MATH 5321 Exam I February 16, 1998

Notation.

$UHP = \{ z : \operatorname{Im} z > 0 \}$ $\mathcal{A}(G) = \{ f : f$ is analytic on $G \}$

1. Let f be a rational function and let P_f denote the pole set of f. Prove that if $\deg(f) \leq -2$ and $P_f \cap \mathbb{R} = \varnothing$, then $\int_{-\infty}^{\infty} f = 2\pi i \sum_{a_k \in P_f \cap UHP} \operatorname{Res}(f, a_k)$.

2. Let γ be a rectifiable curve and ϕ a continuous function on $\{\gamma\}$ (into $\mathbb{C}$). Define $F(z) = \int_\gamma \frac{\phi(w)}{(w-z)^2}\, dw$. Prove that F is continous on $\mathbb{C} \setminus \{\gamma\}$ (into $\mathbb{C}$).

3. Let G be a simply connected region and let $f \in \mathcal{A}(G)$ such that f is nonvanishing on G. Prove that there exists a branch of square root of f on G, i.e., prove there exists $g \in \mathcal{A}(G)$ such that $g^2 = f$ on G.

4. Let $f \in \mathcal{A}(\mathbb{C})$. Prove that if f is not a polynomial, then $g = f(1/z)$ has an essential singularity at $z = 0$.

5. Let $S_f = \{ \int_{|z| = \frac{1}{4}} z^n f(z)\, dz \mid n \in Z \}$. Determine for each of the following functions:

(i) Whether S_f is bounded?
(ii) Whether S_f has any accumulation points?;
(iii) If the answer to ii is affirmative, then identify the accumulation points of S_f.

(a) $f(z) = \frac{z}{1+z^2}$ (b) $f(z) = \sin(\pi z)$ (c) $f(z) = \frac{1}{1-2z}$