of f at $z = a$ is . . .

Residue Theorem Let G be a region and let $f \in A\ (G)$ except for isolated singularities $a_1, a_2, \cdots, a_m$. If g is a closed rectifiable curve in G which does not pass through any of the points a_k and if $g \approx 0$ in G, then

$$\frac{1}{2p}\int_g f = \sum_{k=1}^{m} n(g;a_k)\mathrm{Res}(f;a_k).$$

Computation of Residues:

Lemma 1. Suppose f has a pole of order m at $z = a$. Let $g(z) = (z-a)^m f(z)$. Then,

$$\mathrm{Res}(f;a) = \lim_{z\to a}\frac{1}{(m-1)!}g^{(m-1)}(z).$$

Lemma 2. Suppose f has a simple pole at $z = a$. Then, $\mathrm{Res}(f,a) = \lim_{z\to a}(z-a)f(z)$

Lemma 3. Suppose $f = g/h$ where g, h are analytic on a neighborhood of $z = a$. If $g(a) \neq 0$ and h has a simple zero at $z = a$, then, $\mathrm{Res}(f,a) = \dfrac{g(a)}{h'(a)}$.

Problems about computing residues.

Theorems for Calculation of Integrals using the Residue Theorem

See PDF file: Integration Topics

Problems about computing integrals using the Residue Theorem

Chapter 5.3

Definition. Let G be a region. A function f is meromorphic on G if . . .

Argument Principle Let G be a region and let f be meromorphic on G with poles $\{p_1, p_2, \cdots, p_m\}$ and zeros $\{z_1, z_2, \cdots, z_n\}$, counted according to multiplicity. Let g be a closed rectifiable curve in G which does not pass through any of the points p_j and z_k and let $g \approx 0$ in G. Then, $\frac{1}{2p i}\int_g \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} n(g\,; z_k) - \sum_{j=1}^{m} n(g\,; p_j)$.

Corollary Let G be a region and let f be meromorphic on G with poles $\{p_1, p_2, \cdots, p_m\}$ and zeros $\{z_1, z_2, \cdots, z_n\}$, counted according to multiplicity. Let g be a simple closed rectifiable curve in G which is positively oriented which does not pass through any of the points p_j and z_k and let $g \approx 0$ in G. Let $\text{int}(g)$ denote the bounded component of $\mathbb{C} \setminus \{g\}$. Let $P_{f,g}$ and $Z_{f,g}$ denote the cardinality of $\{p_1, p_2, \cdots, p_m\} \cap \text{int}(g)$ and $\{z_1, z_2, \cdots, z_n\} \cap \text{int}(g)$, respectively. Then, $\frac{1}{2p i}\int_g \frac{f'(z)}{f(z)}\,dz = Z_{f,g} - P_{f,g}$.

Corollary Let G be a region and let f be analytic on G with zeros $\{z_1, z_2, \cdots, z_n\}$, counted according to multiplicity. Let g be a simple closed rectifiable curve in G which is positively oriented which does not pass through any of the points z_k and let $g \approx 0$ in G. Let $\text{int}(g)$ denote the bounded component of $\mathbb{C} \setminus \{g\}$. Let $Z_{f,g}$ denote the cardinality of $\{z_1, z_2, \cdots, z_n\} \cap \text{int}(g)$. Then, $\frac{1}{2p i}\int_g \frac{f'(z)}{f(z)}\,dz = Z_{f,g}$.

Problems about counting the number of zeros of a given function in a given region using Argument Principle

Rouche's Theorem (Ver #1) Let G be a region and let f and g be mereomorphic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros or poles on $C(a,r)$. Let Z_f, Z_g, P_f, P_g denote the cardinality of the zeros of f and g and the cardinality of the poles of f and g inside $C(a,r)$, respectively. If $|f(z)+g(z)| < |f(z)| + |g(z)|$ on $C(a,r)$, then $Z_f - P_f = Z_g - P_g$.

Corollary Let G be a region and let f and g be analytic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros on $C(a,r)$. Let z_f, z_g denote the cardinality of the zeros of f and g inside $C(a,r)$, respectively. If $|f(z)+g(z)| < |f(z)| + |g(z)|$ on $C(a,r)$, then $z_f = z_g$.

Rouche's Theorem (Ver #2) Let G be a region and let f and g be mereomorphic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros or poles on $C(a,r)$. Let $Z_f, Z_{f+g}, P_f, P_{f+g}$ denote the cardinality of the zeros of f and g and the cardinality of the poles of f and $f+g$ inside $C(a,r)$, respectively. If $|g(z)| < |f(z)|$ on $C(a,r)$, then $Z_f - P_f = Z_{f+g} - P_{f+g}$.

Corollary Let G be a region and let f and g be analytic on G. Let $\overline{B(a,r)} \subset G$ such that f and g have no zeros on $C(a,r)$. Let Z_f, Z_g denote the cardinality of the zeros of f and g inside $C(a,r)$, respectively. If $|f(z)+g(z)| < |f(z)| + |g(z)|$ on $C(a,r)$, then $Z_f = Z_{f+g}$.

<u>Problems</u> about counting the number of zeros of a given function in a given region using Rouche's Theorem

Chapter 6.1

<u>Maximum Modulus Theorem</u> <u>(Ver #1)</u> Let G be a region and let $f \in \mathcal{A}(G)$. If there exists $a \in G$ such that $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

<u>Maximum Modulus Theorem</u> <u>(Ver #2)</u> Let G be a bounded region and let $f \in \mathcal{A}(G) \cap \mathcal{C}(\overline{G})$. Then, $\max_{z \in \overline{G}} |f(z)| = \max_{z \in \partial G} |f(z)|$.

<u>Definition</u>. Let G be a region and let $f : G \to \mathbb{R}$. Let $a \in \partial G$ or $a = \infty$. Then, $\limsup_{z \to a} f(z) = \ ...$

Definition. Let G be a region. Then, $\partial_\infty G = \ ...$

Maximum Modulus Theorem (Ver #3) Let G be a region and let $f \in \mathcal{A}(G)$. Suppose there exists a constant M such that $\limsup_{z \to a} |f(z)| \leq M$ for all $a \in \partial_\infty G$. Then, $|f(z)| \leq M$ for all $z \in G$.

Definition. Let D denote the open unit disk (centered at 0) and let D_r denote open disk of radius r (centered at 0).

Supplement Maximum Modulus

Theorem For $f \in \mathcal{A}(D)$ let

a) $M(r,f) = \max_{|z|=r} |f(z)|$ b) $A(r,f) = \max_{|z|=r} \text{Re } f(z)$

c) $I_p(r,f) = \frac{1}{2p} \int_{-p}^{p} |f(re^{iq})|^p \, dq$

Then,

(i) $M(r, f)$ is a strictly increasing function of r on (0,1) unless f is constant.

(i) $A(r, f)$ is a strictly increasing function of r on (0,1) unless f is constant.

(i) $I_p(r, f)$ is a strictly increasing function of r on (0,1) unless f is constant, for $p \in \mathbb{N}$

Parseval's Identity Let $f \in A\ (D)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$. For $0 < r < 1$,

$$I_2(r, f) = \frac{1}{2p} \int_{-p}^{p} | f(re^{iq})|^2 \, dq = \sum_{n=0}^{\infty} |a_n|^2 \, r^{2n}$$

Chapter 6.2

Let D denote the open unit disk (centered at 0) and let D_r denote open disk of radius r (centered at 0).

<u>**Schwarz's Lemma**</u> (Ver #1) Let $f \in A\ (D)$ such that (a) $| f(z)| \le 1$ for all $z \in D$ and (b) $f(0) = 0$. Then, (i) $| f(z)| \le |z|$ for all $z \in D$ and (ii) $| f'(0)| \le 1$. Moreover, equality occurs in (i) or (ii) if and only if $f(z) = z\, z$ for some z, $|z| = 1$.

<u>**Schwarz's Lemma**</u> (Ver #2) Let $f \in A\ (D)$ such that (a) $f(D) \subset D$ and (b) $f(0) = 0$. Then, (i) $f(D_r) \subset D_r$ and (ii) $| f'(0)| \le 1$. Moreover, equality occurs in (i) or (ii) if and only if $f(z) = z\, z$ for some $z \in \partial D$.

Proposition. For $a \in D$ let $j_a(z) = \dfrac{z - a}{1 - \bar{a}z}$. Then, $j_a \in A\ (D)$, $j_a : D \to D$ and, j_a is one-to-one and onto. Further, for $|z| = 1$, $|j_a(z)| = 1$ and $j_a'(0) = 1 - |a|^2$, $j_a'(a) = (1 - |a|^2)^{-1}$.

Proposition Let $f \in A(D),\ f: D \to D$. For $a \in D$ let $f(a) = a$. Then, $| f'(a)| \le \dfrac{1 - |a|^2}{1 - |a|^2}$. Equality occurs if and only if $f(z) = j_{-a}(z\, j_a(z))$ for some $z \notin \partial D$.

Theorem Let $f \in A\ (D)$, $f : D \to D$ such that f is one-to-one and onto. Then, there exists $a \in D$ and $z \in \partial D$ such that $f = z\, j_a$.

Supplement Subordination

<u>Definition</u> Let $f, g \in \mathcal{A}\ (D)$. Then, f is subordinate to g ($f \prec g$) if . . .

Theorem. Let $f \prec g$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$. Then,

i) $f(0) = g(0),\ |f'(0)| \le |g'(0)|$

ii) $f(D) \subset g(D)$

iii) $f(D_r) \subset g(D_r)$

iv) $M(r, f) \le M(r, g)$

v) $I_p(r, f) \le I_p(r, g)$

vi) $\sum_{k=0}^{m} |a_k|^2 \le \sum_{k=0}^{m} |b_k|^2$, $m = 0, 1, 2, 3, \cdots$

vii) $\max_{z \in \overline{D_r}} (1 - |z|^2)\,|f'(z)| \le \max_{z \in \overline{D_r}} (1 - |z|^2)\,|g'(z)|$

Proposition. Let $f, g \in \mathcal{A}\ (D)$. Suppose (a) $f(0) = g(0)$, (b) $f(D) \subset g(D)$, (c) g is one-to-one. Then, $f \prec g$.

Example. Let $\mathcal{P} = \{ f \in \mathcal{A}\ (D) : \mathrm{Re}\, f(z) > 0 \text{ for all } z \in D,\ f(0) = 1\}$. Let $p(z) = \frac{1+z}{1-z}$. Then, $f \in \mathcal{P}$ implies $f \prec p$.

Chapter 10.1

Definition. Let G be a region and let $f : G \to \mathbb{R}$. f is harmonic on G if . . .

Definition. Let G be a region and let $f : G \to \mathbb{R}$. f satisfies the Mean Value Property (MVP) on G if . . .

<u>**Proposition**</u>. Let G be a region. Let f be harmonic on G. Then, f satisfies the MVP on G.

<u>**Maximum Principle** (Ver. #1)</u> Let G be a region and let $u : G \to R$ satisfy the MVP on G. If there exists $a \in G$ such that $u(a) \ge u(z)$ for all $z \in G$, then u is constant.

Maximum Principle (Ver. #2) Let G be a region and let $u, v : G \to \mathbb{R}$ be bounded functions satisfying the MVP on G. Suppose for each $a \in \partial_\infty G$ that $\limsup_{z \to a} u(z) \le \limsup_{z \to a} v(z)$, then either $u(z) < v(z)$ for all $z \in G$ or else $u \equiv v$.

<u>**Minimum Principle** (Ver. #1)</u> Let G be a region and let $u : G \to R$ satisfy the MVP on G. If there exists $a \in G$ such that $u(a) \le u(z)$ for all $z \in G$, then u is constant.

Chapter 10.2

Definition. The Poisson kernel $P_r(q) = \ . \ . \ .$

Proposition $P_r(q) = \mathrm{Re}\dfrac{1+re^{iq}}{1-re^{iq}} = \dfrac{1-r^2}{1-2r\cos(q)+r^2} = \mathrm{Re}\dfrac{e^{iq}+r}{e^{iq}-r}$

Proposition. The Poisson kernel satisfies

(i) $\dfrac{1}{2p}\displaystyle\int_{-p}^{p} P_r(q)\,dq = 1\,,\ 0<r<1$

(ii) $P_r(q) > 0\,,\ P_r(q) = P_r(-q)$

(iii) $P_r(q) < P_r(d)$ for $0<d<|q|<p$, i.e., $P_r(q)$ is strictly decreasing on $(0,p)$

(iv) $\lim\limits_{r\to 1^-} P_r(d) = 0 \quad for \quad each \ \ d\,,\ 0<d\le p$

Theorem Let $f \in C\ (\partial D)\,,\ \ f:\partial D \to \mathbb{R}$. Then, there exists $u \in C\ (\overline{D}) \cap Har\ (D)$ such that

(i) $u(e^{iq}) = f(e^{iq})$

(ii) u is unique

Corollary Let $u \in C\ (\overline{D}) \cap Har\ (D)$. Then, $u(re^{iq}) = \dfrac{1}{2p}\displaystyle\int_{-p}^{p} P_r(q-t)u(e^{it})\,dt\,,\ 0<r<1$

Corollary Let $h\in C(C(a,r)),\ h:C(a,r)\to\mathbb{R}$. Then, there exists $w\in C(\overline{B(a,r)})\cap Har(B(a,r))$ such that $w(z)=h(z)\ on\ C(a,r)$.

Corollary Let $u\in C(\overline{B(a,r)})\cap Har(B(a,r))$. Then,

$$u(a+re^{iq})=\frac{1}{2p}\int_{-p}^{p}\frac{r^2-r^2}{r^2-2rr\cos(q-t)+r^2}u(a+re^{it})\,dt\,,\ 0<r<r$$

Harnack's Inequality Let $u \in C\ (\overline{B(a,r)}) \cap Har\ (B(a,r))\,,\ u(z)\ge 0$. Then,

$$\frac{r-r}{r+r}u(a) \le u(a+re^{iq}) \le \frac{r+r}{r-r}u(a)$$

Chapter 7.1

Definition. Let G be a region and let (Ω, d) be a complete metric space. Then, $C(G,\Omega) = \ldots$

Proposition. Let G be a region. Then there exists a sequence of subsets $\{K_n\}$ of G such that

(i) $K_n \subset\subset G$

(ii) $K_n \subset \operatorname{int}(K_{n+1})$

(iii) $\bigcup_{n=1}^{\infty} K_n = G$

(iv) $K \subset\subset G$ implies $K \subset K_n$ for some $n \in \mathbb{N}$

Lemma If (S,d) is a metric space, then (S,m) is a metric space, where $m(s,t) = \dfrac{d(s,t)}{1+d(s,t)}$. A set O is open in (S,d) if and only if O is open in (S,m) .

Definition. For $K \subset\subset G$ and $f,g \in C(G,\Omega)$, let $r_K(f,g) = \sup_{z\in K} d(f(z), g(z))$,

$s_K(f,g) = \dfrac{r_K(f,g)}{1+r_K(f,g)}$, $B_{r_K}(f,d) = \{g : r_K(f,g) < d\}$.

Definition. For $\{K_n\}$ a compact exhaustion of a region G and for $f,g \in C(G,\Omega)$ let $r(f,g) = \ldots$

Proposition. $(C(G,\Omega), r)$ is a metric space.

Lemma 1.7 (i) Given $e > 0$ there exists $d > 0$ and $K \subset\subset G$ such that for $f,g \in C(G,\Omega)$

$$r_K(f,g) < d \;\Rightarrow\; r(f,g) < e$$

(ii) Given $d > 0$ and there exists $e > 0$ such that

$$r(f,g) < e \;\Rightarrow\; r_K(f,g) < d$$

Lemma 1.10 (i) A set $O \subset C(G,\Omega)$ is open if and only if for each $f \in O$ there exists $d > 0$ and $K \subset\subset G$ such that $O \supset B_{r_K}(f,d)$

(ii) A sequence $\{f_n\} \subset C(G,\Omega)$ converges to f (in the r metric) if and only if for each $K \subset\subset G$ $\{f_n\}$ converges to f in the r_K metric.

Proposition $(C(G,\Omega), r)$ is a complete metric space.

Definition. A set $F \subset C(G,\Omega)$ is normal . . .

Proposition. A set $F \subset C(G,\Omega)$ is normal if and only if $\overline{F}$ is compact.

Proposition. A set $F \subset C(G,\Omega)$ is normal if and only if for each $d > 0$ and $K \subset\subset G$ there exist functions $f_1, f_2, \cdots, f_n \in F$ such that $F \subset \bigcup_{k=1}^{n} B_{r_K}(f_k, d)$.

Definition. A set $F \subset C(G,\Omega)$ is equicontinuous at a point $z_0 \in G$ if . . .

Definition. A set $F \subset C(G,\Omega)$ is equicontinuous on a set $E \subset G$ if . . .

Proposition. Suppose a set $F \subset C(G,\Omega)$ is equicontinuous at each point of G. Then, F is equicontinuous on each $K \subset\subset G$.

Arzela-Ascoli Theorem A set $F \subset C(G,\Omega)$ is normal if and only if (a) for each $z \in G$ the orbit of z under F, *i.e.*, $\{f(z) : f \in F\}$, has compact closure and (b) $F \subset C(G,\Omega)$ is equicontinuous at each point of G.