Take Home Examination Math 5321 March 25, 1998

Due: March 30th 100 Points

Notation:

$B(a,r) = \{ z : |z - a| < r \}$ $C(a,r) = \{ z : |z - a| = r \}$

$D = B(0,1)$ Q_1 = First Quadrant UHP = Upper-half Plane

$\mathcal{A}(G) = \{ f : f$ is analytic on $G\}$ $\mathcal{M}(G) = \{ f : f$ is meromorphic on $G\}$

For $f \in \mathcal{A}(B(a,R))$ and $0 < r < R$

$$M_a(r) = \max_{z \in C(a,r)} |f(z)| \qquad A_a(r) = \max_{z \in C(a,r)} \operatorname{Re} f(z)|$$

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5. Let $f \in \mathcal{A}(B(a,R))$ be such that $f(a) = 0$ and $|f(z)| \le M$ on $B(a,R)$. Show that $|f(z)| \le \frac{M}{R} |z - a|$ with equality if and only if $f(z) = \lambda (z - a)$, where $|\lambda| = \frac{M}{R}$. (This is a generalization of Schwarz's Lemma.)

2. Find the number of zeros of f in G:

   a. $f(z) = z^7 + 6z^3 + 7$, $G = Q_1$

   b. $f(z) = z^4 + 3iz^2 + z - 2 + i$, $G = UHP$

3. Let $B_{1,1} = \{ f \mid f \in A(D),\ f(0) = 1,\ and\ f(D) \subset B(1,1) \}$. For $f \in B_{1,1}$, let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on D.

   a. Show that if $f \in B_{1,1}$, then $|a_n| \le 1$, for n = 1, 2, ...

b. Show that if $f \in B_{1,1}$, then $|a_n| \le \sqrt{1 - |a_1|^2}$ for $n = 2, 3, 4, \ldots$ (which, of course, implies that if $|a_1| = 1$, then $a_n = 0$ for $n = 2, 3, 4, \ldots$).

4. a. Show for $a \in D$ that

$$\int_0^{2\pi} \log |1 + ae^{i\theta}|\, d\theta = 0$$

b. Show for $a \in D$ and $|a| < r < 1$ that

$$\frac{1}{2\pi}\int_0^{2\pi} \log |a - re^{i\theta}|\, d\theta = \log r$$

c. Let $f \in \mathcal{M}(B(0,R))$ with $f(0) \ne 0$. Let Z_f denote the zero set of f and P_f the pole set of f. Let $0 < r < R$ be such that $Z_f \cap C(0,r) = \varnothing$ and $P_f \cap C(0,r) = \varnothing$. Let $\{a_1, a_2, \ldots, a_n\} = Z_f \cap B(0,r)$ and $\{b_1, b_2, \ldots, b_m\} = P_f \cap B(0,r)$. Show that

$$\frac{1}{2\pi}\int_0^{2\pi} \log |f(re^{i\theta})|\, d\theta = \log |f(0)| + \sum_{k=1}^{n} \log \frac{r}{|a_k|} - \sum_{l=1}^{m} \log \frac{r}{|b_l|}$$

5. Let $f \in \mathcal{A}(B(a,R))$ such that $f(a) = 0$. Show that g defined by $g(r) = \dfrac{M_a(r)}{r}$ is an increasing function of r and that if g is non-constant, then g is strictly increasing.

6. Let $f \in \mathcal{A}(B(a,R))$ such that $f(a) = 0$. Show that $M_a(r) \le \dfrac{2r}{R-r} A_a(R)$.

Hint: Consider $g(z) = \dfrac{f(z)}{2A_a(R) - f(z)}$ and apply Schwarz's Lemma.