MATH 5321 | Exam I | 16 February 2009

Take Home

Due: February 23

Answer the problems on separate paper. You do not need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **Show all relevant supporting steps!**

1. (8 pts)

(a) Give an example of a region G such that (i) G is starlike with respect to 0 **and** (ii) there does not exist a point $a \neq 0$ such that G is starlike with respect to a

(b) A region G is said to be *convex in the imaginary direction* if for any vertical line $l_a = \{z \;:\; \operatorname{Re} z = a\}$ the intersection of G with l_a is a single interval (possibly empty, possibly unbounded). Give an example of a region G which is convex in the imaginary direction **but** not convex.

2. (18 pts) On page 88, it is shown that "~" is an equivalence relation on closed rectifiable curves. Consider the following two-way table:

	Exactly 1 Equivalence Class under "~"	Exactly 2 Equivalence Classes under "~"	Infinite Number of Equivalence Classes under "~"
Open Set which is a Region			
Open Set which is not a Region			

For each of the six possible classifications listed in the table either give an example which meets the classification criteria or determine that no such example exists.

3. (24 pts) Find the Laurent series expansion for $f(z) = \dfrac{z^2 - z + 1}{(z-2)^2(z-1)}$ on:

(a) $ann(0;1,2)$; (b) $ann(1;0;1)$; (c) $ann(1;1,2)$; (d) $ann(i;\sqrt{2},\sqrt{5})$

4. (24 pts) Identify each of the finite isolated singularities for each of the following functions. Determine whether the singularities are removable singularities, poles or essential singularities. If the singularity is removable, determine what value the function should be assigned at the singularity to analytically extend the function at the singularity. If the singularity is a pole, determine the order of the pole and, if the pole occurs at $z = 0$, then determine the singular part of the function at 0.

(a) $f(z) = \dfrac{z+1}{z^2(z^2-1)}$ (b) $f(z) = \dfrac{\log(1+z)}{z(\exp(iz)-1)}$

(c) $f(z) = \dfrac{\sin(2\pi z)}{z^4(z^2-1)}$ (d) $f(z) = \dfrac{\exp(2\pi i / z)-1}{z(z^2-1)}$

5. (14 pts) Each of the following functions has a pole at $z = 0$. Determine the order of the pole at $z = 0$ and the determine the residue of the function at $z = 0$.

(a) $f(z) = \dfrac{z^3 \sin(2z^3)}{\cos(z^4)-1}$ (b) $f(z) = \dfrac{(z\sin(3z))^2}{(1-\cos(z^2))^2}$

6. (16 pts) Compute the value of each of the following integrals using the Residue Theorem:

a. $\displaystyle\int_0^\infty \frac{\cos 3x}{(x^2+1)(x^2+4)}\,dx$ b. $\displaystyle\int_0^\infty \frac{x^2}{(x^2+9)^2}\,dx$

For each integral justify that your computation is a valid application of an appropriate theorem.