Math 5321 Exam II 14 Apr 2014

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

Notation: Let G be a region in $\mathbb{C}$. Then, $\mathcal{A}(G) = \{f : f \text{ is analytic on } G\}$.

1. (20 pts) Prove the following theorem: Let G be a region in $\mathbb{C}$ and let $f \in \mathcal{A}(G)$. If there exists $a \in G$ such $|f(a)| \geq |f(z)|$ for all $z \in G$, then f is constant.

2. (21 pts) Give the definition for each of the following:
   - Let G be a region in $\mathbb{C}$ and let (Ω, d) be a complete metric space. Then, $\mathcal{C}(G, \Omega) = \cdots$
   - Let G be a region in $\mathbb{C}$ and let $f : G \to \mathbb{R}$. f is harmonic on G if $\cdots$
   - Let G be a region in $\mathbb{C}$ and let $K \subset\subset G$. Then, for $f, g \in \mathcal{C}(G, \Omega)$, $\rho_K(f, g) = \cdots$

3. (20 pts) Let G be a region in $\mathbb{C}$ and let $\mathcal{F} \subset \mathcal{C}(G, \Omega)$.
   - State the definition for $\mathcal{F}$ being normal
   - State three (3) equivalent conditions for $\mathcal{F}$ being normal

4. (20 pts) Find the number of zeros of $p(z) = z^5 - 15z^4 + 5z^3 - 5z^2 + 50z - 17$ in the annulus $ann(0; 1, 2)$

5. (20 pts) Let $\mathbb{D} = \{z : |z| < 1\}$ and let $H = \{x + iy : x > 0\}$. Let $\mathcal{F} = \{f \in \mathcal{A}(\mathbb{D}) : f(\mathbb{D}) \subset H, f(0) = 1\}$. Prove that $\max_{f \in \mathcal{F}} |f'(0)| \leq 2$.