Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

Notation: Let $G$ be a region in $\mathbb{C}$. Then, $\mathcal{A}(G) = \{f: f$ is analytic on $G\}$.

1. (20 pts) Prove the following theorem: Let $G$ be a region in $\mathbb{C}$ and let $f \in \mathcal{A}(G)$. If there exists $a \in G$ such $|f(a)| \geq |f(z)|$ for all $z \in G$, then $f$ is constant.

2. (21 pts) Give the definition for each of the following:
   - Let $G$ be a region in $\mathbb{C}$ and let $(\Omega, d)$ be a complete metric space. Then, $\mathcal{C}(G, \Omega) =$ \ldots
   - Let $G$ be a region in $\mathbb{C}$ and let $f : G \to \mathbb{R}$. $f$ is harmonic on $G$ if \ldots
   - Let $G$ be a region in $\mathbb{C}$ and let $K \subset \subset G$. Then, for $f, g \in \mathcal{C}(G, \Omega)$, $\rho_K(f, g) =$ \ldots

3. (20 pts) Let $G$ be a region in $\mathbb{C}$ and let $\mathcal{F} \subset \mathcal{C}(G, \Omega)$.
   - State the definition for $\mathcal{F}$ being normal
   - State three (3) equivalent conditions for $\mathcal{F}$ being normal

4. (20 pts) Find the number of zeros of $p(z) = z^5 - 15z^4 + 5z^3 - 5z^2 + 50z - 17$ in the annulus $\text{ann}(0; 1, 2)$

5. (20 pts) Let $\mathbb{D} = \{z : |z| < 1\}$ and let $H = \{x + iy : x > 0\}$. Let $\mathcal{F} = \{f \in \mathcal{A}(\mathbb{D}) : f(\mathbb{D}) \subset H, f(0) = 1\}$. Prove that $\max_{f \in \mathcal{F}} |f'(0)| \leq 2$. 