MATH 5321 Exam II April 7, 2005

Take Home

Due: April 15

Answer the problems on separate paper. You do <u>not</u> need to rewrite the problem statements on your answer sheets. Work carefully. Do your own work. **<u>Show all relevant supporting steps!</u>**

1. Compute the residue at $z = 0$ for each of the following functions:

   a. $f(z) = \dfrac{\sin 3z^2}{\cos p z^2 - 1}$ b. $f(z) = (1 + 2z - z^2 + 2z^3 - z^4 + 2z^5 + z^6)e^{-2/z^2}$

2. Compute the value of each of the following integrals using the Residue Theorem:

   a. $\displaystyle\int_0^\infty \frac{\cos 2x}{(x^2+1)(x^2+4)}\,dx$ b. $\displaystyle\int_0^\infty \frac{\sin 2x}{x(x^2+1)^2}\,dx$

3. Using either the Argument Principle or Rouche's Theorem determine how many solutions $z^4 + z^3 = 2z^2 - 2z - 4$ has in the first quadrant.

4. Let $f \in \mathcal{A}\ (B(0,r))$. Suppose that on $B(0,r)$ that $\text{Re } f(z) < A$. If $f(0) = 0$, then show that for $z \in B(0,r)$ that $|f(z)| \le \dfrac{2A|z|}{r - |z|}$.

5. Let D denote the open unit disk (centered at 0) and let UHP denote the upper half-plane. Let $F = \{f \in \mathcal{A}\ (D) : f : D \to UHP,\ f(0) = 2i\}$. Find the value of $a = \max\limits_{f \in F} |f'(0)|$.

6. Suppose that u is harmonic on $\mathbb{C}$ and that for each $z \in \mathbb{C}$ that $u(z) > -2$. Prove that u is constant.

7. Let G be a region. Let u, v be harmonic on G.
   a. Prove that if u, v are harmonic conjugates, then the product uv is again harmonic on G.
   b. Give an example to show that u, v are not harmonic conjugates, then the product uv need not be harmonic on G.

8. Let G be a region. We say that $u \in \mathcal{C}\ (G, \mathbb{R})$ is subharmonic on G if for each $a \in G$ that $u(a) \le \dfrac{1}{2p}\displaystyle\int_{-p}^{p} u(a + re^{iq})\,dq$ for r sufficiently small. Prove that if u is subharmonic on G, then u satisfies the Maximum Principle (Ver. #1) on G, i.e., if there exists $a \in G$ such that $u(a) \ge u(z)$ for all $z \in G$, then u is constant on G.

9. Prove Dini's Theorem (Problem 6 on page 150).