MATH 5321 Exam I 28 February 2013

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

1. (10 pts) Give the definition for each of the following:

   a. Let $\gamma_0, \gamma_1 : [0,1] \to G$ be two closed rectifiable curves lying in a region G;

      then γ_0 is *homotopic* to γ_1 in G if . . .

   b. A function f has an *isolated singularity* at $z = a$ if . . .

2. (16 pts) State each of the following theorems, giving attention to the correct hypotheses for each:

   a. Open Mapping Theorem

   b. Residue Theorem

3. (21 pts) Identify each of the finite isolated singularities for each of the following functions. Determine whether the singularities are removable singularities, poles or essential singularities. If the singularity is removable, determine what value the function should be assigned at the singularity to analytically extend the function at the singularity. If the singularity is a pole, determine the order of the pole.

   a. $f(z) = \dfrac{z^6 - 1}{z^4 - 1}$ b. $f(z) = \dfrac{1}{e^z - 1} - \dfrac{1}{z}$

   c. $f(z) = z^2 \sin\dfrac{1}{z}$ d. $f(z) = \dfrac{z^2 - 2z}{z(1-z)(2-z)^2}$

4. (21 pts) Each of the following functions has exactly one isolated singularity inside the disk $B(1;\tfrac{3}{2})$. Determine the reside of the function at that isolated singularity.

   a. $f(z) = \dfrac{\sqrt{6z}}{\cos(z)}$ b. $f(z) = \dfrac{(1-z^4)e^{-1/z}}{3z}$

   c. $f(z) = \dfrac{\log(z)}{(z-1)^3}$

5. (20 pts) Prove the following corollary to Cauchy's Theorem (Version #4):

Corollary. Let G be a region in $\mathbb{C}$ which is simply connected and let $f \in \mathscr{A}(G)$. Then f has a primitive on G.

6. (12 pts) Classify each of the following regions at to whether they are convex or not convex, starlike (with respect to some point $a \in G$) or not starlike (with respect to any point $a \in G$) , simply connected or not simply connected.

N.B. Mark each cell in the table as Y or N, i.e., do not leave any cell in the table blank.

Notation:

- (a,b) denotes the (open) straight line segment between a and b
- [a,b] denotes the (closed) straight line segment between a and b.
- ((a, b, c, d)) denotes the (open) interior of the quadralateral with vertices a, b, c, d .
- Q_1 denotes the first quadrant.

Use the table on the next page to report your answers

a. $G_a = B(0,1) \setminus \{(-1,-\frac{1}{2}] \cup [\frac{1}{2},1)\}$

b. $G_b = B(0,1) \setminus [-\tfrac{1}{2}, \tfrac{1}{2}]$

c. $G_c = B(0,1) \setminus \overline{B(\tfrac{1}{2};\tfrac{1}{2})}$

d. $G_d = B(0,4) \setminus B(\dfrac{7}{6}, \dfrac{25}{6})$

e. $G_e = B(0,1) \cup B(2,2)$

f. $G_f = Q_1 \setminus \{[2, 2+i] \cup [2i, 2i+1]\}$

g. $G_g = ((0,\, 3+i,\, 2+2i,\, i)) \cup (0,i) \cup ((0,\, i,\, -2+2i,\, -4+i))$

h. $G_h = H_1 \cap H_2 \cap H_3$ where $H_1 = \{z : \operatorname{Re} z > -\tfrac{1}{2}\}$, $H_2 = \{z : \operatorname{Im} z > -1\}$, $H_3 = \{z : \operatorname{Re}((1-i)z) < 2\}$

Name ______________________________ (Return this sheet with your other problems)

	Convex	Starlike	Simply Connected		Convex	Starlike	Simply Connected
G_a				G_e			
G_b				G_f			
G_c				G_g			
G_d				G_h			